On the Nature of $a_k^* a_k$ and the Emergence of the Born's Rule

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This paper is intended to show that a review in the concept of the game theoretical utility, the revised utility to be applied to the definition of the utility of a wave function representing an object subsystem relative to its observer subsystem, both within an isolated system, leads to the emergence of the Max Born's rule as a profit under a von Neumann's good measure game.

POSITION OF THE PROBLEM

The quantum mechanics is constructed upon a crucial pillar connecting the quantum and classical worlds: the Max Born's postulate. The essence of this postulate, or simply Born's rule, is:

- A quantum object $\sum_{\forall k} a_k \phi_k$, represented in an orthonormal basis $\{\phi_k\}$ that consists of eingenstates of a physical quantity $\mathbf{A}$, *instantaneously* collapses onto an eingenstate (or subspace) of $\mathbf{A}$ (say ϕ_k). This occurs with probability $a_k^* a_k$, when $\mathbf{A}$ is measured, and when the eigenvalue α_k of $\mathbf{A}$ is obtained.

There are a lot of tastes to write the Born's rule, but the above is sufficient to our purposes. Under the canonical formulation of the quantum mechanics, both, the collapse and its probability, do not permit the investigation of a mechanism regarding the probabilistical character encrusted to the collapse, since both are *ad hoc* assumptions which are pillars of the theory. The probability is inherent to the collapse.

Instead of that axiomatic *ultimatum*, this paper focuses on a new approach to obtain the Born's rule, i.e., it is concerned with the answers to the following questions:

- Why $a_k^* a_k$ emerges as an usefull scale of measure connecting a quantum object to its observer?
- Why this usefull scale is perceived as probabilities of collapse onto eigenstates ϕ_k?

We will show that the definition of the game theoretical utility, often misunderstood as a probabilistic defined object inherent to the axiomatic structure of the theory of games due to von Neumann and Oskar Morgenstern, leads to the first answer. A crucial question will emerge: What is the utility of a wave function representing a quantum object? Answering this question, the connection between utility and $a_k^* a_k$ appears, and the first question turns out to be answered.

Here, one may argue that the game theoretical approach was already done by David Deutsch [1]. The point is: the approach used in this paper is non-circular, it does not have inherent taulology. The utility interpretation used here is free of circularity, representing an essential step forward in the decoherence problem: some state of affairs regarding hidden utilitary mechanism used by nature, connecting the quantum and the classical, seems to be a seminal door to a new interpretation of the wave function collapse. Probabilities will emerge as a consequence of the randomness inherent to the registers in the memory configuration of the observer, but the utilitary scale obtained here, $a_k^* a_k$, will be free of an aprioristical interpretation. In other words: we will show that the unique thing usefull to the interpretation of the reality of a quantum object is connected to the quantities $a_k^* a_k$; we will show that $a_k^* a_k$ is also the utility of collapse onto an eigenstate ϕ_k of the physical quantity $\mathbf{A}$; the connection between $a_k^* a_k$ and the probability of collapse will emerge in virtue of the randomness inherent to the memory configuration of the observer under a von Neumann's good measure game (and the second of the above questions will be answered). In virtue of these facts, the Born's rule will become clear and will emerge consequently.

[2] Early approaches that aimed at an understanding of probabilities in the [3] relative-state framework in terms of relative frequencies have been shown to be circular, tautological. Initially, decoherence was thought to provide a natural account of the probability concept in a relative-state framework. The idea was to relate the diagonal elements of the decohered reduced density matrix to the collection of possible events and to interpret the corresponding coefficients as relative frequencies of branches. Since decoherence enables one to reidentify the individual localized components of the wave function over time (describing, for example, observers and their measurement outcomes attached to well-defined branches), this leads to an interpretation of the Born probabilities as empirical frequencies. However, this argument cannot yield a noncircular derivation of the Born rule, since the formalism and interpretation of reduced density matrices presume this rule. Attempts to derive probabilities from reduced density matrices are therefore circular [4, 5].

[2] A derivation that is based on the nonprobabilistic axioms of quantum mechanics and on elements of classical decision theory has been presented by Deutsch [1] (see

also the critique by Barnum et al. [6] and the subsequent defense by Gill [7] and Wallace [8]; Saunders [9] embedded Deutsch's derivation into an operational framework). However, it is important to realize that such decision-theoretic approaches are subject to the same charge of circularity pointed out in the previous paragraph. This is so because these approaches first need to define the classical events (outcomes) to which probabilities are to be assigned. If one starts from pure states as Deutsch's derivation does, one would need to (i) justify the identification of the states with the possible events, and (ii) show that the phase relations between these states are irrelevant, i.e., do not influence the decision of the observer. The approach of Deutsch does not address these issues. Thus it tacitly uses environment-induced superselection (which selects the set of events that can be observed) and decoherence (which explains the irrelevance of phase relations from the view of the local observer), while it fails to supply a derivation of these processes in a manner that does not presume Born's rule.

[2] The solution to the problem of understanding the meaning of probabilities and of deriving Born's rule in a relative-state framework must therefore be sought on a much more fundamental level of quantum mechanics. Quantum information theory has established the notion that quantum mechanics can be viewed as a description of what, and how much, information nature is willing to proliferate [10]. For example, a peculiar feature of quantum mechanics is that complete knowledge of a global pure bipartite quantum state does not appear to contain any information about the absolute state of one of the subsystems. This hints at ways how a concept of objective ignorance, and therefore of objective probabilities, may emerge directly from the quantum feature of entanglement without any classical counterpart. This idea has recently been developed under the heading of environment-assisted invariance, or envariance for short, in a series of papers by Zurek [4, 11–13]. The motivation and spirit of this approach is strongly based on decoherence. Given a set of assumptions, envariance leads to a derivation of quantum probabilities and Born's rule.

Although Zurek's approach leads to a derivation of Born's rule, our approach is totally different, since we start from a game theoretical point of view. The derivation in this paper seems to be much more fundamental, since Born's rule emerges as a particular interpretation of reality, i.e., we will show that the relevant mathematical object is the utility and the probabilistic interpretation is a consequence of the von Neumann's good measure postule. In other words: in a scenario between the instants $t = 0$ and $t = \tau$, where τ is the duration of a measure, we cannot conclude without a priori axiom that $a_k^*(t)a_k(t)$ is the probability of collapse, since there is no collapse at all, but $a_k^*(t)a_k(t)$ still remains the unique thing usefull to the observer in describing the quantum reality of the measured object, although a randomness characteristic may be ausent in the observer memory configuration; surely, at $t = \tau$, when von Neumann's good measure takes place, Born's rule emerges as a permitted description - but it is a particular scenario as we will see; we will show (and propose thereafter) that utility $a_k^*a_k$ is always fundamental as a measure scale, but probability may emerge as utility under von Neumann's good measure assumption when it becomes an usefull description of reality.

ON ABSOLUTE EXISTENCE OF ABSTRACT UTILITIES

The corner stone of the game theoretical object, the abstract utility, that provides a non-circular interpretation of its use in obtaining a result regarding an emergence of probability, our key to obtain the Max Born's rule, is the absolute existence of utilities.

Indeed, regarding the absoluteness of the utilities, the main axiom upon which the utility is stablished in game theory [14] asserts:

- ***There is*** a set $\mathbb{U}$ *of abstract objects* $u, v, w, \cdots$. $\mathbb{U}$ *is equipped* with a relation denoted by $u > v$, such that $\forall\ q \in \mathbb{R} \mid 0 < q < 1$, $\mathbb{U}$ *is also equipped* with an operation denoted by:

$$qu + (1-q)\,v = w, \tag{1}$$

i.e., the existence of the abstract utilities $u, v, w, \cdots$ is absolute [17], the equipments (the relation and the operation in the above axiom) are properties of the set $\mathbb{U}$ and, of course, cannot alter the absolute existence of the elements in the set. We cannot interpret the operation as $qu + (1-q)\,v \equiv w$, if it is a property of the set, i.e., such interpretation would be correct as a property of the utilities. Suppose a game theoretical situation in which two different observers take exactly the same utility w regarding a object. They prefer the object exactly in the same sense. Their different memory configurations in virtue of their very own experiences could imply different intuitions, resulting in different relations between their worst and best scenarios to construct the same preference measure w. One could construct $w = 0.2u + 0.8v$. The other one, $w = 0.3m + 0.7n$. The set $\mathbb{U}$ permits these constructions. It is a property of the set. The utilities remain absolute, i.e., they are not defined by the operation $qu + (1-q)\,v \equiv w$, and the true identity $\equiv$ is mistaken. $\mathbb{U}$ provides an operation to the observers that permits the construction of their preferences using worst and better utility blocks, but the same utility obtained, w, via these constructions, was already in the set. It cannot be different, since the axiom does not mention the existence of observers. The remaining axioms of utility in game theory are just concerned with the equipments of $\mathbb{U}$.

ON HEURISTICS OF UTILITY

Here we start to use the calligraphical notations to denote devices and events. A device will be an hypothetical *gedankenexperiment* composed by an irrelevant mechanism that picks out by lottery mutually exclusive outcomes. An event will be understood via its inherent context.

The analysis via heuristic arguments that provides the interpretation of (1) is very often misunderstood, i.e., the real numbers q are often and erroneously thought as being *defined* via probabilities.

In the next two sections, we argue that probabilities are not fundamental in utility theory, but probability may emerge from the real numbers q. One may argue that this fact is irrelevant, since there is no mention to probabilities in the main axiom. We want to asseverate that a fundamental presence of probability in the main axiom would be the source of a tautological derivation of Born's rule from utility theory. We also want to show how probability emerges from q, since it must occur at some point of our derivation as requested by Born's rule.

Let $\mathcal{X}$ be a device that picks out by lottery two mutually exclusive outcomes, the event $\mathcal{C}$ with *a priori* prescribed probability α, or the event $\mathcal{B}$ with consequently prescribed probability $1-\alpha$. These prescriptions in the definition of the event $\mathcal{X}$ suppose an *ad hoc* presence of the probability information regarding the frequencies of the outcomes $\mathcal{C}$ and $\mathcal{B}$ in the complete set of possible outcomes.

Now, consider an observer i. Consider a case in which i is in a situation of certainty ($\alpha = 1$) regarding the event $\mathcal{C}$. After his very own considerations, i finally chooses an element c_i in $\mathbb{U}$. c_i is the abstract utility picked up by i to represent i's sense of goodness regarding event $\mathcal{C}$ in a case of certainty. Now, consider a case in which i is in a situation of certainty ($\alpha = 0$) regarding the event $\mathcal{B}$. After his very own considerations, i finally chooses an element b_i in $\mathbb{U}$. b_i is the abstract utility picked up by i to represent i's sense of goodness regarding event $\mathcal{B}$ in a case of certainty. In other words, there are two elements of $\mathbb{U}$, c_i and b_i, being i's absolute (absolute $\equiv$ in a situation of certainty) abstract utilities regarding events $\mathcal{C}$ and $\mathcal{B}$.

Under circunstancies of a complete avaiable information about the probabilities of $\mathcal{X}$'s outcomes, one would define the i's abstract utility x_i regarding event $\mathcal{X}$:

$$x_i \equiv \alpha c_i + (1-\alpha)\, b_i, \tag{2}$$

as the expectation abstract value.

Now, instead of the above situation, let the observer i be in the very specific situation of a complete information about his order of preferences, without devices, i.e., i uses $\mathbb{U}$ and its structure and i's very own intuitions to pick up the elements b_i, a_i and c_i in $\mathbb{U}$ to stablish i's order of preferences $b_i < a_i < c_i$ regarding the goodness of the events $\mathcal{B}$ and $\mathcal{C}$ above plus an intermediate preferential event $\mathcal{A}$. We asseverate again that, now, there is not any lottery device. As argued in the previous section, i uses worst and better preferential blocks, respectively b_i and c_i in $\mathbb{U}$, as a basis to pick up the utility a_i in $\mathbb{U}$ regarding the intermediate scenario: the event $\mathcal{A}$. As asserted in the main axiom, i uses real numbers as weights, i.e.:

$$a_i = q c_i + q^{'} b_i, \quad q\,,\, q^{'} \in \mathbb{R}. \tag{3}$$

Since $\mathcal{A}$ cannot be worst than $\mathcal{B}$ neither better than $\mathcal{C}$, both $q,\, q^{'} \in (0,1)$. Putting $q + q^{'} = q^{''}$:

$$0 < q < 1 \Rightarrow 0 < q^{''} - q^{'} < 1 \Rightarrow q^{''} - 1 < q^{'} < q^{''} \therefore$$

$$q^{''} = 1. \tag{4}$$

Therefore, the equation (3) becomes:

$$a_i = q c_i + (1-q)\, b_i, \tag{5}$$

i.e., the equation (1) with no mention to probabilities, since neither probabilistic devices nor probabilities are being explicitally supposed.

One may argue that we did not stablish a proof that the q numbers are not really probabilities, i.e., that we cannot guarantee that i does not implicitally mount a_i via his probabilistic intuitions. This one arguing must remember that there is no mention to probabilities in the main axiom. This one arguing must remember that probabilities, in any case, from lottery devices or from i's intuitions, would be emerging externally to the main axiom. The main axiom provides this possibility, since q is a real number in an interval that permits such interpretations. Is there a counter-example showing that probabilities must not necessarily be incrusted when i uses the q weights from his intuitions to construct and pick up utilities in $\mathbb{U}$? The answer is yes: Daniel Kahneman showed this important characteristic of utilitary choices, becoming a Nobel prize laureate in economics (please, read footnote 16 again).

We conclude this section asseverating that if one starts from game theoretical utility theory, one is not aprioristically carrying incrusted probabilities, but probabilities may emerge as we will show in the next section.

WHEN q BECOMES α?

In the previous section, the events $\mathcal{A}$ and $\mathcal{X}$ were enunciated in totally different essences. E.g., suppose $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{B}$ as different vehicles to go from city 1 to city 2 whose way between these cities is 100 kilometers long by road. Suppose $\mathcal{C}$ as a carriage and $\mathcal{B}$ as a BMW automobile. Remember that $\mathcal{X}$ still remains the device enunciated in the previous section. Now, suppose that i must choose between $\mathcal{A}$ or $\mathcal{X}$ to go from city 1 to 2.

The absolute utility of $\mathcal{A}$ is given by (5). However, (2) cannot be *aprioristically* tought as a definition of the absolute utility of $\mathcal{X}$, since the internal probabilities in $\mathcal{X}$, α and $1-\alpha$, are being externally imposed/prescribed.

Incidentally, if the vehicle $\mathcal{A}$ and the lottery device $\mathcal{X}$ were equally preferred by i, then, by (5) and (2), $a_i = x_i \Rightarrow q = \alpha$, i.e., q would incidentally be interpreted as a probability α - but just in an incidental and *a posteriori* manner. An aprioristical interpretation of the q numbers in the game theoretical utility as being probabilities must be rejected in virtue of the Kahneman's Nobel prize - and q remains a real number in $(0,1)$ with no aprioristical interpretation.

Furthermore, a different value to the prescribed α, say $\alpha^{'} \neq \alpha$, would generate a case in which an *a posteriori* probabilistic interpretation could not occur, since $\mathcal{A}$ and $\mathcal{X}$ would not be equally preferable by i. Also, a different observer, say $j \neq i$, would not necessarily be in a congruous position of total ignorance when j was to choose $\mathcal{A}$ or $\mathcal{X}$, with the same externally prescribed α - since j's very own q could be different in virtue of j's very own intuitions.

One would argue that a typical i has an internal picture, a way of reasoning, of a device like $\mathcal{X}$ such that i prescribes by himself a probability q when utilitary reasoning must be done. Unfortunatelly, the Kahneman's Nobel prize does not permit such hypothetical and aprioristically taken mechanism of reasoning. Again, one may argue that the mechanism of reasoning to pick up utilities from $\mathbb{U}$ are being uncorrectly evaluated, i.e., the Kahneman's results are a consequence of untrained observers concluding that the q must be probabilities in an perfect scenario of mathematically well trained observers. Again, the argument is mistaken in virtue of the fact that the untrained observers would be using q as real numbers before, as wrong probabilities, demonstrating that q could not be the correct probability. Since there is not two kinds of probability, once defined, q is fundamentally a real number. One important point must be asseverated: although von Neumann and Morgenstern had used probabilistic heuristical reasoning regarding the hypothetical meaning of the q numbers, which is taught in game theory text books as being a definition to be used in cases of perfect rational observers, von Neumann and Morgenstern did not asseverated this fact in the axiomatic structure. q simply appears as real numbers in $(0,1)$. A strong form to be used only in cases of perfectly rational observers needs the additional hypothesis defining q as probabilities of hypothetical devices (hypothetical lottery situations) to be used by these perfectly rational observers when these perfectly rational observers must mount their utilities. This strong form would reject the massive class of observers that perform preferential choices without being perfectly reasonable, these ones studied by Kahneman. If one naturally assumes q as real number without aprioristical interpretation, one utilizes the utility machinery in a general sense, permitting all classes of observers, in a metatheoretical spirit, since Kahneman's results must be in agreement with the theory, since Occam's razor recommends no additional hypothesis and, fundamentally, just because there is no mention to probabilities in the main axiom.

Concluding this section and asseverating: the q numbers in (1) are simply real numbers in $(0,1)$ that may, *of course*, *eventually* emerge as probabilities in cases in which q is related to *ad hoc hypothetical* lottery situations/devices generating ignorant feeling regarding i's utilitary reasoning.

ON UTILITY'S NUMERICAL REPRESENTATION AND THE INHERENT AFFINE TRANSFORMATION GROUP

Von Neumann and Oskar Morgenstern stablished the representability of the abstract utilities by real numbers, by the proven propositions in [14]. Since these theorems are proven under the assertions of the axioms of utility in [14], the numerical representation of the abstract utilities does not have any appeal to probabilities. Indeed, the *one and only* chance to incrust propabilities in the mentionated proofs would be via the q numbers, but we concluded that q is free of *a priori* probabilistic incrustations.

Moreover, in [14], Neumann and Morgenstern also stablished by those proofs that any two such numerical representations, say U_i, $U_i^{'}$, of the same absolute abstract utility u_i of an event $\mathcal{A}$ regarding an observer i, must be connected by an affine transformation such that:

$$U_i^{'}\left(u_i\left(\mathcal{A}\right)\right) = \omega_0 U_i\left(u_i\left(\mathcal{A}\right)\right) + \omega_1, \qquad (6)$$

where $\omega_0 \in \mathbb{R}_+^*$, $\omega_1 \in \mathbb{R}$.

ON RHETORICAL ASPECTS OF THE UTILITY OF A WAVE FUNCTION FOR AN OBSERVER

Let ψ^S (S denotes system) be the wave function of a quantum object. What is the *utility* for an observer i, concerning this wave function as a representation of a quantum object? The level of representation must be a mathematical one, since this is the spirit of the mathematics in physics. By the way, the answer is within the question. Any observer would agree: the utility of ψ^S is the representation of a quantum object. In a mathematical representation of a quantum object, a basis must be choosen. In any case, choosen or given/prescribed, the quantum object is absolute in existence, i.e., remains the same quantum object, remains independent of the basis choosen or given by anything external to it.

Since the quantum object to be mathematically represented is absolute, a choosen basis must be equivalent to a given/prescribed one regarding this mathematical representation of ψ^S, $\forall$ observers i. Hence, the utility of ψ^S's mathemathical representation must be exactly the same as the the utility of ψ^S's mathematical representation given a basis [18].

Now, we reformulate, equivalently, our question: Given a basis, what is the utility of the mathematical representation of ψ^S for an observer? Any observer would agree: The utility is on the knowledge of the set of coefficients of the representation in the given basis, since one would not be able to mathematically construct a quantum object without these coefficients.

ON UTILITY OF A WAVE FUNCTION FOR AN OBSERVER

The numerical utility U of a wave function ψ^S mathematically representing a quantum object S for an observer $\mathbb{O}$ (now, we change the notation regarding observers, $\mathbb{O}$ instead of i, in virtue of some confusion that may arise regarding the use of the lower case typo), being ψ^S decomposed in a complete basis of eingenstates ϕ_k, is:

$$U = U\left(\psi^S\right) = U\left(\sum_{\forall k} a_k \phi_k\right), \tag{7}$$

where the coefficients a_k are the coefficients of the decomposition.

In virtue of our previous rhetorical section, a quantum object can be decomposed in another basis. A basis having an eingenstate ϕ such that $a\phi = \sum_{\forall k} a_k \phi_k$ must be equally good/equivalent, i.e., the numerical utility of a quantum object ψ^S for an observer $\mathbb{O}$ is independent of the prescribed basis:

$$U\left(\psi^S\right) = U\left(\sum_{\forall k} a_k \phi_k\right) = U\left(a\phi\right),\ a\phi = \sum_{\forall k} a_k \phi_k, \tag{8}$$

where a is the only nonzero coefficient in the ϕ basis representation.

Invoking our previous rhetorical section, given a basis $\{\cdots, \phi, \cdots\}$, or $\{\phi_1, \cdots, \phi_k, \cdots\}$, the utility must be on the knowledge of the respective coefficients. In the ϕ_k basis, the relevant information regarding the utility of the ψ^S mathematical representation is on the knowledge of all the coefficients a_k. In the ϕ basis, since this basis automatically excludes the necessity of the knowledge of all the coefficients but one, the relevant information is on the knowledge of the unique a. In other words: both sets $\{a\}$, $\{a_1, \cdots, a_k, \cdots\}$ must have the same content of information for an observer $\mathbb{O}$ for purposes of utility, since both sets are sufficient to construct, to mathematically represent the quantum object by ψ^S. Hence:

$$U\left(\psi^S\right) = U\left(a\right) = U(a_1, a_2, \cdots, a_k, \cdots). \tag{9}$$

Suppose a general case in which the coefficients can be correlated. Since the representation of the wave function still needs all the coefficients to be written, some fortuitous correlation could not influence upon the fact: the utility of the wave function's mathematical representation remains on the knowledge of all the coefficients, in spite of the method used to accomplish this. Hence, correlated or not, the utility of ψ^S represented in the ϕ_k basis must be independent of the order in which *all* the coefficients a_k are being known, since the order in which the *complete* $\sum_k ?_k \phi_k$ is to be filled with $?_k$ is irrelevant, i.e., $U(a_1, a_2, \cdots, a_k, \cdots) = U(a_2, a_k, \cdots, a_1, \cdots) = \cdots$.

The meaning of $U(a_1, a_2, \cdots, a_k, \cdots)$ is such that the goodness, numerically represented by this U, is on the knowledge of all the coefficients a_k, i.e., if all the coefficients become perfectly known, then this goodness is maximal and still is to be numerically represented by $U(a_1, a_2, \cdots, a_k, \cdots)$. The goodness in a game is defined by its rule. If, by some reason, the rule of our game does not permit a perfect signaling, under specific situations, regarding the knowledge of the coefficients - that $U(a_1, a_2, \cdots, a_k, \cdots)$, in spite of the signaling, remains as the avaiable goodness (maybe a non-maximal one) when obtaining all these coefficients of the wave function mathematical representation. The goodness in obtaining a_1 is to be numerically represented by $U(a_1)$. The goodness in obtaining a_1 and a_2 is to be the goodness in obtaining a_1, $U(a_1)$, plus the goodness in obtaining a_2, $U(a_2)$. We are not saying, e.g., that $U(a_1)$ exclusivelly depends on a_1, i.e., we asseverate that a_1 has an abstract utility, numerically represented by a goodness, namely $U(a_1)$, when a_1 is being obtained in spite of the utilized method permitted by the rule (in a game theoretical jargon, in spite of the method means that different strategies under an arbitrary rule are leading to the same kind of profit, to the same invariant profit form, but not necessarily leading to the same value). In virtue of our conclusion in the previous paragraph, the order is irrelevant. Hence, the goodness in obtaining a_1, a_2, $\cdots$, in spite of order, must be $U(a_1)$ plus $U(a_2)$ plus $\cdots$, which is the goodness in obtaining all the coefficients, $U(a_1, a_2, \cdots, a_k, \cdots)$. We conclude that the 'plus' operation must be commutative and associative over the real numbers [14] (see the discussion regarding eq. (6)). The most simple way to put these ideas mathematically is by the assumption that a null utility is zero, i.e., arguing that this algebraic structure regarding the 'plus' operation over the real field emerges as consequence of the fact that such 'plus' operation is the ordinary summation (+) over the real field of the numerically represented utilities. Hence, by these considerations and by (9):

$$U\left(\psi^S\right) = U\left(a\right) = U(a_1, a_2, \cdots, a_k, \cdots) = \sum_{\forall k} U\left(a_k\right). \tag{10}$$

The coefficients of the wave function can be multiplied by an arbitrary complex phase, in virtue of the linearity of the differential wave equation. Hence, the utility of a coefficient must be phase invariant. Such invariance is possible if the utility of a coefficient is the same as the utility of its norm. Hence, (10) becomes:

$$U(a) = U(|a|) = \sum_{\forall k} U(a_k) = \sum_{\forall k} U(|a_k|) \quad \therefore \tag{11}$$

$$U\left(\sqrt{\sum_{\forall k} |a_k|^2}\right) = \sum_{\forall k} U\left(\sqrt{|a_k|^2}\right). \tag{12}$$

Defining a new function $f(x) = U(\sqrt{x})$, (12) becomes:

$$f\left(\sum_{\forall k} |a_k|^2\right) = \sum_{\forall k} f\left(|a_k|^2\right), \tag{13}$$

and f must be linear, i.e., $f(x) = Cx$, C *constant.* Thus:

$$f\left(x^2\right) = Cx^2 = U\left(\sqrt{x^2}\right) = U(x). \tag{14}$$

Hence, for $x = a_k$:

$$U(a_k) = U(|a_k|) = f\left(|a_k|^2\right) = C|a_k|^2 = Ca_k^* a_k, \tag{15}$$

where a_k^* is the complex conjugate of a_k. Hence, $C \in \mathbb{R}$, since U, the numerical representation of the abstract utility $u\left(\psi^S\right)$ must be a real number as argued previously [14].

We conclude, by (15) and (6), that $a_k^* a_k$ can be taken as an utility scale. Indeed, with $\omega_1 = 0$, $\omega_0 = C$, in (6), we obtain the totality of classes (exactly the eq. (15)) of numerical representation of our abstract utility, all equally good. Moreover:

- $a_k^* a_k$ emerges as an utility from utility considerations without any a priori appeal to probabilistic interpretations;
- Since the rule regarding signaling is arbitrary, $a_k^* a_k$ is an absolute goodness in obtaning the coefficient a_k of the wave function representing a quantum object in the form $\sum_{\forall k} a_k \phi_k$, in spite of meaning, in spite of interpretation, in spite of postulate regarding its meaning, i.e., **in spite of rule**.

ON EMERGENCE OF THE MAX BORN'S RULE

Rules...

[3] Now, let a pure quantum isolated system be composed by an internal observer subsystem $\mathbb{O}$, with initial state $\psi^{\mathbb{O}}_{[\cdots]}$, plus n identical internal object subsystems, S_1, S_2, $\cdots$, S_n, being these object subsystems initially in the same state:

$$\psi^{S_1} = \psi^{S_2} = \cdots = \psi^{S_n} = \sum_{\forall k} a_k \phi_k, \tag{16}$$

where the ϕ_k are the eingenfunctions of the same quantity $\mathbf{A}$ to be measured by $\mathbb{O}$.

After r measures, $r \leq n$, in the order S_1, S_2, $\cdots$, S_r, the wave function of the pure quantum isolated system is given by:

$$\psi^{S_1+S_2+\cdots+S_n+\mathbb{O}} = \sum_{i,j,\cdots,k} a_i a_j \cdots a_k \phi_i^{S_1} \phi_j^{S_2} \cdots \phi_k^{S_r} \psi^{S_{r+1}} \cdots \psi^{S_n} \psi^{\mathbb{O}}_{[\cdots,\alpha_i^1,\alpha_j^2,\cdots,\alpha_k^r]}. \tag{17}$$

Since the elements of the superposition in (17):

$$\psi_{i,j,\cdots,k} = a_i a_j \cdots a_k \phi_i^{S_1} \phi_j^{S_2} \cdots \phi_k^{S_r} \times \psi^{S_{r+1}} \cdots \psi^{S_n} \psi^{\mathbb{O}}_{[\cdots,\alpha_i^1,\alpha_j^2,\cdots,\alpha_k^r]} \tag{18}$$

are linearly independent, the respective avaiable reality of the subsystems:

$$\phi_{i,j,\cdots,k}^{S_1+S_2+\cdots+S_r} = \phi_i^{S_1} \phi_j^{S_2} \cdots \phi_k^{S_r}, \tag{19}$$

regarding the observer $\mathbb{O}[\cdots, \alpha_i^1, \alpha_j^2, \cdots, \alpha_k^r]$, becomes the maximally avaiable reality, after r measures. Hence, the entire utility of (17) cannot be avaiable for the observer within the isolated system by measures in its respective branch/world, i.e., it seems that some state of affairs regarding hidden reality takes place. After these r measures, the memory configuration of the observer $\mathbb{O}$ makes him conscious about (19), as demanded by the von Neumann's good measure postulate in [15] [19]. After r of such above measures:

$$\phi_{i,j,\cdots,k}^{S_1,S_2,\cdots,S_r,\cdots,S_n} = \phi_i^{S_1} \phi_j^{S_2} \cdots \phi_k^{S_r} \psi^{S_{r+1}} \times \cdots \psi^{S_n} \psi^{\mathbb{O}}_{[\cdots,\alpha_i^1,\alpha_j^2,\cdots,\alpha_k^r]} \tag{20}$$

becomes an *a fortiori* basis, prescribed by $\mathbb{O}$'s actual reality forced by its memory configuration. $\mathbb{O}$ constructs its own linear independent component reality via this forced basis. The unattainable remaining branches in (17) does not exist in $\mathbb{O}$'s reality in virtue of the linear independence, and, of course, these remaining memory configurations cannot be registered in $\mathbb{O}$'s memory configuration. The linear independence explains why the ramaining branches cannot be observed.

Taking the utility scale previously obtained, the only and one goodness avaiable for $\mathbb{O}$, in constructing its own reality via its branch's/world's coefficient in spite of method used to obtain this coefficient, is given by the product $a^*_{ij\cdots k}a_{ij\cdots k}$ taken from its respective branch's/world's coefficient of (20) in the superposition (17), i.e.:

$$U\left(a_{ij\cdots k}\right)=\left(a_i a_j\cdots a_k\right)^*\left(a_i a_j\cdots a_k\right)=a^*_i a_i\cdots a^*_k a_k. \tag{21}$$

Invoking the last paragraph of the previuos section, we asseverate: the meaning eventually given to (21) does not matter for purposes of utility existence. In spite of method/strategy, within a rule, an utility emerges. Consider the strategy in which the utility is maximal. This must be given/defined by (21). The nature pays an maximal amount of goodness to $\mathbb{O}$ in this situation. Suppose $\mathbb{O}$ rejects, in virtue of its chosen strategy, a part of this maximal utility and takes just a fraction of it. This new utility is also an utility, since the constant C in (15) permits it. In virtue of the arbitrary normalization constant regarding the linearity of the differential wave funtion, one is still able to remain considering this fractional utility as being given by (21). This process leads to the conclusion that anything usefull emerging from a method/strategy (given a rule permitting such method/strategy) used to obtain a wave function coefficient is able to be defined as an utility scale given by (21). If something is usefull, this something must be proportional to that maximal utility.

The rule is the von Neumann's good measure rule. Under this rule, this is the $\mathbb{O}$'s memory configuration after r measures:

$$[\cdots,\alpha^1_i,\alpha^2_j,\cdots,\alpha^r_k]. \tag{22}$$

Also, under von Neumann's good measure rule:

- There is no correlation between the measures made on the identical systems; each memory position is filled in a complete random fashion.

Hence, randomness is a consequence of the rule $\Rightarrow$ the rule legitimates an statistical approach $\Rightarrow$ probability is an usefull number permitted by the rule.

The method/strategy choosen by $\mathbb{O}$ is the observation of the randomness of the registered eigenvalues over its memory's positions. This strategy is permitted by the rule. Surelly, the eigenvalues by themselves does not influentiate upon the fact that the meaning of the utility is absolutely an utility (an usefull quantity), but each eigenvalue points to the respective eigenvector of $\mathbf{A}\Rightarrow$ an emergent quantity related to an eigenvalue (the observed frequencies in $\mathbb{O}$'s memory configuration), this emergent quantity to be defined as an utility, is rigidly connected to its respective eingenstate. The utility in its meaning as an usefull quantity is independent of the choosen/prescribed basis.

In other words, since $U(a_k)=Ca^*_k a_k$ is the utility's numerical absolute scale form in obtaining the a_k coefficient of the wave function in spite of method, if one takes the equally good scale with $C=1$ and defines the method as being the von Neumann's good measure method, such *game* must provide a profit π if the eingenvalue α_k is registered in $\mathbb{O}$'s memory configuration or 0 otherwise (since $\mathbb{O}$ is betting in α_k as the winner result). Hence, the utility in obtaining a_k by this method is, in virtue of equation (1):

$$U(a_k)=a^*_k a_k=q_k\pi+(1-q_k)0=q_k\pi. \tag{23}$$

Since the rule defines a randomness character, the counting of the frequencies by the observer leads to the definition of the expression above as connected to a lottery device and the probabilistic interpretation emerges *a posteriori*, i.e., the real number q_k emerges as the prescribed probability p_k to the von Neumann's good measure device regarding the winner result α_k. Hence, the eq. (23) becomes

$$U(a_k)=a^*_k a_k=p_k\pi\Rightarrow p_k=\frac{a^*_k}{\sqrt{\pi}}\frac{a_k}{\sqrt{\pi}}=a^*_k a_k, \tag{24}$$

where the new wave function coefficient absorbs $1/\sqrt{\pi}$ in virtue of the arbitrary normalization constant.

Clarifying. A player (an observer) knows that a_k has an utility numerically given by $U(a_k)$. The maximal utility in spite of a given game is $U(a_1,a_2,\cdots)=U(\forall a_k)$. The minimum is zero, when void information is reached. Hence, an observer $\mathbb{O}^i$ mounts, in virtue of the main axiom, eq. (1):

$$U(a_k)=q^i_k U(\forall a_k)+(1-q^i_k)0=q^i_k U(\forall a_k), \tag{25}$$

where q^i_k is a real number taken by $\mathbb{O}^i$ in the interval $(0,1)$. Now, one (umpire) defines a von Neumann good measure game paying an amount $\gamma^i_k U(\forall a_k)$ when α_k is registered in $\mathbb{O}^i$'s memory configuration or zero otherwise. In virtue of the randomness of the game, as previously seem, such game becomes a von Neumann's good measure device whose utility regarding the α_k result is given by the expected value:

$$U(\alpha_k)=p_k\gamma^i_k U(\forall a_k)+(1-p_k)0=p_k\gamma^i_k U(\forall a_k), \tag{26}$$

where the γ^i_k is a real number defined by umpire such that $\mathbb{O}^i$ becomes ignorant about the preference between $U(a_k)$ or $U(\alpha_k)$ (hence, γ^i_k must be non-negative, since eq. (25) yelds non-negative real values). The real number γ_k^i reflects the criteria used by a fixed observer/player $\mathbb{O}^i$ regarding its choosen q_k^i weight regarding the utility of a_k (eq. (25)). Hence:

$$U(a_k) = a_k^* a_k = q_k^i U(\forall a_k) = U(\alpha_k) = p_k \gamma_k^i U(\forall a_k) \;\; \therefore$$

$$a_k^* a_k = p_k \gamma_k^i U(\forall a_k) = p_k \gamma_k^i \sum_{\forall k} a_k^* a_k = p_k \gamma_k^i N, \qquad (27)$$

where N is the normalization constant. Summing up:

$$\sum_{\forall k} a_k^* a_k = N = N \sum_{\forall k} p_k = N \sum_{\forall k} p_k \gamma_k^i \therefore$$

$$\sum_{\forall k} (1 - \gamma_k^i) p_k = 0. \qquad (28)$$

Eq. (28) has an infinite number of solutions $\{\gamma_k^i\}$. Which one corresponds to a specific $\mathbb{O}^i$ observer? Since eq. (25) reads:

$$U(a_k) = a_k^* a_k = q_k^i U(\forall a_k) \Rightarrow q_k^i = \frac{a_k^*}{\sqrt{N}} \frac{a_k}{\sqrt{N}}, \qquad (29)$$

the right side of eq. (29) contains the normalized wave function coefficient $\forall\ \mathbb{O}^i$. Hence, all observers must use the same q_k^i, implying that a same solution of eq. (28) reflects a criteria that must be the same for all observers. But, different physical situations would imply new von Neumann's good measure device games with different probabilites p_k. A common solution of eq. (28) would vary over different physical situations, over different sets of $\{p_k\}$ in which each set caracterizes a very specific physical situation. Such behavior is not possible, since observers do not know the probabilities p_k when they ignorantly mount eq. (25) before the game starts. Hence, umpire concludes that γ_k^i must be constant. The only constant satisfying eq. (28) is $\gamma_k^i = 1$. The moral is: if one (umpire) proposes a von Neumann's good measure game to observers fully informed about the form of the utilities of the wave function coefficients, $a_k^* a_k$, in such a manner that the observers are ignorant about the *probability values* of the outcomes regarding the eigenvalues α_k and, also, by umpire's calibration of the amount to be paid at each outcome as being $a_k^* a_k / p_k$ when the outcome is α_k or zero otherwise, then such game has the utility $U(\alpha_k) = U(a_k)$ with an unique possible calibration given by:

$$\gamma_k^i = 1, \qquad (30)$$

and, also, by eqs. (25) and (26) (the amount to be paid: $a_k^* a_k / p_k$, guarantees the relation below):

$$U(\alpha_k) = U(a_k) \Rightarrow q_k^i = \gamma_k^i p_k \overbrace{=}^{\text{eq.}(30)} p_k, \qquad (31)$$

implying that the ignorance of observers force them to prescribe their common weights as probabilistical ones, namely the outcome probabilities and, also, by eq. (25):

$$U(a_k) = a_k^* a_k = q_k^i U(\forall a_k) \overbrace{=}^{\text{eq.}(31)} = p_k U(\forall a_k) = p_k N \therefore$$

$$p_k = \frac{U(a_k)}{N} = \frac{a_k^*}{\sqrt{N}} \frac{a_k}{\sqrt{N}} = U(a_k) = a_k^* a_k, \qquad (32)$$

where the normalized wave function coefficient absorbed the arbitrary normalization constant as well the normalized utility and, finally, a typical player/observer utilizes the probabilities as the only one usefull (utility) quantity in virtue of its ignorance.

In virtue of the above results, the probability p_k of collapse onto an eingenstate ϕ_k of $\mathbf{A}$ or, equivalently, the probability of obtaining the eigenvalue α_k of $\mathbf{A}$ is an utility given by:

$$p_k = a_k^* a_k. \qquad (33)$$

This is the emergence of the Max Born's Rule. Finally, and consequently, the normalization constant regarding the differential wave equation turns out to be 1 and the probability of a branch's reality like (20) is given by (21).

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[17] If one objects to the absoluteness regarding the existence of utilities as abstract objects with no need of a pair event-observer to define these utilities, arguing that the existence must be defined by the observer component of the pair event-observer using the observers' probabilistic intuitions, we must emphasize the fallacy of the objection: (1) - intuitions are constructed by observers using their very own sets of information in their memory configurations, as demonstrated by the Nobel prize laureate in economics - Daniel Kahneman [16], i.e., the relevant point is the proper contents of information avaiable to the observers when constructing their intuitions, *eventually* probabilistic intuitions; (2) - the main axiom does not mention any probability to postulate *the existence* of an abstract utility, does not mention events nor observers. We interpret the main axiom as an objective reality, i.e., a fundamental law of nature whose existence of its elements is to be used by observers in a situation of events in which these observers must contruct preferences via intuitions. Moreover, as we will see: the probability (α, not q) will enter externally to the observers via very particular situations of two combined events in which probabilities are to be externally assigned to these events and, also when, absolute utilities, regarding each of the two combined events, are to be picked up from $\mathbb{U}$ by the observers.

[18] One may argue that ψ^S could be treated in an abstract fashion, in an analogy with tensors in abstract forms, whit no need of a basis. Such argument is void, since measures are to be performed by observers. As we argued, the utility would remain independent of any inserted basis. Indeed, there is an analogy with the the case in which a metric is to be defined in a manifold without changing the nature of the absolutely defined objects in the manifold. A metric insertion captures the existence of observers without destroy the pure essence of the object to be represented in the manifold.

[19] One may argue that the von Neumann's good measure postulate is an additional assumption over the canonical formulation of the quantum mechanics. One must do not forget: by this conjecture/postulate, an effect of interaction object/observer within the isolated system is under consideration. The canonical formulation does not permit conjectures regarding effects like this, since the axiomatic collapse shuts up the question.