Correlated two-photon transport in a one-dimensional waveguide side-coupled to a nonlinear cavity

Jie-Qiao Liao[1] and C. K. Law[1]

[1] *Department of Physics and Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, Hong Kong SAR, People's Republic of China*

We investigate the transport properties of two photons inside a one-dimensional waveguide side-coupled to a single-mode nonlinear cavity. The cavity is filled with a nonlinear Kerr medium. Based on the Laplace transform method, we present analytic solution of quantum states of the transmitted and reflected two photons, which are initially prepared in a Lorentzian wave packet. The solution reveals how quantum correlation between the two photons emerge after the scattering by the nonlinear cavity. In particular, we show that the output wave function of the two photons in position space can be localized in the relative coordinates, which is a feature that may be interpreted as a two-photon bound state in this waveguide-cav

PACS numbers:

I. INTRODUCTION

Creating quantum correlations among photons has been a subject of main interest for studying foundations of quantum theory as well as applications in quantum information science. As direct interactions between photons in free space are extremely weak, generation of correlated photons generally requires nonlinear media. Electromagnetically induced transparency and photon blockade are mechanisms that have been exploited to achieve strongly interacting photons [1–4]. Recently, studies of two-photon scattering from a two-level system inside a one-dimensional (1D) waveguide have also reported various features of photon correlation [5–7]. For example, Shen and Fan [6] have discovered the existence of two-photon bound states, and Roy [7] has indicated an interesting application of the system as a few-photon optical diode. We also note that Shi and Sun [8] have employed a formal scattering theory to study multi-photon transport in a 1D waveguide.

In this paper, we investigate the correlation properties of two photons in a 1D waveguide, which is side-coupled to a nonlinear cavity filled with a Kerr medium (Fig. 1). The nonlinear cavity plays the role of a scatterer. It is worth noting that such a Kerr nonlinearity has also been employed in coupled cavity array systems for studying quantum phase transition [9–14] and nonclassical photon statistics [15–18]. Here we will focus on the transport properties of two photons determined by long time solution of the Schrödinger equation, assuming the initial photons are in wave packet forms. We will present an analytic solution based on the Laplace transform method, which has been applied to related photon-atom scattering problems [19]. From the two-photon transmission and reflection amplitudes, we show how the two scattered photons can be correlated in frequencies and position variables, with the latter revealing photon bunching and anti-bunching effects. Our solution also reveals a two-photon resonance condition when the incident photon energies match the cavity frequency shifted by the Kerr interaction. The behavior of transmission and reflection near the resonance will be discussed.

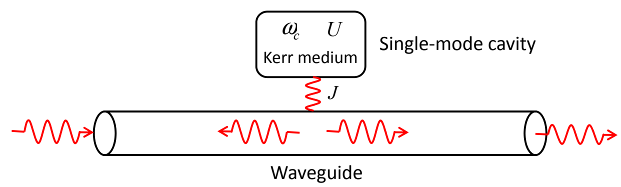

FIG. 1: (Color online). Schematic diagram of the physical setup. A 1D waveguide is coupled to a cavity filled with a Kerr-type nonlinear medium. Photons injected from the left-hand side of the waveguide are scattered by the nonlinear cavity. As a result, photons are reflected or transmitted in the waveguide.

II. PHYSICAL MODEL

The physical model under investigation consists of an infinite long 1D waveguide and a nonlinear cavity located at the origin (Fig. 1). We consider a single-mode field in the cavity, which couples to right- and left-propagating fields of the waveguide via the side coupling [20, 21] so that photons can tunnel between the waveguide and the nonlinear cavity. The Hamiltonian (with $\hbar = 1$) of the system is given by

$$\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \frac{U}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \int_0^\infty dk\omega_k(\hat{r}_k^\dagger\hat{r}_k + \hat{l}_k^\dagger\hat{l}_k) + J\int_0^\infty dk\left[\hat{a}^\dagger(\hat{r}_k + \hat{l}_k) + (\hat{r}_k^\dagger + \hat{l}_k^\dagger)\hat{a}\right]. \quad (1)$$

Here $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators associated with the cavity mode with the resonant frequency ω_c. The second term in Eq. (1) describes the Kerr nonlinear interaction with a strength U. The Hamiltonian of free fields propagating in the waveguide is described by the third term, where $\hat{l}_k$ ($\hat{l}_k^\dagger$) and $\hat{r}_k$ ($\hat{r}_k^\dagger$)

are, respectively, the annihilation (creation) operators for left- and right-propagating waves with wave number k and frequency ω_k. These operators satisfy the commutation relations

$$[\hat{l}_k, \hat{l}^\dagger_{k'}] = [\hat{r}_k, \hat{r}^\dagger_{k'}] = \delta(k-k'), \quad [\hat{l}_k, \hat{r}^\dagger_{k'}] = 0. \tag{2}$$

Finally, the last term in the Hamiltonian (1) represents the coupling between the cavity and the waveguide, where J is the tunneling strength.

For convenience, we introduce even- and odd-parity modes operators of the waveguide,

$$\hat{b}_k \equiv \frac{1}{\sqrt{2}}(\hat{r}_k + \hat{l}_k), \quad \hat{c}_k \equiv \frac{1}{\sqrt{2}}(\hat{r}_k - \hat{l}_k), \tag{3}$$

so that Hamiltonian (1) can be rewritten as

$$\hat{H} = \hat{H}^{(o)} + \hat{H}^{(e)} \tag{4}$$

with

$$\hat{H}^{(o)} = \int_0^\infty dk \omega_k \hat{c}^\dagger_k \hat{c}_k, \tag{5a}$$

$$\hat{H}^{(e)} = \omega_c \hat{a}^\dagger \hat{a} + \frac{U}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \int_0^\infty dk\omega_k \hat{b}^\dagger_k \hat{b}_k + g\int_0^\infty dk(\hat{a}^\dagger \hat{b}_k + \hat{b}^\dagger_k \hat{a}). \tag{5b}$$

Here $g \equiv \sqrt{2}J$ is introduced. We see that the interaction involves only even modes, and photons in the odd modes evolve freely in the waveguide. Therefore we shall focus on the calculation the transport properties of the photons in even modes.

In the rotating frame with respect to $\hat{H}^{(e)}_0 = \omega_c \hat{a}^\dagger \hat{a} + \omega_c \int_0^\infty dk \hat{b}^\dagger_k \hat{b}_k$, the Hamiltonian $\hat{H}^{(e)}$ can be simplified to

$$\hat{H}^{(e)}_I = \frac{U}{2}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \int_0^\infty dk\Delta_k \hat{b}^\dagger_k \hat{b}_k + g\int_0^\infty dk(\hat{a}^\dagger \hat{b}_k + \hat{b}^\dagger_k \hat{a}), \tag{6}$$

where $\Delta_k = \omega_k - \omega_c$ is the detuning. In this paper the dispersion relation for the modes in the waveguide is assumed to be linear, i.e., $\omega_k = v_g k$, and we will set the speed of light in the waveguide, $v_g = 1$.

III. SINGLE-PHOTON TRANSPORT

As a preparation for finding the solution of two-photon scattering, we first consider the single-photon problem [22–26]. Note that the Kerr nonlinearity has zero effect for single photon states. The main purpose in this section is to present the single-photon transmission and reflection coefficients, which will appear in the two-photon solution later in the paper.

In the single-excitation subspace, an arbitrary state can be written as

$$|\varphi(t)\rangle = \alpha(t)|1\rangle_c|\o\rangle + \int_0^\infty dk\beta_k(t)|0\rangle_c|1_k\rangle, \tag{7}$$

where $|1\rangle_c|\o\rangle$ stands for the state with one photon in the cavity and no photon in the waveguide, and $|0\rangle_c|1_k\rangle$ denotes the state with a vacuum cavity field and one photon in the kth (even) mode of the waveguide. The time dependent variables $\alpha(t)$ and $\beta_k(t)$ are the respective probability amplitudes.

By the Schrödinger equation $i|\dot{\varphi}(t)\rangle = \hat{H}^{(e)}_I|\varphi(t)\rangle$, we have

$$\dot{\alpha}(t) = -ig\int_0^\infty dk\beta_k(t), \tag{8a}$$

$$\dot{\beta}_k(t) = -i\Delta_k\beta_k(t) - ig\alpha(t). \tag{8b}$$

By performing the Laplace transform defined by $\tilde{f}(s) \equiv \int_0^\infty f(t)e^{-st}dt$, Eq. (8) becomes

$$s\tilde{\alpha}(s) - \alpha(0) = -ig\int_0^\infty dk\tilde{\beta}_k(s), \tag{9a}$$

$$s\tilde{\beta}_k(s) - \beta_k(0) = -i\Delta_k\tilde{\beta}_k(s) - ig\tilde{\alpha}(s), \tag{9b}$$

where $\alpha(0)$ and $\beta_k(0)$ are the initial values of the probability amplitudes.

Assuming initially the cavity is in the vacuum state and an incident single photon in the waveguide is prepared in a wave packet with a Lorentzian spectrum, the initial condition reads

$$\alpha(0) = 0, \quad \beta_k(0) = \frac{G_1}{\Delta_k - \delta + i\epsilon}, \tag{10}$$

where δ and ϵ are the detuning and spectral width of the photon, and $G_1 = \sqrt{\epsilon/\pi}$ is a normalization constant. The choice of $\beta_k(0)$ in Eq. (10) has the advantage that analytic solutions can be obtained conveniently. In addition, by noting that $\epsilon \to 0$ corresponds to the monochromatic limit, an incident wave packet of a general form can be constructed by coherent superpositions of Lorentzian wave packets of various frequencies.

After some calculations, we obtain

$$\tilde{\alpha}(s) = \frac{1}{s+\frac{\gamma}{2}}\frac{2\pi i g G_1}{\delta - i(s+\epsilon)}, \tag{11a}$$

$$\tilde{\beta}_k(s) = \frac{G_1}{s+i\Delta_k}\left(\frac{1}{\Delta_k - \delta + i\epsilon} + \frac{1}{s+\frac{\gamma}{2}}\frac{2\gamma}{\delta - i(s+\epsilon)}\right). \tag{11b}$$

Note that in obtaining Eq. (11), we have made the approximation: $\int_0^\infty \frac{g^2}{s+i\Delta_k}dk \approx \int_{-\infty}^\infty \frac{g^2}{s+i\Delta_k}d\Delta_k = \gamma/2$, where $\gamma = 2\pi g^2$.

Taking the inverse Laplace transform of Eq. (11), in the long time limit, $\gamma t/2 \to \infty$ and $\epsilon t \to \infty$, we have

$$\alpha(t\to\infty) = 0, \quad \beta_k(t\to\infty) = \bar{t}_k\beta_k(0)e^{-i\Delta_k t}, \tag{12}$$

where

$$\bar{t}_k = \frac{\Delta_k - i\gamma/2}{\Delta_k + i\gamma/2}. \tag{13}$$

Eq. (12) shows that the scattering process results in a phase shift θ_k for a single photon with wave vector k, where the phase shift is defined by $\exp(i\theta_k) = \bar{t}_k$.

In terms of the left- and right-propagation modes, if we assume a photon packet is incident onto the cavity from the left, then the initial state can be written as

$$\begin{aligned} |\varphi(0)\rangle &= \int_0^\infty dk\beta_k(0)\hat{r}_k^\dagger|\emptyset\rangle \\ &= \frac{1}{\sqrt{2}}\int_0^\infty dk\beta_k(0)(\hat{b}_k^\dagger + \hat{c}_k^\dagger)|\emptyset\rangle. \end{aligned} \tag{14}$$

In the long-time limit, the wave function becomes,

$$|\varphi(t\to\infty)\rangle = \int_0^\infty dk\beta_k(0)e^{-i\Delta_k t}(t_k\hat{r}_k^\dagger + r_k\hat{l}_k^\dagger)|\emptyset\rangle, \tag{15}$$

where the transmission and reflection amplitudes are defined as

$$t_k = \frac{\Delta_k}{\Delta_k + i\gamma/2}, \qquad r_k = \frac{-i\gamma/2}{\Delta_k + i\gamma/2}. \tag{16}$$

A similar result has been obtained for the case that a single photon is scattered by a two-level system in a 1D waveguide [22], namely, the transmission amplitudes t_k is zero at the exact resonance. This effect was also reported in Ref. [20] for side coupling with a classical field.

IV. CORRELATED TWO-PHOTON TRANSPORT

A. Equations of motion and solution

We now turn to the two-photon scattering problem. Since the total excitation number operator of the system is a conserved quantity, we can restrict the calculation to the two-excitation subspace. An arbitrary state in this subspace has the form:

$$\begin{aligned} |\Phi(t)\rangle &= A(t)|2\rangle_c|\varnothing\rangle + \int_0^\infty dkB_k(t)|1\rangle_c|1_k\rangle \\ &+ \int_0^\infty dp\int_0^p dqC_{p,q}(t)|0\rangle_c|1_p,1_q\rangle, \end{aligned} \tag{17}$$

where $|2\rangle_c|\varnothing\rangle$ is the state of two photons in the nonlinear cavity and no photon in the waveguide, and $|1\rangle_c|1_k\rangle$ is the state with one photon in the cavity and one photon with wave number k in the waveguide. The last term represents the state with no photon in the cavity and two photons with wave numbers p and q in the waveguide. $A(t)$, $B_k(t)$, and $C_{p,q}(t)$ denote the respective probability amplitudes.

By the Schrödinger equation, the probability amplitudes are governed by:

$$\dot{A}(t) = -iUA(t) - i\sqrt{2}g\int_0^\infty dkB_k(t), \tag{18a}$$

$$\dot{B}_k(t) = -i\Delta_k B_k(t) - i\sqrt{2}gA(t) - ig\int_0^\infty dpC_{p,k}(t), \tag{18b}$$

$$\dot{C}_{p,q}(t) = -i(\Delta_p + \Delta_q)C_{p,q}(t) - ig(B_p(t) + B_q(t)). \tag{18c}$$

We assume that the two injected photons are initially prepared in a Lorentzian wave packet, the initial condition of the system reads,

$$A(0) = 0, \quad B_k(0) = 0, \tag{19a}$$

$$\begin{aligned} C_{p,q}(0) = &G_2\left(\frac{1}{\Delta_p - \delta_1 + i\epsilon}\frac{1}{\Delta_q - \delta_2 + i\epsilon}\right. \\ &\left.+\frac{1}{\Delta_q - \delta_1 + i\epsilon}\frac{1}{\Delta_p - \delta_2 + i\epsilon}\right), \end{aligned} \tag{19b}$$

with the normalization constant

$$G_2 = \frac{\epsilon}{\sqrt{2}\pi}\left(1 + \frac{4\epsilon^2}{(\delta_1 - \delta_2)^2 + 4\epsilon^2}\right)^{-1/2}. \tag{20}$$

Here δ_j and ϵ_j ($j = 1, 2$) are parameters defining the detunings and spectral widths of the two photons. Note that $C_{p,q}$ has been symmetrized in Eq. (19b) because of the bosonic character of photons.

We are interested in the asymptotic solution of $C_{p,q}(t)$ in the long time limit. After a lengthly calculation (see Appendix A), we obtain for $t \gg \gamma^{-1}$ and ϵ^{-1},

$$C_{p,q}(t) = \left(\bar{t}_p\bar{t}_q C_{p,q}(0) + B_{p,q}\right)e^{-i(\Delta_p+\Delta_q)t}, \tag{21}$$

where $\bar{t}_p$ and $\bar{t}_q$ are defined in Eq. (13). The expression of $B_{p,q}$ is given by

$$\begin{aligned} B_{p,q} &= \frac{-2UG_2\gamma^2}{\left(\Delta_p + i\frac{\gamma}{2}\right)\left(\Delta_q + i\frac{\gamma}{2}\right)\left(\Delta_p + \Delta_q - U + i\gamma\right)} \\ &\times\frac{1}{(\Delta_p + \Delta_q - \delta_1 - \delta_2 + 2i\epsilon)} \\ &\times\left[\frac{1}{\left(\Delta_p + \Delta_q - \delta_1 + i\epsilon + i\frac{\gamma}{2}\right)}\right. \\ &\left.+\frac{1}{\left(\Delta_p + \Delta_q - \delta_2 + i\epsilon + i\frac{\gamma}{2}\right)}\right]. \end{aligned} \tag{22}$$

From Eqs. (21) and (22), we notice that the term $B_{p,q}$ is a non-factorizable function of p and q, implying a correlation between the two output photons. The $B_{p,q}$ has a nominator proportional to the strength of Kerr nonlinearity U in the cavity. In the case $U = 0$, Eq. (21) reduces to a simple expression $C_{p,q}(\infty) = \bar{t}_p\bar{t}_q C_{p,q}(0)\exp[-i(\Delta_p + \Delta_q)t]$, describing two independent scattered photons.

B. Two-photon correlation in frequency variables

Let us express the results in terms of the left- and right-propagating modes. Assuming the two photons are injected from the left-hand side of the waveguide, then the initial wave function can be written as

$$|\psi(0)\rangle = \int_0^\infty \int_0^\infty dpdqC_{p,q}(0)\,\hat{r}_p^\dagger \hat{r}_q^\dagger\,|\emptyset\rangle \tag{23}$$

According to Eq. (3) and the solution (21), we obtain the long-time wave function, up to an overall phase factor $\exp[-i(\Delta_p+\Delta_q)t]$, as

$$\begin{aligned}|\psi(t\to\infty)\rangle &= \int_0^\infty \int_0^\infty dpdq(C_{p,q}^{rr}\hat{r}_p^\dagger\hat{r}_q^\dagger + C_{p,q}^{ll}\hat{l}_p^\dagger\hat{l}_q^\dagger)|\emptyset\rangle \\ &+ \int_0^\infty \int_0^\infty dpdq(C_{p,q}^{rl}\hat{r}_p^\dagger\hat{l}_q^\dagger + C_{p,q}^{lr}\hat{l}_p^\dagger\hat{r}_q^\dagger)|\emptyset\rangle,\end{aligned} \tag{24}$$

where

$$C_{p,q}^{rr} = t_p t_q C_{p,q}(0) + \frac{1}{4}B_{p,q}, \tag{25a}$$

$$C_{p,q}^{ll} = r_p r_q C_{p,q}(0) + \frac{1}{4}B_{p,q}, \tag{25b}$$

$$C_{p,q}^{rl} = t_p r_q C_{p,q}(0) + \frac{1}{4}B_{p,q}, \tag{25c}$$

$$C_{p,q}^{lr} = t_q r_p C_{p,q}(0) + \frac{1}{4}B_{p,q}. \tag{25d}$$

Here $C_{p,q}^{rr}$ and $C_{p,q}^{ll}$ are, respectively, the two-photon transmission and two-photon reflection amplitudes, which correspond to the processes that two photons with wave numbers p and q are transmitted into the right-propagation mode or reflected into the left-propagation mode. In addition, $C_{p,q}^{rl}$ ($C_{p,q}^{lr}$) relates to the process that the photon with wave number p (q) is transmitted into the right-propagation mode and the photon with wave number q (p) is reflected into the left-propagation mode.

We point out two interesting situations revealing the strong correlation of output photons in frequency domain. The first situation is achieved by injecting two identical photons with $\delta_1 = \delta_2 = 0$ and a narrow spectral width $\epsilon \ll \gamma$. This corresponds to the case when the peak frequency of the photons coincide with the resonant cavity. In this case the two photons are mainly reflected and uncorrelated [Fig. 2(a)], but if they are transmitted, they are strongly correlated [Fig. 2(b)]. This can be seen by the fact that $t_p = t_q = 0$ at zero detuning, and hence the transmission of both photons is dominated by the $B_{p,q}$ term. In other words, the two-photon transmission near $\delta_1 = \delta_2 = 0$ is almost entirely due to the nonlinearity in the cavity. Such a pair of transmitted photons are frequency correlated with the two-photon transmission probability concentrated along the line $\Delta_p + \Delta_q = 0$ [Fig. 2(b)]. The uncertainty of frequencies of individual transmitted photons is of the order of γ, whereas the uncertainty of sum of frequencies of both photons is of

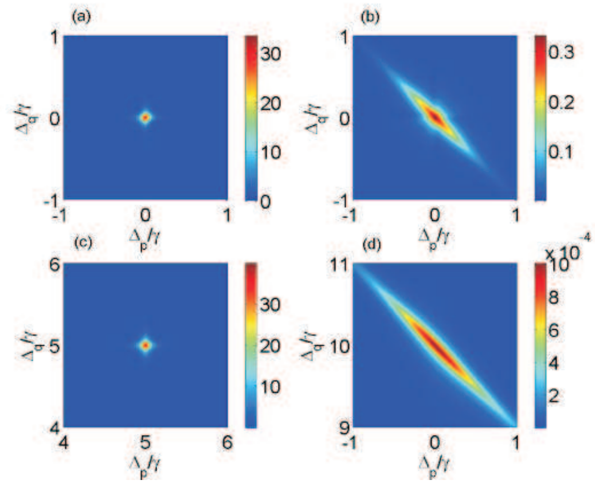

FIG. 2: (Color online). (a) and (b) are plots of $|\gamma C_{p,q}^{ll}|^2$ and $|\gamma C_{p,q}^{rr}|^2$, respectively, when $\delta_1 = \delta_2 = 0$. (c) and (d) are plots of $|\gamma C_{p,q}^{rr}|^2$ and $|\gamma C_{p,q}^{ll}|^2$, respectively, when $\delta_1 = \delta_2 = U/2$. Other parameters are set as $U/\gamma = 10$ and $\epsilon/\gamma = 0.05$.

the order of ϵ. The smaller the ϵ, the narrower is the distribution.

The second situation of interest is two-photon resonance occuring when the sum of energies of the two incident photons equals to that of a cavity containing two photons, i.e., $\delta_1 + \delta_2 = U$. In this case the photons can jointly enter the cavity. We show in Figs. 2(c) and (d) an example with $\delta_1 = \delta_2 = U/2 \gg \gamma$, where the frequency correlation appears more effectively in the reflected amplitude $C_{p,q}^{rr}$, since $r_j \approx 0$ ($j = p, q$). This is shown in the narrow distribution in Fig. 2(d). The transmission part [Fig. 2(c)], although they carry most of the probabilities, are almost uncorrelated.

C. Two-photon correlation in position variables

We now discuss the spatial features of the output photons. For simplicity, but without loss of the generality, we consider the monochromatic limit $\epsilon \to 0$ of incident photons. The two-photon transmission amplitude projected in position space reads (see Appendix B)

$$\langle x_1, x_2|\psi_{rr}\rangle \approx -16\pi^2\mathcal{M}G_2 e^{iE(x_c-t)}\theta(t-x_c)\phi_{rr}(x), \tag{26}$$

with

$$\begin{aligned}\phi_{rr}(x) &= t_{\delta_1}t_{\delta_2}\cos(\delta x) - \frac{U}{E-U+i\gamma} \\ &\times \frac{\gamma^2}{(E+i\gamma)^2-4\delta^2}e^{\frac{(iE-\gamma)}{2}|x|}.\end{aligned} \tag{27}$$

Here we have defined $x_c = (x_1+x_2)/2$ and $x = x_1 - x_2$ for the center-of-mass and relative coordinates respectively, and $E = \delta_1 + \delta_2$ and $\delta = (\delta_1 - \delta_2)/2$. We note that Eq. (26) is a product of center-of-mass wave function and relative wave function $\phi_{rr}(x)$, with $\exp[iE(x_c - t)]\theta(t - x_c)$

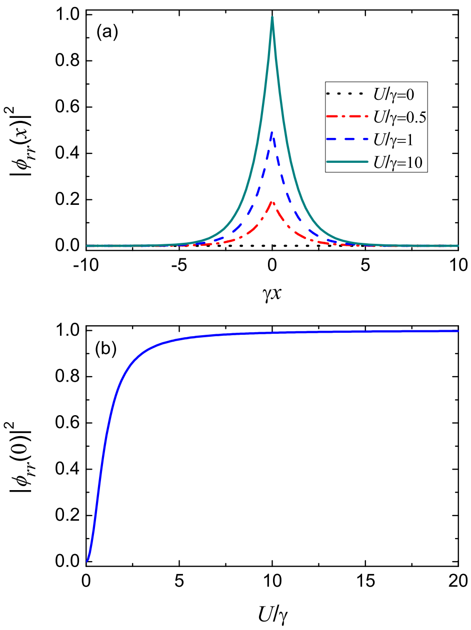

FIG. 3: (Color online). Spatial features of two-photon transmission at $\delta_1 = \delta_2 = 0$. (a) $|\phi_{rr}(x)|^2$ as a function of scaled relative coordinate γx, for various values of the scaled Kerr parameter U/γ. (b) $|\phi_{rr}(0)|^2$ as a function of the scaled Kerr parameter U/γ.

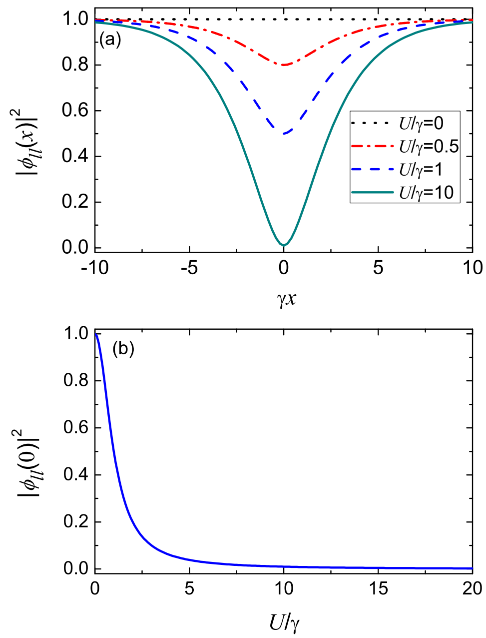

FIG. 4: (Color online). Spatial features of two-photon reflection at $\delta_1 = \delta_2 = 0$. (a) $|\phi_{ll}(x)|^2$ as a function of scaled relative coordinate γx, for various values of the scaled Kerr parameter U/γ. (b) $|\phi_{ll}(0)|^2$ as a function of the scaled Kerr parameter U/γ.

describing the center-of-mass motion of the two transmitted photons. The second term of $\phi_{rr}(x)$, which is an localized function around $x = 0$ with a width γ^{-1}. We remark that a similar feature was reported in Ref. [6] in a photon-atom scattering problem, where the exponential decaying function is connected to the existence of photon bound states.

To reveal spatial correlations, we take $\delta_1 = \delta_2 = 0$ so that the first term of $\phi_{rr}(x)$ can be suppressed. This is shown in Fig. 3(a) for various values of U. Note that $|\phi_{rr}(x)|^2$ is proportional to the joint probability of photons with a separation x, therefore the decaying feature corresponds to photon bunching. In particular, the joint probability of having both transmitted photons at the same position increases with U, but it saturates when $U \gg \gamma$ [Fig. 3(b)]. For the two-photon reflection amplitude in position space, we carry out a similar calculation and obtain,

$$\langle x_1, x_2|\psi_{ll}\rangle \approx -16\pi^2 \mathcal{N} G_2 e^{-iE(x_c+t)}\theta(t+x_c)\phi_{ll}(x), \quad (28)$$

with

$$\phi_{ll}(x) = r_{\delta_1} r_{\delta_2} \cos(\delta x) - \frac{U}{E-U+i\gamma} \times \frac{\gamma^2}{(E+i\gamma)^2 - 4\delta^2} e^{\frac{(iE-\gamma)}{2}|x|}. \quad (29)$$

At $\delta_1 = \delta_2 = \delta = 0$, the second term causes a dip in $|\phi_{ll}(x)|^2$ at $x = 0$ [Fig. 4(a)], which is a signature of photon antibunching as the reflected photons repel each other. As U increases, the joint probability of having both reflected photons at the same position decreases [Fig. 4(b)], which is in contrast to the transmission part.

Finally we describe the effects of two-photon resonance around $\delta_1 + \delta_2 = U$ discussed in the previous subsection. For simplicity we again consider the case $\delta_1 = \delta_2$ here. In Fig. 5, we illustrate the dependence of the relative two-photon wave function on $E = \delta_1 + \delta_2$. The effect of two-photon resonance is most apparent in Fig. 5(a), where the reflected two-photon wave function is strongly localized around $x = 0$ when $E = U$. Away from the resonance, the reflected two-photon wave function exhibits an oscillatory pattern in x [Fig. 5(c)], which is controlled by the two-photon detuning $E - U$. We also plot the transmitted two-photon wave function in Figs. 5(b) and (d), in which similar oscillatory patterns are observed.

V. CONCLUSIONS

In conclusion, we have presented an analytic solution of two-photon scattering inside a one-dimensional waveguide, which is side-coupled to a Kerr-type nonlinear cavity. The system provides a scheme to realize correlated two-photon transport. The Kerr nonlinearity is found to correlate photons in frequency variables such that $\Delta_p + \Delta_q$ is a constant, which is a constraint by the energy conservation. In position space, we have shown that the Kerr nonlinearity can cause the two photons 'stick' together with an average separation distance of the order of $v_g \gamma^{-1}$. We may interpret the result as a two-photon bound state, because of the exponential decaying shape

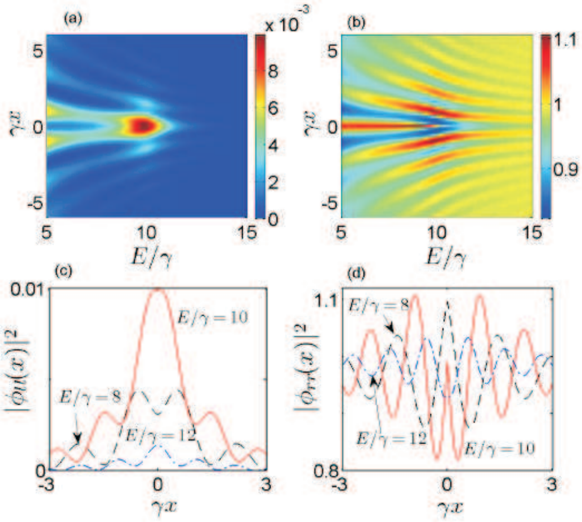

FIG. 5: (Color online). Dependence of spatial features on $E = \delta_1 + \delta_2$. (a) $|\phi_{ll}(x)|^2$ and (b) $|\phi_{rr}(x)|^2$. Examples at particular values of E are shown in (c) and (d). In these figures, we use $U = 10\gamma$ and $\delta = 0$.

of relative wave function. However, because of the interference with single photon processes described by the first term in Eq. (25), features of photon correlation may only be observed efficiently in certain directions. Finally, we note that recent studies of the related topic have considered using a single atom as a scatterer [6, 27]. However, in view of recent progresses of achieving a giant Kerr nonlinearity [1–3], our work suggests that a nonlinear cavity may be an alternative regarding the correlated two-photon transport problem.

Acknowledgments

This work is supported by the Research Grants Council of Hong Kong, Special Administrative Region of China (Project No. 401408).

Appendix A: Solution of Eq. (18) by the Laplace transform method

In this Appendix, we give a detailed derivation for solution of Eq. (18) which governs the transport of two photons in the waveguide. We use the Laplace transform method to solve these equations. Under the initial condition (19), Eq. (18) becomes

$$(s + iU)\tilde{A}(s) = -i\sqrt{2}g \int_0^\infty dk \tilde{B}_k(s), \tag{A1a}$$

$$(s + i\Delta_k)\tilde{B}_k(s) = -i\sqrt{2}g\tilde{A}(s) - ig \int_0^\infty dp \tilde{C}_{p,k}(s), \tag{A1b}$$

$$[s + i(\Delta_p + \Delta_q)]\tilde{C}_{p,q}(s) = C_{p,q}(0) - ig(\tilde{B}_p(s) + \tilde{B}_q(s)). \tag{A1c}$$

Substitution of Eqs. (A1a) and (A1c) into Eq. (A1b), and make use of the initial condition, we obtain the equation for variable $\tilde{B}_k(s)$ as

$$\begin{aligned}[\Delta_k - i(s + \gamma/2)]\tilde{B}_k(s) &= \int_{-\infty}^{\infty} \left(\frac{2g^2}{U - is} + \frac{g^2}{\Delta_p + \Delta_k - is} \right) \tilde{B}_p(s)\, d\Delta_p \\ &+2\pi g G_2 \left(\frac{1}{\Delta_k + \delta_1 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_2 + i\epsilon} + \frac{1}{\Delta_k + \delta_2 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_1 + i\epsilon} \right). \end{aligned} \tag{A2}$$

where we have made the approximation $\int_0^\infty dp \frac{g^2}{s+i(\Delta_p+\Delta_k)} \approx \gamma/2$.

The solution of $\tilde{B}_k(s)$ Eq. (A2), by inspection, takes the form:

$$\tilde{B}_k(s) = \frac{2\pi g G_2}{\Delta_k - i(s+\gamma/2)} \left(\frac{1}{\Delta_k + \delta_1 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_2 + i\epsilon} + \frac{1}{\Delta_k + \delta_2 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_1 + i\epsilon} \right) (1 + \tilde{F}_k(s)), \tag{A3}$$

with

$$\begin{aligned}\tilde{F}_k(s) &= -i\gamma \left(\frac{1}{\Delta_k + \delta_1 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_2 + i\epsilon} + \frac{1}{\Delta_k + \delta_2 - i(s+\epsilon)} \frac{1}{\Delta_k - \delta_1 + i\epsilon} \right)^{-1} \\ &\times \left[\left(\frac{2}{U - is - i\gamma} + \frac{1}{\delta_2 + \Delta_k - i(s+\epsilon)} \right) \frac{1}{\delta_2 - i\left(s + \epsilon + \frac{\gamma}{2}\right)} \frac{1}{\delta_1 + \delta_2 - i(s + 2\epsilon)} \right. \\ &+ \left. \left(\frac{2}{U - is - i\gamma} + \frac{1}{\delta_1 + \Delta_k - i(s+\epsilon)} \right) \frac{1}{\delta_1 - i\left(s + \epsilon + \frac{\gamma}{2}\right)} \frac{1}{\delta_1 + \delta_2 - i(s + 2\epsilon)} \right]. \end{aligned} \tag{A4}$$

Then from Eq. (A1c) we obtain the following expression

$$\begin{aligned}
\tilde{C}_{p,q}(s) \;=\; & \frac{G_2}{s+i(\Delta_p+\Delta_q)}\left[\frac{i\gamma}{s+\frac{\gamma}{2}+i\Delta_p}\left(\frac{1}{s+\epsilon+i(\Delta_p+\delta_1)}\frac{1}{\Delta_p-\delta_2+i\epsilon}+\frac{1}{s+\epsilon+i(\Delta_p+\delta_2)}\frac{1}{\Delta_p-\delta_1+i\epsilon}\right)\right.\\
& +\frac{i\gamma}{s+\frac{\gamma}{2}+i\Delta_q}\left(\frac{1}{s+\epsilon+i(\Delta_q+\delta_1)}\frac{1}{\Delta_q-\delta_2+i\epsilon}+\frac{1}{s+\epsilon+i(\Delta_q+\delta_2)}\frac{1}{\Delta_q-\delta_1+i\epsilon}\right)\\
& -\frac{2\gamma^2}{s+\gamma+iU}\frac{1}{s+2\epsilon+i\left(\delta_1+\delta_2\right)}\left(\frac{1}{s+\epsilon+\frac{\gamma}{2}+i\delta_1}+\frac{1}{s+\epsilon+\frac{\gamma}{2}+i\delta_2}\right)\left(\frac{1}{s+\frac{\gamma}{2}+i\Delta_p}+\frac{1}{s+\frac{\gamma}{2}+i\Delta_q}\right)\\
& -\frac{\gamma^2}{s+2\epsilon+i(\delta_1+\delta_2)}\frac{1}{s+\epsilon+\frac{\gamma}{2}+i\delta_1}\left(\frac{1}{s+\epsilon+i(\delta_1+\Delta_p)}\frac{1}{s+\frac{\gamma}{2}+i\Delta_p}+\frac{1}{s+\epsilon+i(\delta_1+\Delta_q)}\frac{1}{s+\frac{\gamma}{2}+i\Delta_q}\right)\\
& -\frac{\gamma^2}{s+2\epsilon+i(\delta_1+\delta_2)}\frac{1}{s+\epsilon+\frac{\gamma}{2}+i\delta_2}\left(\frac{1}{s+\epsilon+i(\delta_2+\Delta_p)}\frac{1}{s+\frac{\gamma}{2}+i\Delta_p}+\frac{1}{s+\epsilon+i(\delta_2+\Delta_q)}\frac{1}{s+\frac{\gamma}{2}+i\Delta_q}\right)\\
& \left.+\left(\frac{1}{\Delta_p-\delta_1+i\epsilon}\frac{1}{\Delta_q-\delta_2+i\epsilon}+\frac{1}{\Delta_q-\delta_1+i\epsilon}\frac{1}{\Delta_p-\delta_2+i\epsilon}\right)\right]. \qquad \text{(A5)}
\end{aligned}$$

Until now, we have obtained the expression for $\tilde{C}_{p,q}(s)$. Then we can get the expression for the probability amplitude $C_{p,q}(t)$ by performing inverse Laplace transform of $\tilde{C}_{p,q}(s)$. In particular, since we are interested in the output state of the two photons, here we only present the long-time solution of $C_{p,q}(t\to\infty)$ as

$$C_{p,q}(t\to\infty)=(\bar{t}_p\bar{t}_q C_{p,q}(0)+B_{p,q})e^{-i(\Delta_p+\Delta_q)t}, \qquad \text{(A6)}$$

where $\bar{t}_p$ and $\bar{t}_q$ have been defined in Eq. (13), and the expression for the correlation term is

$$\begin{aligned}
B_{p,q} \;=\; & \frac{-2UG_2\gamma^2}{\left(\Delta_p+i\frac{\gamma}{2}\right)\left(\Delta_q+i\frac{\gamma}{2}\right)\left(\Delta_p+\Delta_q-U+i\gamma\right)}\frac{1}{\left(\Delta_p+\Delta_q-\delta_1-\delta_2+2i\epsilon\right)}\\
& \times\left[\frac{1}{\left(\Delta_p+\Delta_q-\delta_1+i\epsilon+i\frac{\gamma}{2}\right)}+\frac{1}{\left(\Delta_p+\Delta_q-\delta_2+i\epsilon+i\frac{\gamma}{2}\right)}\right]. \qquad \text{(A7)}
\end{aligned}$$

Appendix B: Derivation of two-photon output state in position space

In this appendix, we derive the wave function of two-photon output state (24) in position space. For the two-photon transmission process, the corresponding wave function in position space can be written as

$$\begin{aligned}
\langle x_1,x_2|\psi_{rr}\rangle \;=\; & \int_0^\infty\int_0^\infty dpdq C_{p,q}^{rr}\langle x_1,x_2|\hat{r}_p^\dagger\hat{r}_q^\dagger|\emptyset\rangle\\
\approx\; & \mathcal{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(t_pt_qC_{p,q}(0)+B_{p,q}/4)e^{-i(\Delta_p+\Delta_q)t}e^{i\Delta_px_1}e^{i\Delta_qx_2}d\Delta_pd\Delta_q+x_1\leftrightarrow x_2. \qquad \text{(B1)}
\end{aligned}$$

In Eq. (B1), symmetrization of the two photons has been taken into account by introducing $\langle x_1,x_2|\hat{r}_p^\dagger\hat{r}_q^\dagger|\emptyset\rangle = \mathcal{M}(e^{i\Delta_px_1}e^{i\Delta_qx_2}+e^{i\Delta_px_2}e^{i\Delta_qx_1})$. According to the initial condition given in Eq. (19b), we can get the expression for the independent transport part as

$$\begin{aligned}
& \mathcal{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}t_pt_qC_{p,q}\left(0\right)e^{-i(\Delta_p+\Delta_q)t}e^{i\Delta_px_1}e^{i\Delta_qx_2}d\Delta_pd\Delta_q\\
& =-8\pi^2\mathcal{M}G_2t_{\delta_1-i\epsilon}t_{\delta_2-i\epsilon}e^{(iE+2\epsilon)(x_c-t)}\cos(\delta x)\theta(t-x_c), \qquad \text{(B2)}
\end{aligned}$$

where we introduce the center-of-mass coordinator $x_c=(x_1+x_2)/2$, the relative coordinator $x=x_1-x_2$, the total momentum $E=\delta_1+\delta_2$, and the relative momentum $\delta=(\delta_1-\delta_2)/2$. The $\theta(x)$ is the Heaviside step function and $t_{\delta_1-i\epsilon}$ is defined in Eq. (16). Note that here we have taken the approximation $\exp[\gamma(x_1-t)/2]\to 0$ under the assumption of $\gamma/2\gg\epsilon$.

According to Eq. (A7), the Fourier transform of the correlation part B_{pq} can be written as

$$\mathcal{M}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{B_{p,q}}{4}e^{-i(\Delta_p+\Delta_q)t}e^{i\Delta_px_1}e^{i\Delta_qx_2}d\Delta_pd\Delta_q=A_1+A_2, \qquad \text{(B3)}$$

with

$$A_l = -\frac{1}{2}\mathcal{M}UG_2\gamma^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(\Delta_q+i\gamma/2)}\frac{1}{(\Delta_p+i\gamma/2)}\frac{1}{(\Delta_p+\Delta_q-U+i\gamma)}\frac{1}{(\Delta_p+\Delta_q-\delta_1-\delta_2+2i\epsilon)} \times\frac{1}{(\Delta_p+\Delta_q-\delta_l+i\epsilon+i\gamma/2)}e^{i\Delta_p(x_1-t)}d\Delta_p e^{i\Delta_q(x_2-t)}d\Delta_q, \quad \text{(B4)}$$

for $l=1,2$. The Fourier transform of the correlation part can be obtained as

$$A_1+A_2 = \frac{8\pi^2\mathcal{M}G_2U}{(E-U-2i\epsilon+i\gamma)}\frac{\gamma^2}{(E+i\gamma-i2\epsilon)^2-4\delta^2}e^{(iE+2\epsilon)(x_c-t)}e^{(iE+2\epsilon-\gamma)\frac{|x|}{2}}\theta(t-x_c). \quad \text{(B5)}$$

According to Eqs. (B2) and (B5), the second term in Eq. (B1) can be obtained by making the replacement $x_c\to x_c$ and $x\to -x$. Then

$$\langle x_1,x_2|\psi_{rr}\rangle = -16\pi^2\mathcal{M}G_2e^{(iE+2\epsilon)(x_c-t)}\theta(t-x_c)\phi_{rr}(x), \quad \text{(B6)}$$

with

$$\phi_{rr}(x) = t_{\delta_1-i\epsilon}t_{\delta_2-i\epsilon}\cos(\delta x)-\frac{U}{E-U-2i\epsilon+i\gamma}\frac{\gamma^2}{(E+i\gamma-i2\epsilon)^2-4\delta^2}e^{\frac{(iE+2\epsilon-\gamma)}{2}|x|}. \quad \text{(B7)}$$

Using the same method, we can obtain the wave function for the two-photon reflection state,

$$\langle x_1,x_2|\psi_{ll}\rangle = \int_0^{\infty}\int_0^{\infty}dpdqC_{p,q}^{ll}\langle x_1,x_2|\hat{l}_p^{\dagger}\hat{l}_q^{\dagger}|\emptyset\rangle \approx -16\pi^2\mathcal{N}G_2e^{-(iE+2\epsilon)(x_c+t)}\theta(t+x_c)\phi_{ll}(x), \quad \text{(B8)}$$

with

$$\phi_{ll}(x) = r_{\delta_1-i\epsilon}r_{\delta_2-i\epsilon}\cos(\delta x)-\frac{U}{E-U-2i\epsilon+i\gamma}\frac{\gamma^2}{(E+i\gamma-i2\epsilon)^2-4\delta^2}e^{\frac{(iE+2\epsilon-\gamma)}{2}|x|}, \quad \text{(B9)}$$

where $\mathcal{N}$ is defined by $\langle x_1,x_2|\hat{l}_p^{\dagger}\hat{l}_q^{\dagger}|\emptyset\rangle=\mathcal{N}(e^{-i\Delta_p x_1}e^{-i\Delta_q x_2}+e^{-i\Delta_p x_2}e^{-i\Delta_q x_1})$.

[1] H. Schmidt and A. Imamoğlu, Opt. Lett. **21**, 1936 (1996).
[2] A. Imamoğlu, H. Schmidt, G. Woods, and M. Deutsch, Phys. Rev. Lett. **79**, 1467 (1997).
[3] S. E. Harris and Y. Yamamoto, Phys. Rev. Lett. **81**, 3611 (1998).
[4] K. M. Birnbaum, A. Boca, R. Miller, A. D. Boozer, T. E. Northup, and H. J. Kimble, Nature (London) **436**, 87 (2005).
[5] K. Kojima, H. F. Hofmann, S. Takeuchi, and K. Sasaki, Phys. Rev. A **68**, 013803 (2003)
[6] J. T. Shen and S. Fan, Phys. Rev. Lett. **98**, 153003 (2007); Phys. Rev. A **76**, 062709 (2007).
[7] D. Roy, Phys. Rev. B **81**, 155117 (2010).
[8] T. Shi and C. P. Sun, Phys. Rev. B **79**, 205111 (2009).
[9] M. J. Hartmann, F. G. S. L. Brandao, and M. B. Plenio, Nat. Phys. **2**, 849 (2006); Phys. Rev. Lett. **99**, 160501 (2007); Laser and Photon Rev. **2**, No. 6, 527 (2008).
[10] A. D. Greentree, C. Tahan, J. H. Cole, and L. C. L. Hollenberg, Nat. Phys. **2**, 856 (2006).
[11] D. G. Angelakis, M. F. Santos, and S. Bose, Phys. Rev. A **76**, 031805(R) (2007).
[12] M. J. Hartmann, Phys. Rev. Lett., **104**, 113601 (2010).
[13] N. Na, S. Utsunomiya, L. Tian, and Y. Yamamoto, Phys. Rev. A **77**, 031803(R) (2008).
[14] A. Tomadin, V. Giovannetti, R. Fazio, D. Gerace, I. Carusotto, H. E. Türeci, and A. Imamoğlu, Phys. Rev. A **81**, 061801(R) (2010).
[15] D. Gerace, H. E. Türeci, A. Imamoğlu, V. Giovannetti, and R. Fazio, Nat. Phys. **5**, 281 (2009).
[16] T. C. H. Liew and V. Savona, Phys. Rev. Lett. **104**, 183601 (2010).
[17] S. Ferretti, L. C. Andreani, H. E. Türeci, and D. Gerace, Phys. Rev. A **82**, 013841 (2010).
[18] M. Bamba, A. Imamoğlu, I. Carusotto, and C. Ciuti, arXiv:1007.1605.
[19] T. S. Tsoi, Master Thesis, The Chinese University of Hong Kong, 2009.
[20] Y. Xu, Y. Li, R. K. Lee, and A. Yariv, Phys. Rev. E **62**, 7389 (2000).
[21] M. F. Yanik, W. Suh, Z. Wang, and S. Fan, Phys. Rev. Lett. **93**, 233903 (2004).
[22] J. T. Shen and S. Fan, Opt. Lett. **30**, 2001 (2005);

Phys. Rev. Lett. **95**, 213001 (2005); Phys. Rev. A **79**, 023837(2009); **79**, 023838 (2009).
[23] D. E. Chang, A. S. Sørensen, E. A. Demler, and M. D. Lukin, Nat. Phys. **3**, 807 (2007).
[24] T. S. Tsoi and C. K. Law, Phys. Rev. A **78**, 063832 (2008); Phys. Rev. A **80**, 033823 (2009).
[25] L. Zhou, Z. R. Gong, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. Lett. **101**, 100501 (2008); Z. R. Gong, H. Ian, L. Zhou, and C. P. Sun, Phys. Rev. A **78**, 053806 (2008); Y. Chang, Z. R. Gong, and C. P. Sun, arXiv:1005.2274.
[26] J. Q. Liao, J. F. Huang, Y. X. Liu, L. M. Kuang, and C. P. Sun, Phys. Rev. A **80**, 014301 (2009); J. Q. Liao, Z. R. Gong, L. Zhou, Y. X. Liu, C. P. Sun, and F. Nori, Phys. Rev. A **81**, 042304 (2010).
[27] T. Shi, S. Fan, and C. P. Sun, arXiv:1009.2828.