Stable-1/2 Bridges and Insurance:
a Bayesian approach to non-life reserving

Edward Hoyle[*]   Lane P. Hughston[*]   Andrea Macrina[†‡]

September 23, 2010

Abstract

We develop a non-life reserving model using a stable-1/2 random bridge to simulate the accumulation of paid claims, allowing for an arbitrary choice of *a priori* distribution for the ultimate loss. Taking a Bayesian approach to the reserving problem, we derive the process of the conditional distribution of the ultimate loss. The 'best-estimate ultimate loss process' is given by the conditional expectation of the ultimate loss. We derive explicit expressions for the best-estimate ultimate loss process, and for expected recoveries arising from aggregate excess-of-loss reinsurance treaties. Use of a deterministic time change allows for the matching of any initial (increasing) development pattern for the paid claims. We show that these methods are well-suited to the modelling of claims where there is a non-trivial probability of catastrophic loss. The generalized inverse-Gaussian (GIG) distribution is shown to be a natural choice for the *a priori* ultimate loss distribution. For particular GIG parameter choices, the best-estimate ultimate loss process can be written as a rational function of the paid-claims process. We extend the model to include a second paid-claims process, and allow the two processes to be dependent. The results obtained can be applied to the modelling of multiple lines of business or multiple origin years. The multi-dimensional model has the property that the dimensionality of calculations remains low, regardless of the number of paid-claims processes. An algorithm is provided for the simulation of the paid-claims processes.

[*]Department of Mathematics, Imperial College London, London SW7 2AZ, UK
[†]Department of Mathematics, King's College London, London WC2R 2LS, UK
[‡]Institute of Economic Research, Kyoto University, Kyoto 606-8501, Japan

1

1 Introduction

The term 'stable process' refers here to a strictly stable process with index $\alpha \in (0, 2)$; thus, we are excluding the case of Brownian motion ($\alpha = 2$). The use of stable processes for the modelling of prices in financial markets was proposed by Mandelbrot [26] in connection with his analysis of cotton futures. In the tails, the Lévy densities of stable processes exhibit power-law decay. As a result, the behaviour of stable processes is wild, and their trajectories exhibit frequent large jumps. The variance of a stable random variable is infinite. If $\alpha \leq 1$, the expectation either does not exist or is infinite. This heavy-tailed behaviour makes stable processes ill-suited to some applications in finance, such as forecasting and option pricing.

To overcome some of the drawbacks of the stable processes, so-called tempered stable processes have been introduced (see Cont & Tankov [14], for example, for details). A tempered stable process is a pure-jump Lévy process, and its Lévy density is the exponentially dampened Lévy density of a stable process. The exponential dampening of the Lévy density improves the integrability of the process to the extent that all the moments of a tempered stable process exist. Tempered stable processes do not possess the time-scaling property of stable processes.

In this paper we apply stable-1/2 random bridges to the modelling of cumulative losses. The techniques presented can equally be applied to cumulative gains. The integrability of a stable-1/2 random bridge depends on the integrability of its terminal distribution. At some fixed future time, the nth moment of the process is finite if and only if the nth moment of its terminal value is finite. Thus a stable-1/2 random bridge with an integrable terminal distribution can be considered to be a dampened stable-1/2 subordinator. In fact, the stable-1/2 random bridge is a generalisation of the tempered stable-1/2 subordinator. If the Lévy density of a stable-1/2 subordinator is exponentially dampened, the resulting process is an IG process. We shall see that the IG process is a special case of a stable-1/2 random bridge.

We look in detail at the non-life reserving problem. An insurance company will incur losses when certain events occur. An event may be, for example, a period of high wind, the flooding of a river, or a motor accident. The losses are the costs associated with recompensing policy-holders who have been disadvantaged by an event. These costs might, for example, cover repairs to property, replacement of damaged items, loss of business, medical care, and so on. Although a loss is incurred by the insurance company on the date of an event (the 'loss date' or 'accident date'), payment is generally not made immediately. Delays will occur because loss is not always immediately reported to the company, because the full extent of the costs takes time to emerge, because the insurance company's obligation to pay takes time to establish, and so forth.

In return for covering policy-holder risk, the insurance company receives premiums. The premiums received over a fixed period should, typically, be sufficient to cover the losses the company incurs over that period. Since losses can take years to pay in full, the company sets aside some of the premiums to cover future payments; these are called 'reserves'. If the reserves are set too low, the company may struggle to cover its

liabilities, leading to insolvency. Large upward moves in the reserves due to a worsening in the expected future development of liabilities can cause similar problems. If the reserves are set too high, the company may be accused by shareholders or regulators of inappropriately withholding profits. Hence, it is the interest of the company to forecast its ultimate liability as accurately as possible when deciding the level of reserves to set.

We use a stable-1/2 random bridge to model the paid-claims process (i.e. cumulative amount paid to-date) of an insurance company. The losses contributing to the paid-claims process are assumed to have occurred in a fixed interval of time. Sometimes claims-handling information about individual losses is known, such as that contained in police or loss-adjuster reports. In the model presented here, such information is disregarded, and the paid-claims process is regarded as providing all relevant information. We derive the conditional distribution of the company's total liability given the paid-claims process. We then estimate recoveries from reinsurance treaties on the total liability. The expressions arising in such estimates are similar to the expectations encountered in the pricing of call spreads on stock prices.

We shall examine the upper-tail of the conditional distribution of the ultimate liability, and find that it is as heavy as the *a priori* tail. This has an interesting interpretation in the case when the insurer is exposed to a catastrophic loss. At time $t < T$, the probability of a catastrophic loss occurring in the interval $[t, T]$ decreases as t approaches T. However, in some sense, the *size* of a catastrophic loss does not decrease as t approaches T, since the tail of the conditional distribution of the cumulative loss does not thin.

When the *a priori* total loss distribution is a generalized inverse-Gaussian distribution, we find that the model is particularly tractable. We present a family of special cases where the expected total loss can be expressed as a rational function of the current value of the paid-claims process. That is, each member of the family is a martingale that can be written as a rational function of an increasing process.

The model can be extended to include more than one paid-claims process. We consider the case where there are two processes that are not independent, and which have different activity parameters. We then have two ultimate losses to estimate. We provide expressions for the expected values of the ultimate losses given both paid-claims processes. The numerical computations required to evaluate these expressions are no more difficult than those of the one-dimensional case. We demonstrate how to calculate the *a priori* correlation between the ultimate liabilities. The correlation can be used as a calibration tool when modelling cumulative losses arising from related lines of business (e.g. personal motor and commercial motor business).

We also describe how to simulate sample paths of the stable-1/2 random bridge, and how to use a deterministic time-change to adjust the model when the paid-claims process is expected to develop non-linearly.

2 Preliminaries

We fix a probability space $(\Omega, \mathbb{Q}, \mathcal{F})$, and assume that all processes and filtrations under consideration are càdlàg. Unless otherwise stated, when discussing a stochastic process we assume that the process takes values in $\mathbb{R}$ and begins at time 0, and the filtration is that generated by the process itself. We work with a finite time horizon $[0, T]$.

2.1 Lévy processes and stable processes

A Lévy process is a stochastically-continuous process that starts from the value 0, and has stationary, independent increments (see Bertoin [7], Kyprianou [24], and Sato [32]). An increasing Lévy process is called a 'subordinator'. If a Lévy process $\{S_t^\alpha\}$ satisfies the scaling property

$$\{k^{-1/\alpha} S_{kt}\}_{t \geq 0} \stackrel{\text{law}}{=} \{S_t\}_{t \geq 0} \qquad \text{for } k > 0, \tag{1}$$

then we say that it is a (strictly) stable processes with index α, or a stable-α process. It can be shown that for (1) to hold we must have $\alpha \in (0, 2]$. A stable-2 process is a scaled Weiner process, and a stable-1 process is a Cauchy process.

If $\{S_t^\alpha\}$ is a stable subordinator, the Laplace transform of S_t^α exists and is given by

$$\mathbb{E}[e^{-\lambda S_t^\alpha}] = \exp(-\kappa t \lambda^\alpha) \qquad \text{for } \lambda \geq 0, \tag{2}$$

where $\kappa > 0$, and α must be further restricted to $\alpha \in (0, 1)$.

2.2 Stable-1/2 subordinator

Let $\{S_t\}$ be a stable-1/2 subordinator . The Laplace transform of S_t is

$$\mathbb{E}[e^{-\lambda S_t}] = \exp\left\{ -\frac{ct\sqrt{\lambda}}{\sqrt{2}} \right\} \qquad (\lambda \geq 0), \tag{3}$$

for some $c > 0$. Then $\{S_t\}$ satisfies the scaling property $\{k^{-2} S_{kt}\} \stackrel{\text{law}}{=} \{S_t\}$, for $k > 0$. The random variable S_t has a 'Lévy distribution' with density

$$f_t(x) = \mathbb{1}_{\{x>0\}} \frac{ct}{\sqrt{2\pi}\, x^{3/2}} \exp\left(-\frac{1}{2} \frac{c^2 t^2}{x} \right). \tag{4}$$

We call c the 'activity parameter' of $\{S_t\}$. The density (4) is bounded for all $t > 0$ and is strictly positive for all $x > 0$. Integrating (4) yields the distribution function

$$\int_0^x f_t(y)\, \mathrm{d}y = 2\, \Phi\left[-ctx^{-1/2} \right], \tag{5}$$

where $\Phi[x]$ is the standard normal distribution function. The random variable S_t has infinite mean, indeed $\mathbb{E}[S_t^p] < \infty$ if and only if $p < 1/2$. The density of $1/S_t$ is

$$x \mapsto \mathbb{1}_{\{x>0\}} \frac{ct}{\sqrt{2}\,\Gamma[1/2]} x^{-1/2} \exp\left(-\frac{1}{2}c^2 t^2 x\right). \tag{6}$$

Thus the increments of $\{S_t\}$ are distributed as reciprocals of gamma random variables. For $\{W_t\}$ a Weiner process, define the exceedence times $\{\tau_t\}_{t\geq 0}$ by

$$\tau_t = \inf\{s : W_s > ct\}. \tag{7}$$

From Feller [19, X.7], we then have

$$\{S_t\} \overset{\text{law}}{=} \{\tau_t\}. \tag{8}$$

3 Stable-1/2 random bridge

Lévy random bridges are defined in Hoyle *et al.* [22], where some of their properties are derived. We call the random bridge of a stable-1/2 subordinator a 'stable-1/2 random bridge'. We shall state results about stable-1/2 random bridges, and refer to [22] for further details.

3.1 Stable-1/2 bridge

Fix $z > 0$ and let $\{S_{tT}^{(z)}\}_{0 \leq t \leq T}$ be a bridge of the process $\{S_t\}$ to the value z and time T. That is, $\{S_{tT}^{(z)}\}$ is a stable-1/2 subordinator conditioned to arrive (and terminate) at z at time T. Lévy bridges are Markov processes, and the transition law of $\{S_{tT}^{(z)}\}$ is

$$\mathbb{Q}\left[S_{tT}^{(z)} \in \mathrm{d}y \,\Big|\, S_{sT}^{(z)} = x\right] = f_{t-s,T-s}(y - x; z - x)\,\mathrm{d}y, \tag{9}$$

where

$$f_{tT}(y; z) = \frac{f_t(y) f_{T-t}(z - y)}{f_T(z)} \tag{10}$$

$$= \mathbb{1}_{\{0<y\leq z\}} \frac{1}{\sqrt{2\pi}} \frac{ct(T-t)}{T} \frac{\exp\left(-\frac{1}{2}\frac{c^2(Ty-tz)^2}{yz(z-y)}\right)}{(y - y^2/z)^{3/2}}. \tag{11}$$

Note that $f_{tT}(y; z)$ is the density function of the random variable $S_{tT}^{(z)}$. This density is bounded, and has bounded support, so

$$\mathbb{E}\left[\left(S_{tT}^{(z)}\right)^p\right] < \infty \qquad \text{for } p > 0. \tag{12}$$

Integrating the density (11) yields the following distribution function for $y \in [0, z]$:

$$F_{tT}(y; z) = \Phi\left[\frac{c(Ty - tz)}{\sqrt{yz(z-y)}}\right] + \left(1 - \frac{2t}{T}\right) \mathrm{e}^{2c^2 t(T-t)/z} \, \Phi\left[\frac{c((2t-T)y - tz)}{\sqrt{yz(z-y)}}\right]. \tag{13}$$

5

Remark 3.1. *When $t = T/2$, the second term in the distribution function (13) vanishes. The distribution function is analytically invertible, and we obtain the following identity, where Z is a standard normal random variable:*

$$S_{T/2,T}^{(z)} \stackrel{law}{=} \frac{1}{2} z \left(1 + \frac{Z}{\sqrt{c^2 T^2/z + Z^2}} \right). \tag{14}$$

.

Proposition 3.2. *For fixed $k > 0$, the stable-1/2 bridge $\{S_{tT}^{(z)}\}$ satisfies the scaling property*

$$\left\{ S_{tT}^{(z)} \right\}_{0 \le t \le T} \stackrel{law}{=} \left\{ k^{-2} S_{kt,kT}^{(k^2 z)} \right\}_{0 \le t \le T}. \tag{15}$$

Proof. The transition probabilities of the processes are given by

$$\mathbb{Q}\left[S_{tT}^{(z)} \le y \,\Big|\, S_{sT}^{(z)} = x \right] = F_{t-s,T-s}(y - x; z - x), \tag{16}$$

$$\mathbb{Q}\left[k^{-2} S_{kt,kT}^{(k^2 z)} \le y \,\Big|\, k^{-2} S_{ks,kT}^{(k^2 z)} = x \right] = F_{k(t-s),k(T-s)}(k^2 y - k^2 x; k^2 z - k^2 x), \tag{17}$$

for $0 \le s < t < T$. Substituting for $F_{tT}(y; z)$ given in (13) shows that these probabilities are equal. $\qquad\square$

Proposition 3.3.

$$\{S_{tT}^{(z)}\} \stackrel{law}{\longrightarrow} \{\tfrac{t}{T} z\} \qquad as \ c \to \infty. \tag{18}$$

Proof. Fix $z > 0$. It is sufficient to show that

$$\lim_{c \to \infty} F_{tT}(y; z) = \mathbb{1}_{\{Ty \ge tz\}} \qquad \text{for Lebesgue-a.e. } y \in (0, z), \tag{19}$$

since this is equivalent to

$$\lim_{c \to \infty} \mathbb{Q}\left[\left| S_{tT}^{(z)} - \tfrac{t}{T} z \right| < \varepsilon \right] = 1 \qquad \text{for all } t \in [0, T] \text{ and any } \varepsilon > 0. \tag{20}$$

Define α by

$$\alpha = -\frac{(2t - T)y - tz}{\sqrt{yz(z - y)}}, \tag{21}$$

and note that $\alpha > 0$ for $y \in (0, z)$. The inequality [1, 7.1.13] states that

$$\mathrm{e}^{x^2} \int_x^\infty \mathrm{e}^{-t^2} \, \mathrm{d}t \le \frac{1}{x + \sqrt{x^2 + 4/\pi}} \qquad (x > 0), \tag{22}$$

from which we deduce

$$\mathrm{e}^{2c^2 t(T-t)/z} \Phi[-\alpha c] \le \mathrm{e}^{2c^2 t(T-t)/z} \sqrt{\frac{2}{\pi}} \frac{\mathrm{e}^{-\alpha^2 c^2/2}}{\alpha c + \sqrt{\alpha^2 c^2 + 2/\pi}}$$

$$= \sqrt{\frac{2}{\pi}} \frac{\exp\left(-c^2 \frac{(Ty-tz)^2}{2y(z-y)z} \right)}{\alpha c + \sqrt{\alpha^2 c^2 + 2/\pi}}. \tag{23}$$

6

Since the left-hand side of (23) is positive, we see that

$$\lim_{c\to\infty} e^{2c^2 t(T-t)/z} \Phi[-\alpha c] = 0. \tag{24}$$

Then we have

$$\lim_{c\to\infty} F_{tT}(y;z) = \lim_{c\to\infty} \Phi\left[\frac{c(Ty-tz)}{\sqrt{yz(z-y)}}\right] + \left(1 - \frac{2t}{T}\right) \lim_{c\to\infty} e^{2c^2 t(T-t)/z} \Phi[-\alpha c]$$

$$= \mathbb{1}_{\{Ty-tz\geq 0\}} - \tfrac{1}{2}\mathbb{1}_{\{Ty=tz\}}, \tag{25}$$

which completes the proof. $\qquad\qquad\square$

We define the incomplete first moment $M_{tT}(y;z)$ of $S_{tT}^{(z)}$ by

$$M_{tT}(y;z) = \int_0^y u\, f_{tT}(u;z)\, du \qquad (0 \leq y \leq z). \tag{26}$$

Straightforward use of calculus gives

$$M_{tT}(y;z) = \frac{t}{T}z \left\{ \Phi\left[\frac{c(Ty-tz)}{\sqrt{yz(z-y)}}\right] - e^{2c^2 t(T-t)/z}\, \Phi\left[\frac{c((2t-T)y-tz)}{\sqrt{yz(z-y)}}\right] \right\}. \tag{27}$$

We can also calculate the second moment of $S_{tT}^{(z)}$. The result is

$$\mathbb{E}\left[\left(S_{tT}^{(z)}\right)^2\right] = \frac{t}{T}z^2 \left\{ 1 - c(T-t)e^{\frac{c^2 T^2}{2z}} \sqrt{\frac{2\pi}{z}}\, \Phi\left[-cTz^{-1/2}\right] \right\}. \tag{28}$$

It then follows from equation (9) that for $0 \leq s < t < T$ we have

$$\mathbb{E}\left[S_{tT}^{(z)} \,\middle|\, S_{sT}^{(z)} = x\right] = \frac{T-t}{T-s}x + \frac{t-s}{T-s}z, \tag{29}$$

and

$$\mathbb{E}\left[\left(S_{tT}^{(z)}\right)^2 \,\middle|\, S_{sT}^{(z)} = x\right]$$

$$= \frac{t-s}{T-s}(z-x)^2 \left\{ 1 - c(T-t)e^{\frac{c^2(T-s)^2}{2(z-x)}} \sqrt{\frac{2\pi}{(z-x)}}\, \Phi\left[-c\frac{T-s}{\sqrt{z-x}}\right] \right\}. \tag{30}$$

7

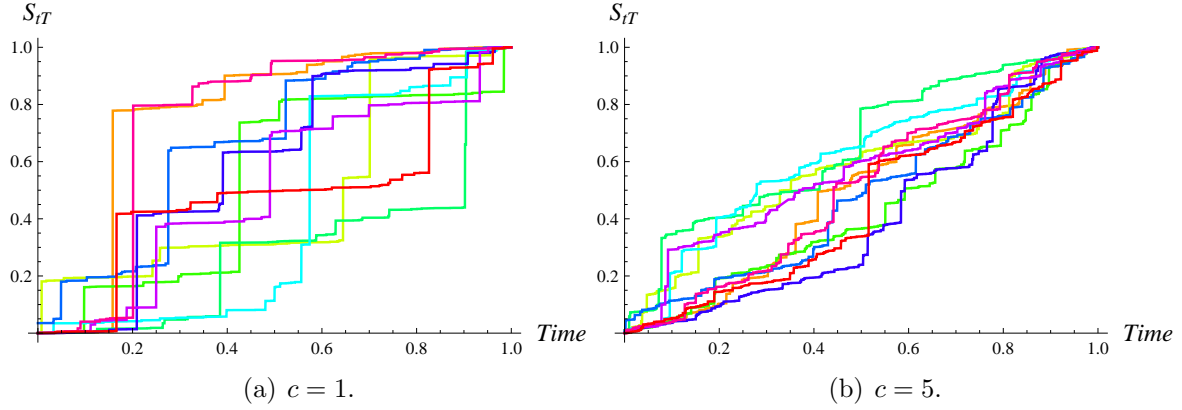$$\text{(a) } c = 1. \qquad\qquad\qquad \text{(b) } c = 5.$$

Figure 1: Simulations of the stable-1/2 bridge demonstrating the influence of the activity parameter c. Qualitatively speaking, increasing the value of c decreases the frequency of large jumps, and increases the frequency of small jumps.

3.2 Stable-1/2 random bridge

Let ν be a probability law on $\mathbb{R}_+$. We say that $\{\xi_{tT}\}_{0 \le t \le T}$ is a stable-1/2 random bridge with terminal law ν if the following conditions are satisfied:

1. ξ_{TT} has law ν.

2. There exists a stable-1/2 subordinator $\{S_t\}$ such that

$$\mathbb{Q}\left[\xi_{t_1,T} \le x_1, \ldots, \xi_{t_n,T} \le x_n \,|\, \xi_{TT} = z\right] = \mathbb{Q}\left[S_{t_1} \le x_1, \ldots, S_{t_n} \le x_n \,|\, S_T = z\right], \tag{31}$$

   for every $n \in \mathbb{N}_+$, every $0 < t_1 < \cdots < t_n < T$, every $(x_1, \ldots, x_n) \in \mathbb{R}^n$, and ν-a.e. z.

Hence, if the bridge laws of $\{\xi_{tT}\}$ are the bridge laws of a stable-1/2 subordinator, then $\{\xi_{tT}\}$ is a stable-1/2 random bridge. It is useful to think of $\{\xi_{tT}\}$ as a stable-1/2 bridge to a random variable with law ν.

The finite-dimensional distributions of $\{\xi_{tT}\}$ are

$$\mathbb{Q}\left[\xi_{t_1,T} \in dx_1, \ldots, \xi_{t_n,T} \in dx_n, \xi_{TT} \in dz\right] = \prod_{i=1}^n \left[f_{t_i - t_{i-1}}(x_i - x_{i-1}) \, dx_i\right] \psi_{t_n}(dz; x_n), \tag{32}$$

where the (un-normalised) measure $\psi_t(dz; \xi)$ is given by

$$\psi_0(dz; \xi) = \nu(dz), \tag{33}$$

$$\psi_t(dz; \xi) = \frac{f_{T-t}(z - \xi)}{f_T(z)} \nu(dz)$$

$$= \mathbb{1}_{\{z > \xi\}} \frac{\left(1 - \frac{t}{T}\right)}{\left(1 - \frac{\xi}{z}\right)^{3/2}} \exp\left\{\frac{1}{2}\left(\frac{c^2 T^2}{z} - \frac{c^2(T-t)^2}{z - \xi}\right)\right\} \nu(dz), \tag{34}$$

8

for $0 < t < T$. It can be shown that $\{\xi_{tT}\}$ is a Markov process with transition law

$$
\begin{aligned}
\mathbb{Q}[\xi_{tT} \in \mathrm{d}y \,|\, \xi_{sT} = x] &= \frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} f_{t-s}(y - x) \,\mathrm{d}y, \\
\mathbb{Q}[\xi_{TT} \in \mathrm{d}y \,|\, \xi_{sT} = x] &= \frac{\psi_s(\mathrm{d}y; x)}{\psi_s(\mathbb{R}; x)},
\end{aligned}
\tag{35}
$$

for $0 \le s < t < T$.

Now fix a time $s < T$ and define a process $\{\eta_{tT}\}_{s \le t \le T}$ by

$$
\eta_{tT} = \xi_{tT} - \xi_{sT}.
\tag{36}
$$

Then $\{\eta_{tT}\}$ is a stable-1/2 bridge with terminal law $f_{T-s}(x)\psi_{T-s}(\mathbb{R}; x) \,\mathrm{d}x$. Furthermore, given ξ_{sT}, $\{\eta_{tT}\}$ is a stable-1/2 bridge with terminal law

$$
\nu^*(A) = \frac{\psi_s(A + \xi_{sT}; \xi_{sT})}{\psi_s(\mathbb{R}; \xi_{sT})},
\tag{37}
$$

where $A + y$ denotes the set $\{x : x - y \in A\}$. We call this the 'dynamic consistency' property of stable-1/2 bridges. The financial significance of the dynamic consistency condition is discussed in [9], [11], and [22].

4   Insurance model

We approach the non-life insurance claims reserving problem by modelling a paid-claims process by a stable-1/2 random bridge. We shall look at the problem of calculating the reserves required to cover the losses arising from a single line of business when we observe the paid-claims process. Arjas [2] and Norberg [30, 31] provide general descriptions of the problem. England & Verrall [18] and Würthrich & Merz [33] survey some of the existing actuarial models. Bühlmann [12] and Mikosch [29] contain related topics. The present work ties in with that of Brody *et al.* [11] who use a gamma random bridge process to model a cumulative loss or gain.

The method we use has a flavour of the Bornhuetter-Ferguson model from actuarial science [8] (see also [18]). In implementing the Bornhuetter-Ferguson model, one begins with an *a priori* estimate for the *ultimate loss* (the total cumulative loss arising from the underwritten risks). Periodically, this estimate is revised using a chain-ladder technique to take into account the *a priori* estimate and the development of the total paid (or reported) claims to date.

In the proposed model, we assume an *a priori* distribution for the ultimate loss. By conditioning on the development of the paid-claims process, we revise the ultimate loss distribution using Bayesian methods. In this way we continuously update the conditional distribution for the total loss. This is as opposed to the deterministic Bornhuetter-Ferguson model in which only a point estimate is updated. Knowledge

of the conditional distribution allows one to calculate confidence intervals around the expected loss, and to calculate expected reinsurance recoveries.

Credibility theory uses Bayesian methods to calculate insurance premiums (see, for example, Bühlmann & Gisler [13]). In a typical set-up, the premium a policyholder must pay for insurance cover is a functional of their future claims distribution. The policyholder's future-claims distribution is parameterised by a rating factor Θ, whose value is unknown, but for which we have an *a priori* distribution. Observation of the policyholder's claims history then leads to an updating of the distribution of Θ, and hence an updating of their premium for future cover. Würthrich & Merz [33] discuss Bayesian reserving models based on credibility theory. The two main differences between credibility reserving models and the present work are (i) they only update in discrete time, and (ii) the ultimate-loss distribution is updated indirectly through a 'rating' variable Θ. The second difference means that it is not straightforward to allow an arbitrary *a priori* ultimate-loss distribution.

The main assumptions of the stable-1/2 bridge model are the following:

1. The claims arising from the line of business have *run off* at time T. That is, at time T all claims have been settled, and the ultimate loss U_T is known.

2. U_T has *a priori* law ν such that $U_T > 0$ and $\mathbb{E}[U_T^2] < \infty$.

3. The paid-claims process $\{\xi_{tT}\}$ is a stable-1/2 random bridge, and $\xi_{TT} = U_T$.

4. The *best estimate* of the ultimate loss is $U_{tT} = \mathbb{E}\left[U_T \,\middle|\, \mathcal{F}_t^\xi\right]$, where $\{\mathcal{F}_t^\xi\}$ is the natural filtration of $\{\xi_{tT}\}$.

A few remarks should be made about assumption 4. First, using the natural filtration of $\{\xi_{tT}\}$ as the reserving filtration means that the paid-claims process is the *only* source of information about the ultimate loss once the measure ν is set. We do not consider the situation where we have access to information about claims that have been reported but not yet paid in full (such as case estimates). Second, the expectation is taken with respect to $\mathbb{Q}$, which may or may not be the 'real-world' measure. Let us call $\mathbb{Q}$ the *actuarial measure*. When reserving, practitioners routinely discount data before modelling. Discounting may adjust the data for the time-value of money or for the effects of claims inflation. Claims inflation, and interest rates, though understood to be stochastic, usually only provide a small amount of uncertainty to the forecasting of the ultimate loss, relative to the uncertainty surrounding the frequency and (discounted) sizes of insurance claims. Furthermore, it is often for practical purposes reasonable to assume that claims inflation and interest rates are independent of claim frequency and size. Hence, a stochastic reserving model may lose little from the assumption that interest rates and inflation rates are deterministic. We make this assumption, and further assume that the paid-claims process has been discounted for the effects of interest and inflation.

5 Estimating the ultimate loss

The time-t conditional law of U_T is

$$
\begin{aligned}
\nu_t(\mathrm{d}z) &= \frac{\psi_t(\mathrm{d}z; \xi_{tT})}{\psi_t(\mathbb{R}; \xi_{tT})} \\
&= \frac{\mathbb{1}_{\{z>\xi_{tT}\}} \left(\frac{z}{z-\xi_{tT}}\right)^{3/2} \exp\left(-\frac{c^2}{2}\left(\frac{(T-t)^2}{z-\xi_{tT}} - \frac{T^2}{z}\right)\right) \nu(\mathrm{d}z)}{\int_{\xi_{tT}}^{\infty} \left(\frac{u}{u-\xi_{tT}}\right)^{3/2} \exp\left(-\frac{c^2}{2}\left(\frac{(T-t)^2}{u-\xi_{tT}} - \frac{T^2}{u}\right)\right) \nu(\mathrm{d}u)}.
\end{aligned}
\tag{38}
$$

The best-estimate ultimate loss is then

$$
U_{tT} = \int_{\xi_{tT}}^{\infty} z \, \nu_t(\mathrm{d}z).
\tag{39}
$$

At time $t < T$, the total amount of claims yet to be paid is $U_T - \xi_{tT}$. The amount that the insurance company sets aside to cover this unknown amount is called the *reserve*. The expectation of the total future payments is called the *best-estimate reserve*, and can be expressed by

$$
R_{tT} = U_{tT} - \xi_{tT}.
\tag{40}
$$

For prudence, the reserve may be greater than the best-estimate reserve. However, for regulatory reasons it is sometimes required that the best-estimate reserve is reported. The variance of the total future payments is the variance of the ultimate loss, which is

$$
\operatorname{Var}\left[U_T - \xi_{tT} \,\Big|\, \mathcal{F}_t^{\xi}\right] = \operatorname{Var}\left[U_T \,\Big|\, \mathcal{F}_t^{\xi}\right] = \int_{\xi_{tT}}^{\infty} (z - U_{tT})^2 \, \nu_t(\mathrm{d}z).
\tag{41}
$$

6 The paid-claims process

We give the first two conditional moments of the paid-claims process. Using equations (29) and (30), and a straightforward conditioning argument, we have

$$
\mathbb{E}\left[\xi_{tT} \,\big|\, \mathcal{F}_s^{\xi}\right] = \frac{T-t}{T-s}\xi_{sT} + \frac{t-s}{T-s}U_{sT},
\tag{42}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\xi_{tT}^2\right] &= \frac{t}{T}\int_0^{\infty} z^2 \left\{1 - c(T-t)\mathrm{e}^{\frac{c^2T^2}{2z}}\sqrt{\frac{2\pi}{z}}\,\Phi\left[-cTz^{-1/2}\right]\right\} \nu(\mathrm{d}z) \\
&= \frac{t}{T}\mathbb{E}\left[U_T^2\right] - c(T-t)\sqrt{2\pi}\int_0^{\infty} z^{3/2}\,\mathrm{e}^{\frac{c^2T^2}{2z}}\,\Phi\left[-cTz^{-1/2}\right] \nu(\mathrm{d}z).
\end{aligned}
\tag{43}
$$

Equation (42) implies that the paid-claims development is expected to be linear. We return to this point later. Fix $s < T$ and define the relocated process $\{\eta_{tT}\}_{s\le t\le T}$ by

$$
\eta_{tT} = \xi_{tT} - \xi_{sT}.
\tag{44}
$$

11

The dynamic consistency property implies that, given ξ_{sT}, $\{\eta_{tT}\}$ is a stable-1/2 random bridge with marginal law of η_{TT} being $\nu^*(A) = \nu_s(A + \xi_{sT})$. Then we have

$$
\begin{aligned}
\mathbb{E}\left[\xi_{tT}^2 \,\middle|\, \mathcal{F}_s^\xi\right] &= \mathbb{E}\left[\eta_{tT}^2 \,\middle|\, \xi_{sT}\right] + 2\xi_{sT}\,\mathbb{E}\left[\eta_{tT} \,\middle|\, \xi_{sT}\right] + \xi_{sT}^2 \\
&= \frac{T-t}{T-s}\xi_{sT}^2 + \frac{t-s}{T-s}\mathbb{E}\left[U_T^2 \,\middle|\, \xi_{sT}\right] \\
&\quad - c(T-t)\sqrt{2\pi}\int_{\xi_{sT}}^\infty (z - \xi_{sT})^{\frac{3}{2}}\, \mathrm{e}^{\frac{c^2(T-s)^2}{2(z-\xi_{sT})}}\, \Phi\left[-\frac{c(T-s)}{\sqrt{z-\xi_{sT}}}\right]\nu_s(\mathrm{d}z). \quad (45)
\end{aligned}
$$

7 Reinsurance

An insurance company may buy reinsurance to protect against adverse claim development. The *stop-loss* and *aggregate excess-of-loss* treaties are two types of reinsurance that cover some or all of the total amount of claims paid over a fixed threshold. Under a stop-loss treaty, the reinsurance covers all the losses above a prespecified level. If this level is K, then the reinsurance provider pays a total amount $(U_T - K)^+$ to the insurance company. The 'aggregate L excess of K' treaty is a capped stop-loss, and covers the layer $[K, K + L]$. In this case the reinsurance provider pays an amount $(U_T - K)^+ - (U_T - K - L)^+$.

The insurance company typically receives money from the reinsurance provider periodically. The amount received depends on the amount that they have paid on claims to-date. If the insurer has the paid-claims process $\{\xi_{tT}\}$, and receives payments from a stop-loss treaty (at level K) on the fixed dates $t_1 < t_2 < \cdots < t_n = T$, then the amount received on date t_i is

$$
(\xi_{t_i,T} - K)^+ - (\xi_{t_{i-1},T} - K)^+. \quad (46)
$$

The expected value of reinsurance payments such as (46) can be calculated using the following:

Proposition 7.1. *Fix* $t \in (0, T)$. *At time* $s < t$, *the expected exceedence of* ξ_{tT} *over some fixed* $K > 0$ *is*

$$
\begin{aligned}
D_{st} &= \mathbb{E}\left[(\xi_{tT} - K)^+ \,\middle|\, \mathcal{F}_s^\xi\right] \\
&= \frac{T-t}{T-s}\xi_{sT} + \frac{t-s}{T-s}U_{sT} - K \\
&\quad + \mathbb{1}_{\{K > \xi_{sT}\}}(K - \xi_{sT})\int_K^\infty F_{t-s,T-s}(K - \xi_{sT}; z - \xi_{sT})\,\nu_s(\mathrm{d}z) \\
&\quad - \mathbb{1}_{\{K > \xi_{sT}\}}\int_K^\infty M_{t-s,T-s}(K - \xi_{sT}; z - \xi_{sT})\,\nu_s(\mathrm{d}z). \quad (47)
\end{aligned}
$$

Proof. If $K \leq \xi_{sT}$ then

$$
\begin{aligned}
\mathbb{E}\left[(\xi_{tT} - K)^+ \,\middle|\, \mathcal{F}_s^\xi\right] &= \mathbb{E}\left[\xi_{tT} \,\middle|\, \mathcal{F}_s^\xi\right] - K \\
&= \frac{T-t}{T-s}\xi_{sT} + \frac{t-s}{T-s}U_{sT} - K. \quad (48)
\end{aligned}
$$

12

Thus we need only consider the case when $K > \xi_{sT}$. The $\mathcal{F}_s^\xi$-conditional law of ξ_{tT} is

$$\mathbb{Q}[\xi_{tT} \in \mathrm{d}y \,|\, \mathcal{F}_s^\xi] = \frac{\psi_t(\mathbb{R}; \xi_{tT})}{\psi_s(\mathbb{R}; \xi_{sT})} f_{t-s}(y - \xi_{sT}) \,\mathrm{d}y. \tag{49}$$

Hence we have

$$
\begin{aligned}
D_{st} &= \frac{1}{\psi_s(\mathbb{R}; \xi_{sT})} \int_K^\infty (y - K) \psi_t(\mathbb{R}; y) f_{t-s}(y - \xi_{sT}) \,\mathrm{d}y \\
&= \frac{1}{\psi_s(\mathbb{R}; \xi_{sT})} \int_K^\infty (y - K) \int_K^\infty \frac{f_{T-t}(z - y)}{f_T(z)} \,\nu(\mathrm{d}z)\, f_{t-s}(y - \xi_{sT}) \,\mathrm{d}y \\
&= \frac{1}{\psi_s(\mathbb{R}; \xi_{sT})} \int_K^\infty \int_K^z (y - K) \frac{f_{T-t}(z - y) f_{t-s}(y - \xi_{sT})}{f_T(z)} \,\mathrm{d}y\, \nu(\mathrm{d}z) \\
&= \int_K^\infty \int_K^z (y - K) f_{t-s, T-s}(y - \xi_{sT}; z - \xi_{sT}) \,\mathrm{d}y\, \nu_s(\mathrm{d}z). \tag{50}
\end{aligned}
$$

Making the change of variable $x = y - \xi_{sT}$ yields

$$
\begin{aligned}
D_{st} &= \int_K^\infty \int_{K-\xi_{sT}}^{z-\xi_{sT}} (x + \xi_{sT} - K) f_{t-s, T-s}(x; z - \xi_{sT}) \,\mathrm{d}x\, \nu_s(\mathrm{d}z) \\
&= \int_K^\infty \left\{ \frac{t-s}{T-s}(z - \xi_{sT}) - M_{t-s, T-s}(K - \xi_{sT}; z - \xi_{sT}) \right\} \nu_s(\mathrm{d}z) \\
&\quad + (\xi_{sT} - K) \int_K^\infty \left\{ 1 - F_{t-s, T-s}(K - \xi_{sT}; z - \xi_{sT}) \right\} \nu_s(\mathrm{d}z) \\
&= \frac{T-t}{T-s} \xi_{sT} + \frac{t-s}{T-s} U_{sT} - K + \int_K^\infty (K - \xi_{sT}) F_{t-s, T-s}(K - \xi_{sT}; z - \xi_{sT}) \,\nu_s(\mathrm{d}z) \\
&\quad - \int_K^\infty M_{t-s, T-s}(K - \xi_{sT}; z - \xi_{sT}) \,\nu_s(\mathrm{d}z). \tag{51}
\end{aligned}
$$

$\square$

Suppose that the insurance company has limited its liability by entering into a stop-loss reinsurance contract. At time $s \in [0, T)$, the expected reinsurance recovery between times t and u is

$$\mathbb{E}\left[(\xi_{uT} - K)^+ - (\xi_{tT} - K)^+ \,\big|\, \mathcal{F}_s^\xi \right] = D_{su} - D_{st}, \tag{52}$$

for $s < t < u \le T$.

Using a similar method to the calculation of D_{st}, we can calculate the expectation of ξ_{tT} conditional on it exceeding a threshold. For a threshold $\theta > \xi_{sT}$, we find

$$\mathbb{E}[\xi_{tT} \,|\, \xi_{sT}, \xi_{tT} > \theta] = \frac{\frac{T-t}{T-s} \xi_{sT} + \frac{t-s}{T-s} U_{sT} - \int_{\xi_{sT}}^\infty M_{t-s, T-s}(\theta - \xi_{sT}; z - \xi_{sT}) \,\nu_s(\mathrm{d}z)}{1 - \int_{\xi_{sT}}^\infty F_{t-s, T-s}(\theta - \xi_{sT}; z - \xi_{sT}) \,\nu_s(\mathrm{d}z)}. \tag{53}$$

Sometimes called the conditional value-at-risk (CVaR), this expected value is a coherent risk measure, and is a useful tool for risk management (see McNeil *et al.* [28]). Note that CVaR is normally defined as an expected value conditional on a shortfall in profit. Since we are modelling loss, and not profit, the risk we most wish to manage is on the upside. Hence, conditioning on an exceedence is of greater interest.

8   Tail behaviour

In this section we consider how the probability of extreme events is affected by the paid-claims development. Suppose that the line of business we are modelling is exposed to rare but 'catastrophic' large loss events. In this case we assume that the *a priori* distribution of the ultimate loss has a heavy right-tail. If a catastrophic loss could hit the insurance company at any time before run-off, then it is important that any conditional distributions for the ultimate loss retain the heavy-tail property. We shall see that in the stable-1/2 random bridge model, the conditional distributions are as heavy tailed as the *a priori* distribution.

Assume that U_T has a continuous density $p(z)$ which is positive for all z above some threshold. Then the value of U_T is unbounded in the sense that

$$\mathbb{Q}[U_T > x] > 0, \quad \text{for all } x \in \mathbb{R}. \tag{54}$$

Define

$$\text{Tail}_t = \lim_{L \to \infty} \frac{\mathbb{Q}\left[\xi_{TT} > L\right]}{\mathbb{Q}\left[\xi_{TT} - \xi_{tT} > L \mid \xi_{tT}\right]}. \tag{55}$$

If $\text{Tail}_t = \infty$ then the tail of the future-payments distribution at time $t > 0$ is not as heavy as the *a priori* tail. That is, a catastrophic loss at time t is 'smaller' than a catastrophic loss at time 0. If $\text{Tail}_t = 0$ then the tail of the future-payments distribution is greater at time t than *a priori*. If $0 < \text{Tail}_t < \infty$ then the tail is as heavy at time t as *a priori*. Using L'Hôpital's rule, we have

$$
\begin{aligned}
\text{Tail}_t &= \lim_{L \to \infty} \frac{\psi_t(\mathbb{R}; \xi_{tT}) \int_L^\infty p(z)\,\mathrm{d}z}{\int_{L+\xi_{tT}}^\infty \left(\frac{z}{z - \xi_{tT}}\right)^{3/2} \exp\left(-\frac{c^2}{2}\left(\frac{(T-t)^2}{z - \xi_{tT}} - \frac{T^2}{z}\right)\right) p(z)\,\mathrm{d}z} \\
&= \lim_{L \to \infty} \frac{\psi_t(\mathbb{R}; \xi_{tT})\, p(L)}{\left(\frac{L + \xi_{tT}}{L}\right)^{3/2} \exp\left(-\frac{c^2}{2}\left(\frac{(T-t)^2}{L} - \frac{T^2}{L + \xi_{tT}}\right)\right) p(L + \xi_{tT})} \\
&= \psi_t(\mathbb{R}; \xi_{tT}) \lim_{L \to \infty} \frac{p(L)}{p(L + \xi_{tT})},
\end{aligned} \tag{56}
$$

for $t \in (0, T)$. Some examples follow:

- If $p(z) \propto \mathbb{1}_{\{z>0\}} \mathrm{e}^{-z}$ (exponential) then $\text{Tail}_t = \psi_t(\mathbb{R}; \xi_{tT})\, \mathrm{e}^{\xi_{tT}}$.

- If $p(z) \propto \mathbb{1}_{\{z>0\}} \mathrm{e}^{-z^2}$ (half-normal) then $\text{Tail}_t = \psi_t(\mathbb{R}; \xi_{tT})\, \mathrm{e}^{\xi_{tT}^2}$.

- If $p(z) \propto \mathbb{1}_{\{z>0\}} z^{-3/2} \mathrm{e}^{-1/z}$ (Lévy) then $\text{Tail}_t = \psi_t(\mathbb{R}; \xi_{tT})$.

This property has an interesting parallel with the *subexponential* distributions. By definition, X has a subexponential distribution if

$$\lim_{L \to \infty} \frac{\mathbb{Q}\left[\sum_{i=1}^n X_i > L\right]}{\mathbb{Q}\left[X > L\right]} = n, \tag{57}$$

14

where $\{X_i\}_{i=1}^n$ are independent copies of X (see Embrechts *et al.* [17]). We note that

$$\lim_{L\to\infty} \frac{\mathbb{Q}\,[Z_T > L]}{\mathbb{Q}\,[Z_T - Z_t > L \mid Z_t]} = \infty, \tag{58}$$

for $\{Z_t\}$ a Brownian motion, a geometric Brownian motion or a gamma process. If $\{Z_t\}$ is a stable-1/2 subordinator, so the increments of $\{Z_t\}$ are subexponential, then

$$\lim_{L\to\infty} \frac{\mathbb{Q}\,[Z_T > L]}{\mathbb{Q}\,[Z_T - Z_t > L \mid Z_t]} = \frac{T}{T-t}. \tag{59}$$

9   Generalized inverse-Gaussian prior

The generalized inverse-Gaussian (GIG) distribution is a three-parameter family of distributions on the positive half-line (see Jørgensen [23] or Eberlein & v. Hammerstein [16] for further details). The density of the GIG distribution is

$$f_{GIG}(x;\lambda,\delta,\gamma) = \mathbb{1}_{\{x>0\}} \left(\frac{\gamma}{\delta}\right)^{\lambda} \frac{1}{2\,K_\lambda[\gamma\delta]} x^{\lambda-1} \exp\left(-\tfrac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right). \tag{60}$$

Here $K_\nu[z]$ the modified Bessel function (see Abramowitz & Stegun [1, 9.6]). The permitted parameter values are

$$\delta \geq 0, \qquad\qquad \gamma > 0, \qquad\qquad \text{if } \lambda > 0, \tag{61}$$

$$\delta > 0, \qquad\qquad \gamma > 0, \qquad\qquad \text{if } \lambda = 0, \tag{62}$$

$$\delta > 0, \qquad\qquad \gamma \geq 0, \qquad\qquad \text{if } \lambda < 0. \tag{63}$$

For $\lambda > 0$, taking the limit $\delta \to 0^+$ yields the gamma distribution. For $\lambda < 0$, taking the limit $\gamma \to 0^+$ yields the reciprocal-gamma distribution—this includes the Lévy distribution for $\lambda = -1/2$ (recall that the Lévy distribution is the increment distribution of stable-1/2 subordinators). The case $\lambda = -1/2$ and $\gamma > 0$ corresponds to the inverse-Gaussian (IG) distribution. If X has density (60) then the moment $\mu_k = \mathbb{E}[X^k]$ is given by

$$\mu_k = \frac{K_{\lambda+k}[\gamma\delta]}{K_\lambda[\gamma\delta]} \left(\frac{\delta}{\gamma}\right)^k \qquad \text{for } \lambda \in \mathbb{R},\ \delta > 0,\ \gamma > 0, \tag{64}$$

$$\mu_k = \begin{cases} \dfrac{\Gamma[\lambda+k]}{\Gamma[\lambda]} \left(\dfrac{2}{\gamma^2}\right)^k & k > -\lambda \\ \infty & k \leq -\lambda \end{cases} \text{ and } \lambda > 0,\ \delta = 0,\ \gamma > 0, \tag{65}$$

$$\mu_k = \begin{cases} \dfrac{\Gamma[-\lambda-k]}{\Gamma[-\lambda]} \left(\dfrac{\delta^2}{2}\right)^k & k < -\lambda \\ \infty & k \geq -\lambda \end{cases} \text{ and } \lambda < 0,\ \delta > 0,\ \gamma = 0. \tag{66}$$

The following identity is useful [1, 10.2.15]:

$$K_{n+\frac{1}{2}}[z] = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{j=0}^{n} (n+\tfrac{1}{2}, j)(2z)^{-j}, \quad \text{for } n \in \mathbb{N}, \tag{67}$$

where (m, n) is Hankel's symbol,

$$(m, n) = \frac{\Gamma[m + \frac{1}{2} + n]}{n! \, \Gamma[m + \frac{1}{2} - n]}. \tag{68}$$

The inverse-Gaussian (IG) process is a Lévy process with increment density

$$q_t(x) = \mathbb{1}_{\{x>0\}} \frac{ct}{\sqrt{2\pi}} \frac{1}{x^{3/2}} \exp\left(-\frac{\gamma^2}{2x}\left(x - \tfrac{c}{\gamma}t\right)^2\right). \tag{69}$$

We see that $q_t(x) = f_{GIG}(x; -\frac{1}{2}, ct, \gamma)$. The kth moment of $q_t(x)$ is

$$m_t^{(k)} = \sqrt{\frac{2}{\pi}} \gamma e^{\gamma ct} \left(\frac{ct}{\gamma}\right)^{k+\frac{1}{2}} K_{k-1/2}[\gamma ct], \tag{70}$$

for $k > 0$. Using (67), the first four integer moments simplify to

$$m_t^{(1)} = \frac{ct}{\gamma}, \tag{71}$$

$$m_t^{(2)} = \frac{ct}{\gamma^3}(1 + \gamma ct), \tag{72}$$

$$m_t^{(3)} = \frac{ct}{\gamma^5}(3 + 3\gamma ct + \gamma^2 c^2 t^2), \tag{73}$$

$$m_t^{(4)} = \frac{ct}{\gamma^7}(15 + 15\gamma ct + 6\gamma^2 c^2 t^2 + \gamma^3 c^3 t^3). \tag{74}$$

9.1 GIG terminal distribution

We shall see that the GIG distributions constitute a natural class of *a priori* distributions for ultimate loss. With $\gamma > 0$ and $c > 0$ fixed, we examine some properties of a paid-claims process $\{\xi_{tT}\}$ with time-T density $f_{GIG}(z; \lambda, cT, \gamma)$. The transition law is

$$\mathbb{Q}[\xi_{tT} \in dy \,|\, \xi_{sT} = x] = \frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} f_{t-s}(y - x)\, dy, \tag{75}$$

$$\mathbb{Q}[\xi_{TT} \in dy \,|\, \xi_{sT} = x] = \frac{\psi_s(dy; x)}{\psi_s(\mathbb{R}; x)}, \tag{76}$$

where

$$\psi_0(dz; \xi) = f_{GIG}(z; \lambda, cT, \gamma)\, dz, \tag{77}$$

$$\psi_t(dz; \xi) = (1 - \tfrac{t}{T})\mathbb{1}_{\{z>\xi\}} \frac{\exp\left(-\frac{c^2}{2}\left(\frac{(T-t)^2}{z-\xi} - \frac{T^2}{z}\right)\right)}{(1 - \xi/z)^{3/2}} f_{GIG}(z; \lambda, cT, \gamma)\, dz. \tag{78}$$

16

Writing

$$\kappa = \left(\frac{\gamma}{cT}\right)^{\lambda} \frac{1}{2\,K_{\lambda}[\gamma\sqrt{T}]},\tag{79}$$

we have

$$\psi_t(\mathbb{R};y) = \kappa(1 - \tfrac{t}{T})\mathrm{e}^{-\frac{1}{2}\gamma^2 y} \int_y^{\infty} z^{\lambda+\frac{1}{2}}\,\frac{\mathrm{e}^{-\frac{c^2}{2}\frac{(T-t)^2}{z-y}-\frac{1}{2}\gamma^2(z-y)}}{(z-y)^{3/2}}\,\mathrm{d}z$$

$$= \kappa(1 - \tfrac{t}{T})\mathrm{e}^{-\frac{1}{2}\gamma^2 y} \int_0^{\infty} (z+y)^{\lambda+\frac{1}{2}}\,\frac{\mathrm{e}^{-\frac{c^2}{2}\frac{(T-t)^2}{z}-\frac{1}{2}\gamma^2 z}}{z^{3/2}}\,\mathrm{d}z$$

$$= \frac{\kappa\sqrt{2\pi}}{cT}\mathrm{e}^{-\frac{1}{2}\gamma^2 y-\gamma c(T-t)} \int_0^{\infty} (z+y)^{\lambda+\frac{1}{2}}q_{T-t}(z)\,\mathrm{d}z.\tag{80}$$

Given $\xi_{tT} = y$, the best-estimate ultimate loss is

$$U_{tT} = \psi_t(\mathbb{R};y)^{-1} \int_y^{\infty} z\,\psi_t(\mathrm{d}z;y) = \frac{\int_0^{\infty}(z+y)^{\lambda+\frac{3}{2}}q_{T-t}(z)\,\mathrm{d}z}{\int_0^{\infty}(z+y)^{\lambda+\frac{1}{2}}q_{T-t}(z)\,\mathrm{d}z}.\tag{81}$$

9.2 Case $\lambda = -1/2$

When $\lambda = -1/2$ we have

$$\frac{\psi_t(\mathbb{R};y)}{\psi_s(\mathbb{R};x)}f_{t-s}(y-x) = \mathbb{1}_{\{y-x>0\}}\frac{1}{\sqrt{2\pi}}\frac{c(t-s)}{(y-x)^{3/2}}\exp\left(-\frac{\gamma^2}{2}\frac{((y-x)-c(t-s)/\gamma)^2}{y-x}\right)$$

$$= q_{t-s}(y-x).\tag{82}$$

Hence $\{\xi_{tT}\}$ is an IG process. Note that in this case $\{\xi_{tT}\}$ has independent increments.

9.3 Case $\lambda = n - \frac{1}{2}$

We now consider the case where $\lambda = n - \frac{1}{2}$, for $n \in \mathbb{N}_+$. For convenience we write

$$q_t^{(k)}(x) = f_{GIG}(x; k-1/2, ct, \gamma),\tag{83}$$

Hence we have $q_t^{(0)}(x) = q_t(x)$. The transition density of $\{\xi_{tT}\}$ is then

$$\frac{\psi_t(\mathbb{R};y)}{\psi_s(\mathbb{R};x)}f_{t-s}(y-x) = q_{t-s}(y-x)\frac{\int_0^{\infty}(z+y)^n q_{T-t}(z)\,\mathrm{d}z}{\int_0^{\infty}(z+x)^n q_{T-s}(z)\,\mathrm{d}z}$$

$$= q_{t-s}(y-x)\frac{\sum_{k=0}^{n}\binom{n}{k}m_{T-t}^{(n-k)}\,y^k}{\sum_{k=0}^{n}\binom{n}{k}m_{T-s}^{(n-k)}\,x^k}.\tag{84}$$

When $n = 1$ this is

$$\frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} f_{t-s}(y - x) = q_{t-s}(y - x)\frac{y + \frac{c}{\gamma}(T - t)}{x + \frac{c}{\gamma}(T - s)}$$

$$= \left(1 - \frac{c(t - s)}{\gamma x + c(T - s)}\right) q_{t-s}^{(0)}(y - x)$$

$$+ \left(\frac{c(t - s)}{\gamma x + c(T - s)}\right) q_{t-s}^{(1)}(y - x). \qquad (85)$$

Thus the increment density is a weighted sum of GIG densities. We shall now derive a weighted sum representation for general n. We can write

$$\int_0^\infty (z + y)^n \, q_{T-t}(z) \, \mathrm{d}z = \int_0^\infty ((z + x) + (y - x))^n \, q_{T-t}(z) \, \mathrm{d}z$$

$$= \sum_{k=0}^n \binom{n}{k}(y - x)^{n-k} \int_0^\infty (z + x)^k \, q_{T-t}(z) \, \mathrm{d}z$$

$$= \sum_{k=0}^n \binom{n}{k}(y - x)^{n-k} \sum_{j=0}^k \binom{k}{j} m_{T-t}^{(k-j)} x^k. \qquad (86)$$

Then we have

$$\frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} f_{t-s}(y - x) = q_{t-s}(y - x)\frac{\int_0^\infty (z + y)^n q_{T-t}(z) \, \mathrm{d}z}{\int_0^\infty (z + x)^n q_{T-s}(z) \, \mathrm{d}z}$$

$$= q_{t-s}(y - x)\frac{\sum_{k=0}^n \binom{n}{k}(y - x)^{n-k} \sum_{j=0}^k \binom{k}{j} m_{T-t}^{(k-j)} x^j}{\sum_{k=0}^n \binom{n}{k} m_{T-s}^{(n-k)} x^k}. \qquad (87)$$

However, when $k \in \mathbb{N}_0$,

$$\frac{z^k \, q_{t-s}(z)}{q_{t-s}^{(k)}(z)} = \frac{z^k \, f_{GIG}(z; -1/2, c(t - s), \gamma)}{f_{GIG}(z; k - 1/2, c(t - s), \gamma)}$$

$$= \left(\frac{c(t - s)}{\gamma}\right)^k \frac{K_{k-1/2}[\gamma c(t - s)]}{K_{1/2}[\gamma c(t - s)]}$$

$$= m_{t-s}^{(k)}. \qquad (88)$$

Hence we have

$$(y - x)^{n-k} \, q_{t-s}(y - x) = m_{t-s}^{(n-k)} \, q_{t-s}^{(n-k)}(y - x). \qquad (89)$$

Using the identity (89), (87) can be expanded to obtain

$$\frac{\psi_t(\mathbb{R}; y)}{\psi_s(\mathbb{R}; x)} f_{t-s}(y - x) = \sum_{k=0}^n w_{st}^{(k)}(x) \, q_{t-s}^{(k)}(y - x), \qquad (90)$$

18

where

$$w_{st}^{(k)}(x) = \frac{\binom{n}{k} m_{t-s}^{(n-k)} \sum_{j=0}^{k} \binom{k}{j} m_{T-t}^{(k-j)} x^j}{\sum_{j=0}^{n} \binom{n}{j} m_{T-s}^{(n-j)} x^j}. \tag{91}$$

Notice that $w_{st}^{(k)}(x)$ is a rational function whose denominator is a polynomial of order n, and whose numerator is a polynomial of order $k \leq n$. Thus the transition probabilities of $\{\xi_{tT}\}$ depend on the first n integer powers of the current value. The conditional law of the ultimate loss is

$$\frac{\psi_s(\mathrm{d}y; \xi_{sT})}{\psi_s(\mathbb{R}; \xi_{sT})} = \frac{y^n q_{T-s}^{(0)}(y - \xi_{sT})}{\sum_{k=0}^{n} \xi_{sT}^k m_{T-s}^{(n-k)}} \, \mathrm{d}y. \tag{92}$$

We can verify that $\sum_{k=0}^{n} w_{st}^{(k)}(x) = 1$ using the fact that IG densities are closed under convolution. We have

$$q_{T-s}(z) = \int_0^z q_{T-t}(y) q_{t-s}(z - y) \, \mathrm{d}y, \quad \text{for } 0 \leq s < t < T. \tag{93}$$

For fixed $n \in \mathbb{N}_+$, we then have

$$\sum_{k=0}^{n} \binom{n}{k} m_{T-s}^{(n-k)} x^k = \int_0^\infty (z + x)^n q_{T-s}(z) \, \mathrm{d}z$$

$$= \int_0^\infty (z + x)^n \int_0^z q_{T-t}(y) q_{t-s}(z - y) \, \mathrm{d}y \, \mathrm{d}z$$

$$= \int_0^\infty q_{T-t}(y) \int_y^\infty (z + x)^n \, q_{t-s}(z - y) \, \mathrm{d}z \, \mathrm{d}y$$

$$= \int_0^\infty q_{T-t}(y) \int_0^\infty (z + y + x)^n \, q_{t-s}(z) \, \mathrm{d}z \, \mathrm{d}y$$

$$= \int_0^\infty q_{t-s}(y) \left[\sum_{k=0}^{n} \binom{n}{k} m_{t-s}^{(n-k)} (y + x)^k \right] \mathrm{d}y$$

$$= \sum_{k=0}^{n} \binom{n}{k} m_{t-s}^{(n-k)} \int_0^\infty (y + x)^k \, q_{t-s}(y) \, \mathrm{d}y$$

$$= \sum_{k=0}^{n} \binom{n}{k} m_{t-s}^{(n-k)} \sum_{j=0}^{k} \binom{k}{j} m_{T-t}^{(k-j)} x^j, \tag{94}$$

which gives

$$\sum_{k=0}^{n} \frac{\binom{n}{k} m_{t-s}^{(n-k)} \sum_{j=0}^{k} \binom{k}{j} m_{T-t}^{(k-j)} x^j}{\sum_{j=0}^{n} \binom{n}{j} m_{T-s}^{(n-j)} x^j} = 1. \tag{95}$$

9.4 Moments of the paid-claims process

The best-estimate ultimate loss simplifies to

$$U_{tT} = \frac{\sum_{k=0}^{n+1} \binom{n+1}{k} m_{T-t}^{(n+1-k)} \xi_{tT}^k}{\sum_{k=0}^n \binom{n}{k} m_{T-t}^{(n-k)} \xi_{tT}^k}. \tag{96}$$

For example, when $n = 1$ we obtain

$$U_{tT} = \frac{c(T-t)(1 + \gamma c(T-t)) + 2\gamma^2 c(T-t)\xi_{tT} + \gamma^3 \xi_{tT}^2}{\gamma^2 c(T-t) + \gamma^3 \xi_{tT}}. \tag{97}$$

By similar calculations, we have

$$\mathbb{E}[\xi_{TT}^m \,|\, \xi_{tT}] = \frac{\sum_{k=0}^{n+m} \binom{n+m}{k} m_{T-t}^{(n+m-k)} \xi_{tT}^k}{\sum_{k=0}^n \binom{n}{k} m_{T-t}^{(n-k)} \xi_{tT}^k} \quad \text{for } m \in \mathbb{N}_+, \tag{98}$$

and

$$\mathbb{E}\left[e^{\frac{1}{2}\alpha^2 \xi_{TT}} \,\Big|\, \xi_{tT} \right] = \frac{\sum_{k=0}^n \binom{n}{k} \bar{m}_{T-t}^{(n-k)} \xi_{tT}^k}{\sum_{k=0}^n \binom{n}{k} m_{T-t}^{(n-k)} \xi_{tT}^k} \exp\left(\tfrac{1}{2}\alpha^2 \xi_{tT} - (T-t)(\bar{\gamma} - \gamma) \right), \tag{99}$$

for $0 < \alpha < \gamma$, where $\bar{\gamma} = \sqrt{\gamma^2 - \alpha^2}$, and $\bar{m}_t^{(k)}$ is the kth moment of the IG distribution with parameters $\delta = ct$ and $\gamma = \bar{\gamma}$.

10 Exposure adjustment

We have seen that

$$\mathbb{E}[\xi_{tT}] = \frac{t}{T}\mathbb{E}[U_T]; \tag{100}$$

thus in the model so far the development of the paid-claims process is expected to be linear. This is not always the case in practice. In some cases the marginal exposure is (strictly) decreasing as the development approaches run-off. This manifests itself as

$$\frac{\partial^2}{\partial t^2}\mathbb{E}[\xi_{tT}] < 0, \tag{101}$$

for t close to T. A straightforward method to adjust the development pattern is through a time change. We describe the marginal exposure of the insurer through time by a deterministic function $\varepsilon : [0, T] \to \mathbb{R}_+$. The total exposure of the insurer is

$$\int_0^T \varepsilon(s)\,\mathrm{d}s. \tag{102}$$

We define the increasing function $\tau(t)$ by

$$\tau(t) = T\frac{\int_0^t \varepsilon(s)\,\mathrm{d}s}{\int_0^T \varepsilon(s)\,\mathrm{d}s}. \tag{103}$$

By construction $\tau(0) = 0$ and $\tau(T) = T$.

Now let $\tau(t)$ determine the *operational time* in the model. We define the time-changed paid-claims process $\{\xi_{tT}^\tau\}$ by

$$\xi_{tT}^\tau = \xi(\tau(t), T), \tag{104}$$

and set the reserving filtration to be the natural filtration of $\{\xi_{tT}^\tau\}$. Then we have

$$\mathbb{E}[\xi_{tT}^\tau] = \frac{\int_0^t \varepsilon(s)\,\mathrm{d}s}{\int_0^T \varepsilon(s)\,\mathrm{d}s}\mathbb{E}[U_T] \tag{105}$$

and

$$\frac{\partial^2}{\partial t^2}\mathbb{E}[\xi_{tT}^\tau] = \frac{\mathbb{E}[U_T]}{\int_0^T \varepsilon(s)\,\mathrm{d}s}\varepsilon'(t). \tag{106}$$

10.1 Craighead curve

Craighead [15] proposed fitting a Weibull distribution function to the development pattern of paid claims for forecasting the ultimate loss (see also Benjamin & Eagles [6]). In actuarial work, the Weibull distribution function is sometimes referred to as the 'Craighead curve'. To achieve a similar development pattern we can use the Weibull density as the marginal exposure:

$$\varepsilon(t) = \frac{b}{a}\left(x/a\right)^{b-1}\mathrm{e}^{-(x/a)^b}, \tag{107}$$

for $a, b > 0$. Then the time change $\tau(t)$ is the renormalised, truncated Weibull distribution function

$$\tau(t) = T\frac{1 - \mathrm{e}^{-(t/a)^b}}{1 - \mathrm{e}^{-(T/a)^b}}. \tag{108}$$

See Figure 2 for plots of this function. When $b \leq 1$, $\tau'(t)$ is decreasing. Under such a time change, the marginal exposure is decreasing for all $t \in [0, T]$. When $b > 1$, $\tau'(t)$ achieves its maximum at

$$t^* = a\left(\frac{b-1}{b}\right)^{1/b}, \tag{109}$$

and $\tau'(t)$ is decreasing for $t \geq t^*$. Thus, if $T > t^*$ then the marginal exposure is decreasing for $t \in [t^*, T]$. If $T \leq t^*$ then the marginal exposure is increasing for $t \in [0, T]$.
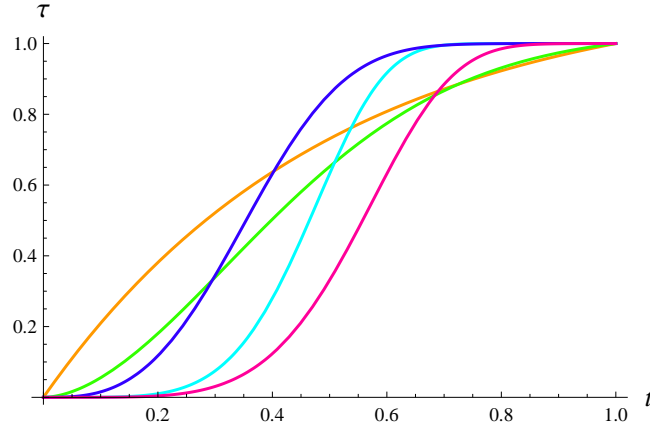
Figure 2: Plots of the truncated Weibull time change for various parameters, and with $T = 1$. The expected paid-claims development of the model will have the same profile as $\tau(t)$ (scaled by $\mathbb{E}[U_T]$). Hence, under one of the above time changes, when t is close to T the marginal exposure falls (i.e. $\frac{\partial^2}{\partial t^2}\mathbb{E}[\xi_{tT}^\tau] < 0$).

11 Simulation

We consider the simulation of sample paths of a stable-$1/2$ random bridge. First, we can generalise (14) to

$$
\left[\xi(\tfrac{s+t}{2}, T) \,\big|\, \xi(s, T) = y, \xi(t, T) = z\right] \stackrel{\text{law}}{=}
$$
$$
y + \frac{1}{2}(z - y)\left(1 + \frac{Z}{\sqrt{c^2(t-s)^2/(z-y) + Z^2}}\right), \quad (110)
$$

where $0 < s < t \leq T$, and Z is a standard normal random variable. We can then generate a discretised sample path $\{\hat\xi(t_i, T)\}_{i=0}^{2^n}$, where $t_i = iT2^{-n}$, by the following recursive algorithm:

1. Generate the variate $\hat\xi(T, T)$ with law ν, and set $\hat\xi(0, T) = 0$.

2. Generate $\hat\xi(\tfrac{T}{2}, T)$ from $\hat\xi(0, T)$ and $\hat\xi(T, T)$ using identity (110).

3. Generate $\hat\xi(\tfrac{T}{4}, T)$ from $\hat\xi(0, T)$ and $\hat\xi(\tfrac{T}{2}, T)$, and then generate $\hat\xi(\tfrac{3T}{4}, T)$ from $\hat\xi(\tfrac{T}{2}, T)$ and $\hat\xi(T, T)$.

4. Generate $\hat\xi(\tfrac{T}{8}, T)$, $\hat\xi(\tfrac{3T}{8}, T)$, $\hat\xi(\tfrac{5T}{8}, T)$, $\hat\xi(\tfrac{7T}{8}, T)$.

5. Etc.

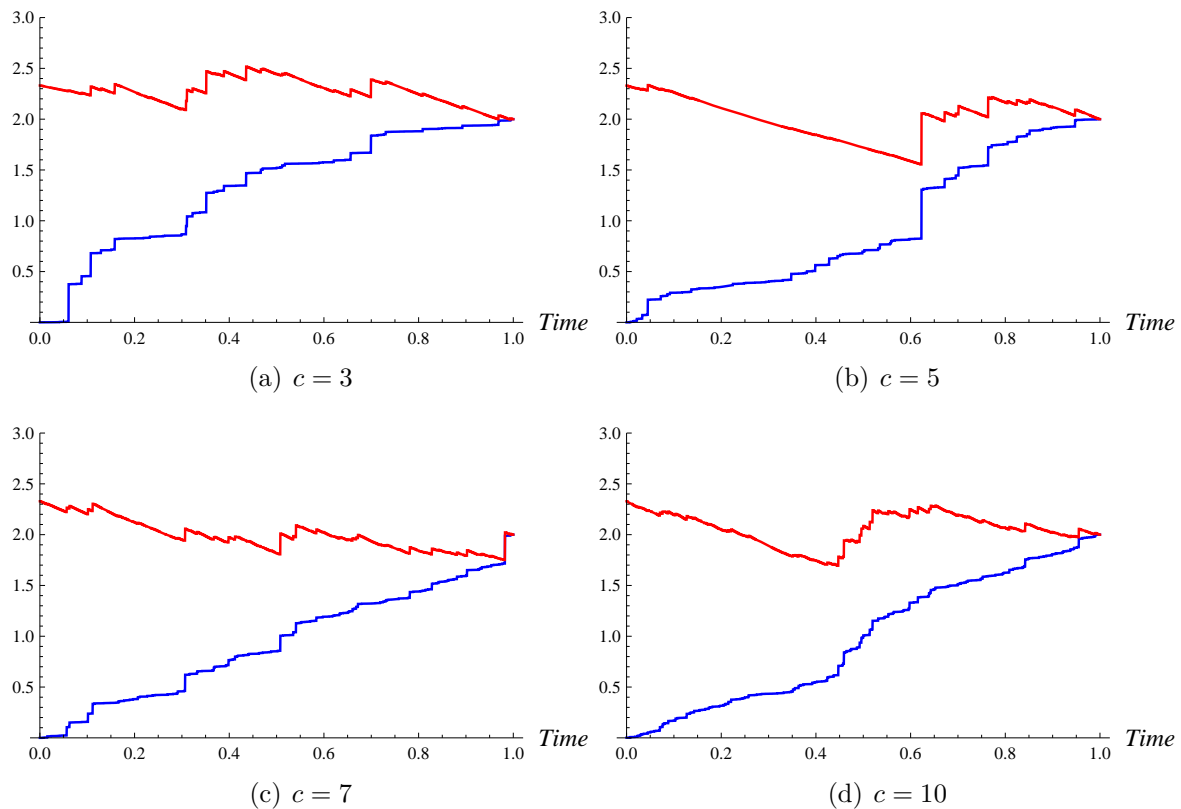See Figure 3 for example simulations.

Figure 3: Simulations of the paid-claims process $\{\xi_{tT}\}$ (bottom line) and the best-estimate process $\{U_{tT}\}$ (top line). Various values of the activity parameter c are used. *A priori*, the ultimate loss U_T has a generalized Pareto distribution (GPD) with density $f_{GPD}(x) = \mathbb{1}_{\{x>1\}} \left(1 + \frac{x-1}{4}\right)^{-5}$. (This is the GPD with scale parameter $\sigma = 1$, location parameter $\mu = 1$, and shape parameter $\xi = 1/4$.)

12   Multiple lines of business

We shall generalise the paid-claims model to achieve two goals. The first is to allow more than one paid-claims process, and allow dependence between the processes. The second is to keep the dimensionality of the calculations low with a view to practicality. The following results can be applied to the modelling of multiple lines of business or multiple origin years when there is dependence between loss processes.

12.1   Two paid-claims processes

We consider a case with two paid-claims processes, but the results can be extended to higher dimensions. In what follows, we set $f_t^c(x) = f_t(x)$ as given by (4), and $f_{tT}^c(x) = f_{tT}(x)$ as given by (11). Here we have introduced the superscript to emphasise the dependence on c. Let $\{S(t, T^*)\}$ be a stable-1/2 random bridge with terminal

density $p(z) = \nu(\mathrm{d}z)/\mathrm{d}z$, and with activity parameter c. Fix a time $T < T^*$, then define two paid-claims processes by

$$\xi_{tT}^{(1)} = S(t, T^*) \qquad\qquad (0 \le t \le T), \qquad (111)$$

$$\xi_{tT}^{(2)} = k^2 S(\lambda t + T, T^*) - k^2 S(T, T^*) \qquad (0 \le t \le T), \qquad (112)$$

where $\lambda = T^*/T - 1$, and $k = c_2/(c\lambda)$ for some $c_2 > 0$. The density of $\xi_{TT}^{(1)}$ is given by

$$
\begin{aligned}
p^{(1)}(x) &= f_T^c(x) \int_0^\infty \frac{f_{T^*-T}^c(z - x)}{f_{T^*}^c(z)} p(z)\, \mathrm{d}z \\
&= \int_0^\infty f_{T,T^*}^c(x; z)\, p(z)\, \mathrm{d}z,
\end{aligned}
\qquad (113)
$$

and the density of $\xi_{TT}^{(2)}$ is

$$
\begin{aligned}
p^{(2)}(x) &= k^{-2} f_{T^*-T}^c(k^{-2}x) \int_0^\infty \frac{f_T^c(z - k^{-2}x)}{f_{T^*}^c(z)} p(z)\, \mathrm{d}z \\
&= k^{-4} f_{T^*-T}^c(k^{-2}x) \int_0^\infty \frac{f_T^c(k^{-2}z - k^{-2}x)}{f_{T^*}^c(k^{-2}z)} p(k^{-2}z)\, \mathrm{d}z \qquad (114) \\
&= k^{-4} \int_0^\infty f_{T^*-T,T^*}^c(k^{-2}x; k^{-2}z)\, p(k^{-2}z)\, \mathrm{d}z \qquad (115) \\
&= k^{-2} \int_0^\infty f_{T,\lambda^{-1}T^*}^{c_2}(x; z)\, p(k^{-2}z)\, \mathrm{d}z. \qquad (116)
\end{aligned}
$$

Here (114) follows after a change of variable, (115) follows from the definition of $f_{tT}(y; z)$ given in (10), and (116) follows from the functional form of $f_{tT}(y; z)$ given in (11). It follows from the dynamic consistency property that $\{\xi_{tT}^{(1)}\}$ is a stable-1/2 random bridge with terminal density $p^{(1)}(z)$ and activity parameter c. Using the dynamic consistency property and the scaling property of stable-1/2 bridges, one can show that $\{\xi_{tT}^{(2)}\}$ is a stable-1/2 bridge with terminal density $p^{(2)}(z)$ and activity parameter c_2. The conditional, joint density of $(\xi_{tT}^{(1)}, k^{-2}\xi_{tT}^{(2)})$ is

$$
\mathbb{Q}\left[\xi_{tT}^{(1)} \in \mathrm{d}y_1, k^{-2}\xi_{tT}^{(2)} \in \mathrm{d}y_2 \,\Big|\, \xi_{sT}^{(1)} = x_1, k^{-2}\xi_{sT}^{(2)} = x_2 \right] =
$$

$$
\left\{ \int_{z=x_1+x_2}^\infty \frac{f_{T^*-(1+\lambda)t}^c(z - (y_1 + y_2))}{f_{T^*-(1+\lambda)s}^c(z - (x_1 + x_2))} p(z)\, \mathrm{d}z \right\} f_{t-s}^c(y_1 - x_1)\, \mathrm{d}y_1\, f_{\lambda(t-s)}^c(y_2 - x_2)\, \mathrm{d}y_2,
$$

$$(117)$$

for $0 \leq s < t \leq T$. Then we have

$$\mathbb{Q}\left[\xi_{tT}^{(1)} + k^{-2}\xi_{tT}^{(2)} \in \mathrm{d}y \,\Big|\, \xi_{sT}^{(1)} = x_1, k^{-2}\xi_{sT}^{(2)} = x_2\right]$$

$$= \left\{\int_{z=x_1+x_2}^{\infty} \frac{f_{T^*-(1+\lambda)t}^c(z-y)}{f_{T^*-(1+\lambda)s}^c(z-(x_1+x_2))} p(z) \,\mathrm{d}z\right\} f_{(1+\lambda)(t-s)}^c(y-(x_1+x_2)) \,\mathrm{d}y$$

$$= \left\{\int_{z=x_1+x_2}^{\infty} f_{(1+\lambda)(t-s),T^*-(1+\lambda)s}^c(z-(x_1+x_2); y-(x_1+x_2)) \, p(z) \,\mathrm{d}z\right\} \mathrm{d}y; \quad (118)$$

and, given $\xi_{sT}^{(1)} = x_1$ and $k^{-2}\xi_{sT}^{(2)} = x_2$, the marginal density of $\xi_{tT}^{(1)}$ is

$$y_1 \mapsto \int_{z=x_1+x_2}^{\infty} f_{t-s,T^*-(1+\lambda)s}^c(y_1-x_1; z-(x_1+x_2)) \, p(z) \,\mathrm{d}z, \quad (119)$$

and the marginal density of $k^{-2}\xi_{tT}^{(2)}$ is

$$y_2 \mapsto \int_{z=x_1+x_2}^{\infty} f_{\lambda(t-s),T^*-(1+\lambda)s}^c(y_2-x_2; z-(x_1+x_2)) \, p(z) \,\mathrm{d}z. \quad (120)$$

12.2 Correlation

The *a priori* correlation between the terminal values is well defined when the second moment of ν is finite. The correlation can be used as a tool in the calibration of the model. Assuming that $\mathbb{E}[S(T^*,T^*)^2] < \infty$, the correlation is defined as

$$\frac{\mathbb{E}\left[\xi_{TT}^{(1)}\xi_{TT}^{(2)}\right] - \mathbb{E}\left[\xi_{TT}^{(1)}\right]\mathbb{E}\left[\xi_{TT}^{(2)}\right]}{\sqrt{\left(\mathbb{E}\left[\left(\xi_{TT}^{(1)}\right)^2\right] - \mathbb{E}\left[\xi_{TT}^{(1)}\right]^2\right)\left(\mathbb{E}\left[\left(\xi_{TT}^{(2)}\right)^2\right] - \mathbb{E}\left[\xi_{TT}^{(2)}\right]^2\right)}}. \quad (121)$$

We shall calculate each of the components of (121) separately. First, we have

$$\mathbb{E}\left[\xi_{TT}^{(1)}\right] = \mathbb{E}[S(T,T^*)] = \frac{T}{T^*}\mathbb{E}[S(T^*,T^*)]. \quad (122)$$

Noting that

$$\xi_{TT}^{(2)} = k^2(S(T^*,T^*) - S(T,T^*))$$
$$\stackrel{\mathrm{law}}{=} k^2 S(T^*-T,T^*), \quad (123)$$

we have

$$\mathbb{E}\left[\xi_{TT}^{(2)}\right] = k^2\mathbb{E}[S(T^*-T,T^*)]$$
$$= k^2\left(1 - \frac{T}{T^*}\right)\mathbb{E}[S(T^*,T^*)]. \quad (124)$$

The second moments of $\xi_{TT}^{(1)}$ and $\xi_{TT}^{(2)}$ follow from (43), and are given by

$$\mathbb{E}\left[\left(\xi_{TT}^{(1)}\right)^2\right] = \frac{T}{T^*}\mathbb{E}\left[S(T^*, T^*)^2\right] - (T^* - T)C_{T^*}, \tag{125}$$

and

$$\mathbb{E}\left[\left(\xi_{TT}^{(2)}\right)^2\right] = k^4\left(1 - \frac{T}{T^*}\right)\mathbb{E}\left[S(T^*, T^*)^2\right] - k^4 T C_{T^*}, \tag{126}$$

where

$$C_{T^*} = c\sqrt{2\pi}\int_0^\infty z^{3/2}\,\mathrm{e}^{\frac{c^2 T^{*2}}{2z}}\,\Phi\left[-cT^* z^{-1/2}\right]\,p(z)\,\mathrm{d}z. \tag{127}$$

The final term required for working out the correlation is the cross moment. This is

$$\mathbb{E}\left[\xi_{TT}^{(1)}\,\xi_{TT}^{(2)}\right] = k^2\mathbb{E}\left[S(T, T^*)\left(S(T^*, T^*) - S(T, T^*)\right)\right]$$
$$= k^2\mathbb{E}\left[S(T, T^*)S(T^*, T^*)\right] - k^2\mathbb{E}\left[S(T, T^*)^2\right]. \tag{128}$$

The first term on the right of (128) is

$$k^2\mathbb{E}\left[S(T, T^*)S(T^*, T^*)\right] = k^2\int_0^\infty\int_0^\infty x\,y\,\frac{f_T^c(x)f_{T^*-T}^c(y - x)}{f_{T^*}^c(y)}\,\mathrm{d}x\,p(y)\,\mathrm{d}y$$
$$= k^2\int_0^\infty\int_0^\infty x\,y\,f_{T,T^*}^c(x; y)\,\mathrm{d}x\,p(y)\,\mathrm{d}y$$
$$= k^2\frac{T}{T^*}\int_0^\infty y^2\,p(y)\,\mathrm{d}y$$
$$= k^2\frac{T}{T^*}\mathbb{E}[S(T^*, T^*)^2]. \tag{129}$$

The second term on the right of (128) is given by (125). Hence we have

$$\mathbb{E}\left[\xi_{TT}^{(1)}\,\xi_{TT}^{(2)}\right] = k^2(T^* - T)C_{T^*}. \tag{130}$$

The expression for the correlation follows by putting equations (122), (124), (125), (126), and (130) together.

12.3 Ultimate loss estimation

We now estimate the terminal values of the paid-claims processes. At time $t < T$, the best-estimate ultimate loss of $\{\xi_{tT}^{(1)}\}$ (or, indeed, $\{\xi_{tT}^{(2)}\}$) depends on the two values $\xi_{tT}^{(1)}$

and $\xi_{tT}^{(2)}$. The best-estimate ultimate loss of $\{\xi_{tT}^{(1)}\}$ is

$$
\begin{aligned}
U_{tT}^{(1)} &= \mathbb{E}\left[\xi_{TT}^{(1)} \,\middle|\, \xi_{tT}^{(1)} = x_1, \xi_{tT}^{(2)} = x_2\right] \\
&= \mathbb{E}\left[S(T,T^*) \,\middle|\, S(t,T^*) = x_1, S(T+\lambda t, T^*) - S(T,T^*) = k^{-2}x_2\right] \\
&= \mathbb{E}\left[S(T+\lambda t, T^*) \,\middle|\, S(t,T^*) = x_1, S(T+\lambda t, T^*) - S(T,T^*) = k^{-2}x_2\right] - k^{-2}x_2 \\
&= \mathbb{E}\left[S(T+\lambda t, T^*) \,\middle|\, S(t,T^*) = x_1, S((1+\lambda)t, T^*) - S(t,T^*) = k^{-2}x_2\right] - k^{-2}x_2
\end{aligned}
$$
$$(131)$$
$$
= \mathbb{E}\left[S(T+\lambda t, T^*) \,\middle|\, S((1+\lambda)t, T^*) = x_1 + k^{-2}x_2\right] - k^{-2}x_2 \tag{132}
$$
$$
\begin{aligned}
&= \frac{T-t}{T^* - (1+\lambda)t}\left(\mathbb{E}\left[S(T^*,T^*) \,\middle|\, S((1+\lambda)t, T^*) = x_1 + k^{-2}x_2\right] - k^{-2}x_2\right) \\
&\qquad + \frac{T^* - (T-t)}{T^* - (1+\lambda)t}x_1.
\end{aligned}
$$
$$(133)$$

Equation (131) holds since reordering the increments of an LRB yields an LRB with same law, (132) follows from the Markov property of LRBs, and (133) follows from (42). We also have

$$
\mathbb{E}\left[S(T^*,T^*) \,\middle|\, S((1+\lambda)t, T^*) = x_1 + k^{-2}x_2\right] = \int_0^\infty z\, p_t(z)\, \mathrm{d}z, \tag{134}
$$

where

$$
\begin{aligned}
p_t(z) = \mathbb{1}_{\{z > x_1 + k^{-2}x_2\}} K^{-1} &\left(\frac{z}{z - (x_1 + k^{-2}x_2)}\right)^{3/2} \\
&\times \exp\left(-\frac{c^2}{2}\left(\frac{(T^* - (1+\lambda)t)^2}{z - (x_1 + k^{-2}x_2)} - \frac{T^{*2}}{z}\right)\right) p(z),
\end{aligned} \quad (135)
$$

and K is a constant chosen to normalise the density. Similarly, the best-estimate ultimate loss of $\{\xi_{tT}^{(2)}\}$ is

$$
\begin{aligned}
U_{tT}^{(2)} = k^2 \frac{T^* - (T-t)}{T^* - (1+\lambda)t}&\left(\mathbb{E}\left[S(T^*,T^*) \,\middle|\, S((1+\lambda)t, T^*) = x_1 + k^{-2}x_2\right] - x_1\right) \\
&+ \frac{T-t}{T^* - (1+\lambda)t}x_2. \quad (136)
\end{aligned}
$$

To compute both $U_{tT}^{(1)}$ and $U_{tT}^{(2)}$ we need to perform at most two one-dimensional integrals (the integral we need is (134), but we note that $p_t(x)$ includes a normalising constant K—which is found be evaluating a second integral). We are saved the complication of performing double integrals.

To extend these results to higher dimensions we can split the 'master' process $\{S_{tT}\}$ into more than two subprocesses. Regardless of the number of subprocesses (i.e. paid-claims processes), all of the best-estimate ultimate losses can be computed by performing at most two one-dimensional integrals. This makes such a multivariate model highly computationally efficient.

Acknowledgements

References

[1] M. Abramowitz & I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables.* Dover, New York, 1964.

[2] E. Arjas. The claims reserving problem in non-life insurance: some structural ideas. *ASTIN Bulletin*, 19:139–152, 1989.

[3] O.E. Barndorff-Nielsen & B. Jørgensen. Some parametric models on the simplex. *Journal of Multivariate Analysis*, 39:106–116, 1991.

[4] F. Baudoin. Conditioned stochastic differential equations: theory, examples and application to finance. *Stochastic Processes and their Applications*, 100:109–145, 2002.

[5] F. Baudoin & L. Nguyen-Ngoc. The financial value of a weak information on a financial market. *Finance and Stochastics*, 8:415–435, 2004.

[6] S. Benjamin & L.M. Eagles. A curve fitting method and a regression method. In *Claims Reserving Manual*, volume 2, D3. Faculty and Institute of Actuaries, 1997. \{www.actuaries.org.uk/__data/assets/pdf_file/0012/24321/crm2-D3.pd%f\}.

[7] J. Bertoin. *Lévy Processes.* Cambridge Univeristy Press, Cambridge, 1996.

[8] R.L. Bornhuetter & R.E. Ferguson. The Actuary and IBNR. *Proceedings of the Casualty Actuarial Society*, 59:181–195, 1972.

[9] D.C. Brody, L.P. Hughston & A. Macrina. Beyond hazard rates: A new framework for credit-risk modelling. In R. Elliot, M. Fu, R. Jarrow & Ju-Yi Yen, editors, *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan.* Birkhäuser/Springer, 2007.

[10] D.C. Brody, L.P. Hughston & A. Macrina. Information-based asset pricing. *International Journal of Theoretical and Applied Finance*, 11:107–142, 2008.

[11] D.C. Brody, L.P. Hughston & A. Macrina. Dam rain and cumulative gain. *Proceedings of the Royal Society A*, 464:1801–1822, 2008.

[12] H. Bühlmann. *Mathematical methods in risk theory.* Springer-Verlag, Heidelberg, 1970.

[13] H. Bühlmann & A. Gisler. *A course in credibility theory and its applications.* Springer, Berlin, 2005.

[14] R. Cont & P. Tankov. *Financial Modelling with Jump Processes.* Chapman & Hall, 2004.

[15] D.H. Craighead. Some aspects of the London reinsurance market in world-wide short-term business. *Journal of the Institiute of Actuaries*, 106:227–297, 1979.

[16] E. Eberlein & E.A. v. Hammerstein. Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes. In R. Dalang, M. Dozzi & F. Russo, editors, *Seminar on Stochastic Analysis, Random Fields and Applications IV*, volume 58 of *Progress in Probability*. Birkhäuser, 2004.

[17] P. Embrechts, C. Klüppelberg & T. Mikosch. *Modelling Extremal Events for Insurance and Finance.* Springer-Verlag, Berlin, 1997.

[18] P.D. England & R.J. Verrall. Stochastic claims reserving. *British Actuarial Journal*, 8(3):443–544, 2002.

[19] W. Feller. *An introduction to probability theory and its applications II.* Wiley, New York, 1971.

[20] P.J. Fitzsimmons. Markov process with identical bridges. *Electronic Journal of Probability*, 3(12):1–12, 1998.

[21] P.J. Fitzsimmons, J. Pitman & M. Yor. Markovian bridges: Construction, palm interpretation, and splicing. *Seminar on Stochastic Processes*, 33:102–133, 1993.

[22] E. Hoyle, L.P. Hughston & A. Macrina. Lévy random bridges and the modelling of financial information. *Working Paper*, 2009. arXiv:0912.3652.

[23] B. Jørgensen. *Statistical Properties of the Generalized Inverse Gaussian Distribution*, volume 9 of *Lecture Notes in Statistics*. Springer, New York, 1982.

[24] A.E. Kyprianou. *Introductory Lectures on the Fluctuations of Lévy Processes with Applications.* Springer-Verlag, Berlin, 2006.

[25] A. Macrina. *An information-based framework for asset pricing: X-factor theory and its applications.* PhD thesis, King's College London, 2006. \{www.mth.kcl.ac.uk/finmath/Macrina/PhD_Macrina.pdf\}.

[26] B.B. Mandelbrot. The variation of certain spectulative prices. *The Journal of Business*, 36:394–419, 1963.

[27] B.B. Mandelbrot. *Fractals and scaling in finance: discontinuity, concentration, risk*. Springer-Verlag, New York, 1997.

[28] A.J. McNeil, R. Frey & P. Embrechts. *Quantitative Risk Management*. Princeton University Press, 2005.

[29] T. Mikosch. *Non-life insurance mathematics: an introduction with stochastic processes*. Springer-Verlag, Berlin, 2004.

[30] R. Norberg. Prediction of outstanding liabilities in non-life insurance. *ASTIN Bulletin*, 23:95–115, 1993.

[31] R. Norberg. Prediction of outstanding liabilities II: model variations and extensions. *ASTIN Bulletin*, 29:5–25, 1999.

[32] K.-I. Sato. *Lévy processes and infintely divisble distributions*. Cambridge Univeristy Press, Cambridge, 1999.

[33] M.V. Würthrich & M. Merz. *Stochastic claims reserving methods in insruance*. Wiley, Chichester, 2008.