CASIMIR FRICTION FORCE AND ENERGY DISSIPATION FOR MOVING HARMONIC OSCILLATORS. II

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Abstract

This paper, the second in a series (the first one appearing as arXiv:1001.2489), continues our investigations of a two-oscillator system in linear relative motion serving as a simplified model of the Casimir friction setup. Time-dependent perturbation theory to first order is used to find the change in energy to second order. The present results agree with, and confirm, our earlier results obtained via different routes.

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1 Introduction

Consider two dielectric or metallic slabs with parallel surfaces, closely separated. If the slabs are set in tangential motion with respect to each other, with constant velocity, they become exposed to a friction force called Casimir friction, an effect that has received considerable attention in the recent past. The physical reason for the effect is that photons transferred between the slabs are subject to Doppler shifts. Such frequency shifts are physically expected to lead to energy dissipation, and hence a friction force. Some papers,

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limited to the period from 2007 onwards, are listed in refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. As one might expect, the problem becomes somewhat singular if the relative velocity of the slabs is assumed constant, from $t = -\infty$ to $t = +\infty$. It is physically advantageous, therefore, to imagine that the interaction is effectively coupled in at a large negative finite time in the past, and coupled out again at a large positive t. This is, of course, in principle the same technique as is often employed in scattering field theory.

It might appear most natural to attack the Casimir friction problem by making use of standard macroscopic electrodynamics for media in constant rectilinear motion. It means use of Maxwell's equations in moving matter endowed with a refractive index n. And as expected, it turns out that most of the mentioned papers are following this kind of approach. A complicating factor in the present case is, however, that there is no obvious rest system of the matter to refer to; none of the slabs plays a privileged role. This contrasts the usual case in phenomenological electrodynamics where the covariant formulation is simply constructed such that the theory reduces to conventional electrodynamics in the uniquely defined rest inertial system (cf., for instance, Refs. [11, 12, 13]).

Faced with this circumstance or difficulty, it lies at hand to search for alternative approaches to the friction problem. On natural possibility is then to make use of statistical mechanical methods, for harmonic oscillators in uniform relative motion. These methods were used by us in a recent work [14], and were used also in previous works [15, 16]. The microscopic method has some advantages in comparison with macroscopic electrodynamics, most notably that the formalism becomes more simple and transparent. Yet, the statistical mechanical approach has the property that it is capable of showing the main features of the problem. The microscopic approach has also been followed recently by Barton (preprint, personal communication). In the present paper we continue our investigations from Ref. [14], along similar lines as there.

Specifically, we shall treat the following aspects of the problem:

- We calculate a general expression for the change ΔE in energy, Eq. (14) below, by means of quantum mechanical perturbation theory. It turns out that the change in energy occurs to second order in the perturbation. Nevertheless, time-dependent perturbation theory to the *first* order is sufficient to find this second order effect. This is because the

phases of the perturbed change in amplitudes, and the initial amplitudes of the eigenstates, are uncorrelated at thermal equilibrium. Thus change in amplitudes of eigenstates will be the square of perturbed amplitudes; i.e. there are no cross-terms. We find that ΔE is positive, corresponding to a friction force. Doubts occasionally raised in the literature about the very existence of the Casimir friction effect [5] are thus from this standpoint laid at rest.

- Making use of the mentioned expression for ΔE we compare the present formalism with that of Ref. [14], the latter result obtained in a quite different way and on a quite different form. In Ref. [14], the linear response via the Kubo formalism was used [15, 16, 17] to calculate the force which in turn could be divided into a reversible and an irreversible part. It is the latter part that is associated with dissipation. A satisfactory feature is that the present derivation, although being quite different from that of Ref. [14], leads to the same physical result.

2 Time-dependent perturbation theory

To fix the notation, we start with perturbation theory for a system at thermal equilibrium. The wave function can be written as

$$\psi = \sum_n a_n \psi_n, \tag{1}$$

where $\psi_n = \psi_n(x)$ are the eigenstates. For simplicity we here let x represent all the coordinates of the system. If ψ is normalized, $\int \psi^* \psi dx = 1$, then $|a_n|^2$ is the probability for the system to be in eigenstate n. At thermal equilibrium this probability is given by the Boltzmann factor

$$P_n = |a_n|^2 = \frac{1}{Z} e^{-\beta E_n}, \tag{2}$$

where E_n is the energy eigenvalue of the state and Z is the partition function

$$Z = \sum_n e^{-\beta E_n}. \tag{3}$$

Let now the Hamiltonian be perturbed by the time-dependent interaction

$$V(t) = -Aq(t), \tag{4}$$

where A is a quantum mechanical operator while $q(t)$ is a scalar function. The A is time independent.

Due to the perturbation the coefficients a_n will change. If the system starts in a state m there are transitions to other states given by a change in a_n

$$\Delta a_n = b_{nm}. \tag{5}$$

The b_{nm} is given by the standard expression

$$b_{nm} = \frac{1}{i\hbar}\int_{-\infty}^{t} V_{nm}(\tau)e^{i\omega_{nm}\tau}d\tau, \tag{6}$$

where

$$V_{nm}(\tau) = \int \psi_n^* \, V(\tau)\psi_m dx = -A_{nm}\, q(\tau),$$

$$A_{nm} = \langle n|A|m\rangle = \int \psi_n^* \, A \, \psi_m dx. \tag{7}$$

Here $\omega_{nm} = \omega_n - \omega_m$, with $\omega_n = E_n/\hbar$.

As mentioned above, we will assume that the perturbation vanishes after some time. Then we will obtain the total change in Δa_n with

$$b_{nm} = -\frac{1}{i\hbar} A_{nm}\, \hat{q}(-\omega_{nm}),$$

$$\hat{q}(\omega) = \int_{-\infty}^{\infty} q(t)e^{-i\omega t}dt, \tag{8}$$

where the hat denotes Fourier transform.

From a general perspective, the system may start in a combination of eigenstates with transitions from several states. (It might be natural in this context to think about the Casimir-Polder setup with molecules traveling close to a dielectric surface. For molecules, in contrast to atoms, the energy levels are closely separated and may thus easily allow transitions. For recent investigations along these lines, cf. Refs. [18, 19].) With this, Eq. (5) will be modified to $\Delta a_n = \sum_{m\neq n} a_m b_{nm}$. Now, the state n does not only receive contributions, but gives away contributions to other states also. The latter must follow from the corresponding increase of probabilities for the other states. Omitting the latter for the moment, the perturbed coefficients are

$$a_{1n} = a_n + \Delta a_n = a_n + \sum_{m\neq n} a_m b_{nm}. \tag{9}$$

The a_n will have complex phase factors, and in thermal equilibrium one must assume the phases of a_n and a_m ($m \neq n$) to be uncorrelated. Thus by thermal average

$$\langle a_n^* a_m \rangle = 0. \tag{10}$$

With this the new probability of the state n becomes

$$P_{1n} = \langle a_{1n}^* a_{1n} \rangle = |a_n|^2 + \sum_{m \neq n} |a_m|^2 B_{nm},$$

$$B_{nm} = b_{nm} b_{nm}^* = |b_{nm}|^2. \tag{11}$$

The last term is the increase in probability from the other states. Likewise, the state n must obey a similar loss of probability to other states to conserve probability. The loss to other states is thus $\sum_{m \neq n} |a_n|^2 B_{mn}$. With Eq. (6) we have $b_{mn} = b_{nm}^*$, by which $B_{mn} = B_{nm}$. The latter equation reflects that the transition probabilities between each pair of states are the same in either direction. With this, the resulting perturbed probability of state n modifies the expression (11) into

$$P_{1n} = |a_n|^2 + \sum_{m \neq n} (|a_m|^2 - |a_n|^2) B_{nm} = P_n + \sum_m (P_m - P_n) B_{nm}. \tag{12}$$

The change in energy can now be evaluated as

$$\Delta E = \sum_n E_n (P_{1n} - P_n) = \sum_{nm} E_n (P_m - P_n) B_{nm}$$

$$= \sum_{nm} (E_n - E_m) P_m B_{nm} + \sum_{nm} (E_m P_m - E_n P_n) B_{nm} = \sum_{nm} (E_n - E_m) P_m B_{nm}. \tag{13}$$

Utilizing the symmetry with respect to n and m in this expression and inserting for P_m from Eq. (2) we find

$$\Delta E = \frac{1}{2} \sum_{nm} (E_n - E_m)(P_m - P_n) B_{nm} = \frac{1}{Z} \sum_{nm} e^{-\frac{1}{2}\beta(E_n + E_m)} \Delta_{nm} \sinh(\frac{1}{2}\beta \Delta_{nm}) B_{nm}, \tag{14}$$

with $\Delta_{nm} = E_n - E_m$, and where from Eqs. (8) and (11)

$$B_{nm} = \frac{1}{\hbar^2} A_{nm} A_{nm}^* \hat{q}(-\omega_{nm}) \hat{q}(\omega_{nm}). \tag{15}$$

Here it is to be noted that $\Delta E \geq 0$. We conclude that whenever a system in thermal equilibrium is disturbed by some external perturbation the energy

always increases (or, is unchanged), i.e., energy is dissipated. The dissipation occurs to second order in the perturbation. To first order there is no dissipation; the changes are adiabatic.

We note in passing the similar nature of the formalism for an electromagnetic field in a dissipative medium: the mean quantity of heat developed per unit time and volume is

$$Q = \omega\varepsilon''(\omega)\langle \mathbf{E}^2 \rangle,$$

where ε'' denotes the imaginary part of the permittivity, $\varepsilon = \varepsilon' + i\varepsilon''$. Irreversibility of the dissipation process implies the condition $\varepsilon''(\omega) > 0$ for positive ω. Cf., for instance, ref. [20], Sec. 80.

3 Energy dissipation from friction force

Recently we evaluated the dissipated energy via the friction force [14]. The $q(t)$ denotes (or can be interpreted to denote) the position

$$x = x(t) = q(t), \tag{16}$$

then A is the operator for a force. By use of the Kubo relation [15, 16, 17] for this situation the resulting force due to the perturbation is

$$F_f = \int_{-\infty}^{t} \phi_{AA}(t - t')q(t')dt', \tag{17}$$

where

$$\phi_{AA}(t) = \frac{1}{i\hbar}\mathrm{Tr}\left\{\rho[\mathrm{A}, \mathrm{A(t)}]\right\}. \tag{18}$$

Here

$$\rho = \frac{e^{-\beta H}}{Z}, \quad \text{with} \quad \mathrm{Z} = \mathrm{Tr}(\mathrm{e}^{-\beta \mathrm{H}}),$$

is the canonical density matrix, and

$$A(t) = e^{itH/\hbar} A\, e^{-itH/\hbar}, \tag{19}$$

with H the Hamiltonian. With velocity

$$v(t) = \dot{x}(t) = \dot{q}(t) \tag{20}$$

the total energy dissipated by the system is

$$\Delta E = -\int_{-\infty}^{\infty} v(t)F_f dt = -\int_{-\infty}^{\infty} \left[\int_{-\infty}^{t} \dot{q}(t)\phi_{AA}(t - t')q(t')dt'\right] dt, \tag{21}$$

which is the same result as in Eq. (27) in Ref. [14]. Now the quantity $q(t)$ need not be a position as given by Eq. (16), but as it can be interpreted as a position in a generalized sense, we concluded in [14] that the result (21) has a broader applicability. We will now show that this is actually the case, by showing that the result (21) is the same as the new result (14) obtained in the present work, by means of time-dependent perturbation theory.

With wave function representation we first have

$$e^{-\beta H} \rightarrow \sum_n \psi_n(x) e^{-\beta E_n} \psi_n^*(x_1), \tag{22}$$

$$\rho AA(t) = \frac{1}{Z} \sum_{nmk} \int \psi_n(x) e^{-\beta E_n} \psi_n^*(x_1) A \psi_m(x_1) e^{i\omega_m t} \psi_m^*(x_2) A \times \psi_k(x_2) e^{-i\omega_k t} \psi_k^*(x_3) dx_1 dx_2. \tag{23}$$

Thus we obtain

$$\mathrm{Tr}(\rho AA(t)) = \frac{1}{Z} \sum_{nm} e^{-\beta E_n} A_{nm} e^{i\omega_m t} A_{mn} e^{-i\omega_n t}, \tag{24}$$

as $\int \psi_k^*(x)\psi_n(x)dx = \delta_{kn}$ ($x_3 = x_1 = x$), and A_{nm} is given by Eq. (7). Likewise we calculate $\mathrm{Tr}(\rho A(t)A)$ by exchange of ω_n and ω_m in Eq. (24). The response function becomes

$$\phi_{AA}(t) = \frac{1}{i\hbar} \mathrm{Tr}\left\{\rho[A, A(t)]\right\} = \frac{1}{i\hbar} \sum_{nm} M_{nm} (e^{-i\omega_{nm} t} - e^{i\omega_{nm} t}), \tag{25}$$

with

$$M_{nm} = -\frac{1}{Z} e^{-\frac{1}{2}\beta(E_n + E_m)} \sinh(\frac{1}{2}\beta\Delta_{nm}) A_{nm} A_{nm}^* \tag{26}$$

(recall that $\Delta_{nm} = E_n - E_m = \hbar\omega_{nm},\ A_{mn} = A_{nm}^*$). The expression for M_{nm} follows if one first exchanges n and m in Eq. (24), then adds the resulting term to it and divides by 2. By inserting Eq. (25) into Eq. (21) one gets the integral

$$I = \int_{t>t'} \int \dot{q}(t) q(t') \left(e^{-i\omega t} e^{i\omega t'} - e^{i\omega t} e^{-i\omega t'}\right) dt' dt, \tag{27}$$

where here $\omega = \omega_{nm}$ $[(E_n - E_m)/\hbar = \Delta_{nm}/\hbar]$. By partial integration and exchange of integration variables t and t' (in the last term below) we get

$$I = i\omega \int_{t>t'} \int q(t) q(t') \left(e^{-i\omega t} e^{i\omega t'} + e^{i\omega t} e^{-i\omega t'}\right) dt' dt$$

$$= i\omega \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(t)q(t')e^{-i\omega t}e^{i\omega t'}dt'dt = i\omega \hat{q}(\omega)\hat{q}(-\omega). \tag{28}$$

By inserting this into Eq. (21) via Eq. (25) we get for the dissipated energy

$$\Delta E = \frac{1}{\hbar} \sum_{nm} M_{nm}\, \omega\, \hat{q}\,(\omega)\, \hat{q}(-\omega). \tag{29}$$

With $\omega = \omega_{nm} = \Delta_{nm}/\hbar$ and M_{nm} given by the expression (26) this is nothing but the result (14) together with (15) obtained by time-dependent perturbation theory. Thus we have been able to derive the same expression for the dissipated energy in two independent ways.

4 Friction between harmonic oscillators

In our recent work [14] the friction between a pair of harmonic oscillators with interaction

$$-Aq(t) = [\psi(\mathbf{r}_0) + \nabla\psi(\mathbf{r}_0) \cdot \mathbf{v}t + ...]x_1x_2 \tag{30}$$

($t > 0$), was evaluated. Here, we will evaluate the energy dissipation by a direct use of Eq. (14) or (29). The first term in Eqs. (30) can be disregarded as it gives a reversible force, distinct from dissipation. Further, we replace t with $te^{-\eta t}$ ($\eta \to 0$) to make the interaction vanish as $t \to \infty$.

For harmonic oscillators one can introduce the usual annihilation and creation operators

$$x_i = \left(\frac{\hbar}{2m_i\omega_i}\right)^{1/2} (a_i + a_i^\dagger) \tag{31}$$

($i = 1, 2$), with properties

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{32}$$

With this the interaction becomes

$$-Aq(t) = \gamma(a_1a_2 + a_1a_2^\dagger + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger)\, te^{-\eta t}, \tag{33}$$

where

$$\gamma = (\frac{1}{2}D\hbar)^{1/2}(\mathbf{v} \cdot \nabla\psi), \quad D = \frac{\hbar}{2m_1m_2\omega_1\omega_2}. \tag{34}$$

Since here only small η ($\to 0$) is considered, the terms $a_1 a_2$ and $a_1^\dagger a_2^\dagger$ will not contribute. Thus we can use

$$A = a_1 a_2^\dagger + a_1^\dagger a_2, \quad \text{and} \quad q(t) = \gamma t e^{-\eta t}. \tag{35}$$

For the matrix elements (7) we then get

$$A_{n_1,n_2,n_1+1,n_2-1} = \langle n_1 n_2 | a_1 a_2^\dagger | n_1+1, n_2-1\rangle = \sqrt{n_1+1}\,\sqrt{n_2},$$

$$A_{n_1,n_2,n_1-1,n_2+1} = \langle n_1 n_2 | a_1^\dagger a_2 | n_1-1, n_2+1\rangle = \sqrt{n_1}\,\sqrt{n_2+1}, \tag{36}$$

while all other elements are zero. The Fourier transform of $q(t)$ is, for $t > 0$,

$$\hat{q}(\omega) = \gamma \int_0^\infty t e^{-\eta t} e^{-i\omega t} = \frac{\gamma}{(\eta + i\omega)^2}, \tag{37}$$

so that, for $\eta \to 0$,

$$\hat{q}(\omega)\hat{q}(-\omega) = \frac{\gamma^2}{(\eta^2+\omega^2)^2} \to \frac{\pi\gamma^2}{2\eta\omega^2}\delta(\omega). \tag{38}$$

here $\omega = \omega_1 - \omega_2$, where ω_1 and ω_2 are the eigenfrequencies of the two oscillators. Further, with $\omega \to 0$ ($m = n \pm 1$)

$$\Delta_{nm} \sinh(\frac{1}{2}\beta\Delta_{nm}) \to \frac{1}{2}\beta\Delta_{nm}^2 = \frac{1}{2}\beta(\pm\hbar\omega)^2 = \frac{1}{2}\beta\hbar^2\omega^2. \tag{39}$$

Then the matrix elements (36) should be squared and averaged by the Boltzmann distribution given by Eq. (14). We have $\langle n_1\rangle \approx \langle n_2\rangle \approx \langle n\rangle$, with $\omega_1 \to \omega_2$, and

$$\langle n\rangle = \frac{\sqrt{x}}{Z}\sum_{n=0}^\infty n x^n = \frac{x}{1-x}, \quad x = e^{-\beta\hbar\omega_1},$$

$$Z = \sqrt{x}\sum_{n=0}^\infty x^n = \frac{\sqrt{x}}{1-x}. \tag{40}$$

Then $\langle n\rangle + 1 = 1/(1-x)$, by which

$$\langle (n_1+1)n_2 + n_1(n_2+1)\rangle = 2(\langle n\rangle + 1)\langle n\rangle$$

$$= \frac{2x}{(1-x)^2} = \frac{1}{2\sinh^2(\frac{1}{2}\beta\hbar\omega_1)}. \tag{41}$$

Inserting in Eqs. (14) and (15) by multiplying together Eqs. (38), (39), and (41) we obtain for the energy dissipation

$$\Delta E = \frac{\pi\beta\hbar^2\gamma^2}{8\eta\sinh^2(\frac{1}{2}\beta\omega_1)}\,\delta(\omega_1-\omega_2). \tag{42}$$

With γ inserted from Eq. (34) this is the same as the result (21) of Ref. [14] with its Eq. (19) for the friction force inserted.

According to Eq. (42), the case of zero temperature ($\beta \to \infty$) yields $\Delta E \to 0$. This result is related to our assumptions, including slowly varying coupling or low velocities, i.e. $\eta \to 0$ in Eqs. (35) and (38). At more rapidly varying coupling or higher velocities also finite frequencies would contribute, leading to a finite energy change and a finite friction force at $T = 0$.

5 Summary

We have evaluated the total energy dissipation for a system perturbed by a time-dependent interaction. This task was achieved by using standard time-dependent perturbation theory to obtain the change in energy to second order in the perturbation. This change is always positive or zero. The result agrees with, and confirms, our previous result of Ref. [14] obtained in a different, and independent, way.

Our theory assumes thermal equilibrium, low velocities, and nonrelativistic mechanics. Photons accordingly are not present in the theory. Photons were included, however, in our earlier study [16].

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