

OPTIMAL DIVIDEND AND INVESTING CONTROL OF A INSURANCE COMPANY WITH HIGHER SOLVENCY CONSTRAINTS

Zongxia Liang

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.

Email: zliang@math.tsinghua.edu.cn

Jianping Huang

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.

Email: jp.huang08@gamil.com

ABSTRACT. This paper considers optimal control problem of a large insurance company under a fixed insolvency probability. The company controls proportional reinsurance rate, dividend pay-outs and investing process to maximize the expected present value of the dividend pay-outs until the time of bankruptcy. This paper aims at describing the optimal return function as well as the optimal policy. As a by-product, the paper theoretically sets a risk-based capital standard to ensure the capital requirement of can cover the total risk.

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1. Introduction

In this paper we consider optimal control problem of a large insurance company in which the dividend pay-outs, investing process and the risk exposure are controlled by management. The investing process in a financial market may contain an element of risk, so it will impact security and solvency of the company (see Theorem 4.1 below). Moreover, the company has a minimal reserve as its guarantee fund to protect insureds and attract sufficient number of policy holders. We assume that the company can only reduce its risk exposure by proportional reinsurance policy for simplicity. The objective of the company is to find a policy, consisting of risk control and dividend payment scheme, which maximizes the expected total discounted dividend pay-outs until the time of bankruptcy. This is a

mixed regular-singular control problem on diffusion model which has been a renewed interest recently, e.g. He and Liang[18] and references therein, Højgaard and Taksar [14, 13, 12], Harrison and Taksar [11], Paulsen and Gjessing [22], Radner and Shepp [24]. Optimizing dividend pay-outs is a classical problem in actuarial mathematics, on which earlier work is given in e.g. Borch[1, 2] and Gerber[9]. We notice that some of these papers seem not to take security and solvency into consideration and so the results therein may not be commonly used in practice because the insurance business is a business *affected with a public interest*, and insureds and policy-holders should be protected against insurer insolvencies (see Williams and Heins[30](1985), Riegel and Miller[26](1963), and Welson and Taylor[29](1959)). The policy, making the company go bankrupt before termination of contract between insurer and policy holders or the policy of low solvency(see [4]), is not the best way and should be prohibited even though it can win the highest profit. Therefore, *one of our motivations* is to consider optimal control problem of a large insurance company under higher solvency and security, and to find the best equilibrium policy between making profit and improving security.

Unfortunately, there are very few results concerning on optimal control problem of a large insurance company based on higher solvency and security. Paulsen[23] studied this kind of optimal controls for diffusion model via properties of return function, some of our results somewhat like that of the [23], but both approaches used are very different. He, Hou and Liang[20] investigated the optimal control problem for linear Brownian model. However, we find that the case treated in the [20] is a trivial case, that is, the company of the model in the [20] will never go to bankruptcy, it is an ideal model in concept, and it indeed does not exist in reality(see Theorem 4.2 below). Because probability of bankruptcy for the model treated in the present paper is very large (see Theorem 4.1 below), our results can not be directly deduced from the [20]. Therefore, to solve these the problems we need to use initiated idea from the [20], stochastic analysis and PDE method to establish a complete setting for further discussing optimal control problem of a large insurance company under higher solvency and security in which the dividend pay-outs, investing process and the risk

exposure are controlled by management. This is *another one of our motivations*. This paper is the first systematic presentation of the topic, and the approach here is rather general, so we anticipate that it can deal with other models. We aim at deriving the optimal return function, the optimal retention rate and dividend payout level. The main result of this paper will be presented in section 3 below. As a by-product, the paper theoretically sets a risk-based capital standard to ensure the capital requirement of can cover the total given risk. Moreover, we also discuss how the risk and minimum reserve requirement affect the optimal reactions of the insurance company by the implicit types of solutions and how the optimal retention ratio and dividend payout level are affected by the changes in the minimum reserve requirement and risk faced by the insurance company.

The paper is organized as follows: In next section 2 we establish a stochastic control model of a large insurance company. In section 3 we present main result of this paper and its economic and financial interpretations, and discuss how the risk and minimum reserve requirement affects the optimal retention ratio and dividend payout level of the insurance company. In section 4 we give analysis on risk of stochastic control model treated in the present paper and study relationships among investment risk, underwriting risk and the insolvency probability. In section 5 we give some numerical samples to portray how the risk and minimum reserve requirement affect dividend payout level of the insurance company. The proofs of theorems and lemmas which study properties of probability of bankruptcy and optimal return function will be given in the appendix.

2. Mathematical model

To give a mathematical formulation of the optimization problem treated in this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ denote a filtered probability space. For the *intuition* of our diffusion model we start from the classical Cramér-Lundberg model of a reserve(risk) process. In this model claims arrive according to a Poisson process N_t with intensity ν on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The size of each claim is U_i . Random variables U_i are i.i.d. and are independent

of the Poisson process N_t with finite first and second moments given by μ and σ^2 respectively. If there is no reinsurance, dividend pay-outs or investments, the reserve (risk) process of insurance company is described by

$$r_t = r_0 + pt - \sum_{i=1}^{N_t} U_i,$$

where p is the premium rate. If $\eta > 0$ denotes the *safety loading*, the p can be calculated via the expected value principle as

$$p = (1 + \eta)v\mu.$$

In a case where the insurance company shares risk with the reinsurance, the sizes of the claims held by the insurer become $U_i^{(a)}$, where a is a (fixed) retention level. For proportional reinsurance, a denotes the fraction of the claim covered by cedent. Consider the case of *cheap reinsurance* for which the reinsuring company uses the same safety loading as the cedent, the reserve process of the cedent is given by

$$r_t^{(a,\eta)} = u + p^{(a,\eta)}t - \sum_{i=1}^{N_t} U_i^{(a)},$$

where $p^{(a,\eta)} = (1 + \eta)v\mathbb{E}\{U_i^{(a)}\}$. Then as $\eta \rightarrow 0$

$$\{\eta r_{t/\eta^2}^{(a,\eta)}\}_{t \geq 0} \xrightarrow{D} BM(\mu(a)t, \sigma^2(a)t) \quad (2.1)$$

in $\mathcal{D}[0, \infty)$ (the space of right continuous functions with left limits endowed with the skorohod topology), where

$$\mu(a) = v\mathbb{E}\{U_i^{(a)}\}, \quad \sigma^2(a) = v\mathbb{E}\{U_i^{(a)}\}^2,$$

and $BM(\mu, \sigma^2)$ stands for Brownian motion with the drift coefficient μ and diffusion coefficient σ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The passage to the limit works well in the presence of a big portfolios. We refer the reader for this fact and for the specifics of the diffusion approximations to Emanuel, Harrison and Taylor[5](1975), Grandell[6](1977), Grandell[7](1978), Grandell[8](1990), Harrison[10](1985), Iglehart[15](1969), and Schmidli[27](1994).

Throughout this paper we consider the retention level to be the control parameter selected at each time t by the insurance company. We denote this value by $a(t)$. If there is no dividend pay-outs or investments, in view

of (2.1), we can assume that in our model the reserve process $\{R_t\}$ of the insurance company is given by

$$dR_t = a(t)\mu dt + a(t)\sigma dW_t^1,$$

where $U_i^{(a)} = aU_i$, $\mu(a) = a\mathbb{E}\{U_i\}$ and $\sigma^2(a) = a^2\sigma^2$. And the reserve invested in a financial asset is the price process $\{P_t\}$ governed by

$$dP_t = rP_t dt + \sigma_p P_t dW_t^2,$$

where $r > 0$, $\sigma_p \geq 0$, $\{W_t^1\}_{t \geq 0}$ and $\{W_t^2\}_{t \geq 0}$ are two independent standard Brownian motions on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The case of $\sigma_p = 0$ corresponds to the situation where only risk free assets, such as bonds or bank accounts are used for investments.

A policy π is a pair of non-negative càdlàg \mathcal{F}_t -adapted processes $\{a_\pi(t), L_t^\pi\}$, where $a_\pi(t)$ corresponds to the risk exposure at time t and L_t^π corresponds to the cumulative amount of dividend pay-outs distributed up to time t . A policy $\pi = \{a_\pi(t), L_t^\pi\}$ is called admissible if $0 \leq a_\pi(t) \leq 1$ and L_t^π is a nonnegative, non-decreasing, right-continuous function. When π is applied, the resulting reserve process is denoted by $\{R_t^\pi\}$. We assume that the initial reserve R_0^π is a deterministic value x . In view of independence of W^1 and W^2 , the dynamics for R_t^π is given by

$$dR_t^\pi = (a_\pi(t)\mu + rR_t^\pi)dt + \sqrt{a_\pi^2(t)\sigma^2 + \sigma_p^2 \cdot (R_t^\pi)^2} dW_t - dL_t^\pi, \quad (2.2)$$

where $\{W_t\}$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Moreover, we suppose that the insurance company has a minimal reserve m as its guarantee fund to protect insureds and attract sufficient number of policy holders, that is, the company needs to keep its reserve above m . The company is considered bankrupt as soon as the reserve falls below m . We define the time of bankruptcy by $\tau_x^\pi = \inf\{t \geq 0 : R_t^\pi \leq m\}$. Obviously, τ_x^π is an \mathcal{F}_t -stopping time.

We denote by Π the set of all admissible policies. For any $b \geq 0$, let $\Pi_b = \{\pi \in \Pi : \int_0^\infty \mathbb{1}_{\{s: R^\pi(s) < b\}} dL_s^\pi = 0\}$. Then it is easy to see that $\Pi = \Pi_0$ and $b_1 > b_2 \Rightarrow \Pi_{b_1} \subset \Pi_{b_2}$. For a given admissible policy π we define the

optimal return function $V(x)$ by

$$\begin{aligned} J(x, \pi) &= \mathbb{E}\left\{\int_0^{\tau_x^\pi} e^{-ct} dL_t^\pi\right\}, \\ V(x, b) &= \sup_{\pi \in \Pi_b} \{J(x, \pi)\}, \end{aligned} \quad (2.3)$$

$$V(x) = \sup_{b \in \mathfrak{B}} \{V(x, b)\} \quad (2.4)$$

and the optimal policy π^* by

$$J(x, \pi^*) = V(x), \quad (2.5)$$

where

$$\mathfrak{B} := \{b : \mathbb{P}[\tau_b^{\pi_b} \leq T] \leq \varepsilon, J(x, \pi_b) = V(x, b) \text{ and } \pi_b \in \Pi_b\},$$

$c > 0$ is a discount rate, $\tau_b^{\pi_b}$ is the time of bankruptcy $\tau_x^{\pi_x}$ when the initial reserve $x = b$ and the control policy is π_b . $1 - \varepsilon$ is the standard of security and less than solvency for given $\varepsilon > 0$.

The main purpose of this paper is to find the optimal return function $V(x)$ and the optimal policy π^ . Throughout this paper we assume that $r \leq c$ in view of $V(x) = \infty$ for $r > c$ (see Højgaard and Taksar [14]).*

3. Main result

In this section we first introduce an auxiliary Hamilton-Jacobi-Bellman (HJB) equation, then we present main result of this paper, finally we give economic and financial interpretations of the main result.

Lemma 3.1. *Let $h \in C^2[m, \infty)$ satisfy the following HJB equation*

$$\max_{a \in [0,1]} \left\{ \frac{1}{2} [\sigma^2 a^2 + \sigma_p^2 x^2] h''(x) + [\mu a + rx] h'(x) - ch(x) \right\} = 0, \quad x \geq m \quad (3.1)$$

with boundary condition $h(m) = 0$. Then

(i) $h'(x) > 0, \forall x \geq m$.

(ii) *There exists a unique $b_0 > x_0$ such that $h''(b_0) = 0$ and $(x - b_0)h''(x) > 0$ for all $x \geq m$ except b_0 , where $x_0 = \frac{\sigma^2(1-a^*)}{\mu}$, α^* is a constant in $(0, 1)$.*

Proof. The proof of this lemma is standard and can be proved by the same way as in the proof of He and Liang [19], Shreve, Lehoczka and Gaver [28] and Paulsen and Gjessing [22]. So we omit it here. \square

Assume that $h(x)$ is a solution of (3.1). Define functions $F_b(x)$ and $a^*(x)$ by

$$F_b(x) = \begin{cases} 0, & 0 \leq x < m, \\ \frac{h(x)}{h'(b)}, & m \leq x \leq b, \\ x - b + F_b(b), & x \geq b \end{cases} \quad (3.2)$$

and

$$a^*(x) = \begin{cases} \lambda x, & 0 \leq x \leq x_0, \\ 1, & x \geq x_0 \end{cases} \quad (3.3)$$

respectively, where $\lambda = \frac{\mu}{\sigma^2(1-\alpha^*)}$. It easily follows that $F_b \in C^2([m, \infty) \setminus \{b\})$. Now we can present the main result of this paper as follows. We will give rigorous proof of the main result in the appendix.

Theorem 3.1. *Let level of risk $\varepsilon \in (0, 1)$ and time horizon T be given.*

(i) *If $\mathbb{P}[\tau_{b_0}^{\pi_{b_0}^*} \leq T] \leq \varepsilon$ then the optimal return function $V(x)$ is $F_{b_0}(x)$ defined by (3.2), and $V(x) = F_{b_0}(x) = V(x, 0) = J(x, \pi_{b_0}^*)$. The optimal policy $\pi_{b_0}^*$ is $\{a^*(R_t^{\pi_{b_0}^*}), L_t^{\pi_{b_0}^*}\}$, where $\{R_t^{\pi_{b_0}^*}, L_t^{\pi_{b_0}^*}\}$ is uniquely determined by the following stochastic differential equation*

$$\begin{cases} dR_t^{\pi_{b_0}^*} = (a^*(R_t^{\pi_{b_0}^*})\mu + rR_t^{\pi_{b_0}^*})dt + \sqrt{(a^*(R_t^{\pi_{b_0}^*}))^2\sigma^2 + \sigma_p^2 \cdot (R_t^{\pi_{b_0}^*})^2} dW_t \\ \quad - dL_t^{\pi_{b_0}^*}, \\ m \leq R_t^{\pi_{b_0}^*} \leq b_0, \\ \int_0^\infty I_{\{t: R_t^{\pi_{b_0}^*} < b_0\}}(t) dL_t^{\pi_{b_0}^*} = 0. \end{cases} \quad (3.4)$$

The solvency of the company is bigger than $1 - \varepsilon$.

(ii) *If $\mathbb{P}[\tau_{b_0}^{\pi_{b_0}^*} \leq T] > \varepsilon$ then there is a unique optimal dividend $b^* (\geq b_0)$ satisfying $\mathbb{P}[\tau_{b^*}^{\pi_{b^*}^*} \leq T] = \varepsilon$. The optimal return function $V(x)$ is $F_{b^*}(x)$ defined by (3.2), that is,*

$$V(x) = F_{b^*}(x) = \sup_{b \in \mathfrak{B}} \{V(x, b)\}, \quad (3.5)$$

where

$$b^* = \min\{b : \mathbb{P}[\tau_b^{\pi_b^*} \leq T] = \varepsilon\} = \min\{b : b \in \mathfrak{B}\} \in \mathfrak{B} \quad (3.6)$$

and

$$\mathfrak{B} := \{b : \mathbb{P}[\tau_b^{\pi_b} \leq T] \leq \varepsilon, J(x, \pi_b) = V(x, b) \text{ and } \pi_b \in \Pi_b\}.$$

Moreover,

$$V(x) = V(x, b^*) = J(x, \pi_{b^*}^*) \quad (3.7)$$

and the optimal policy $\pi_{b^*}^*$ is $\{a^*(R_t^{\pi_{b^*}^*}), L_t^{\pi_{b^*}^*}\}$, where $\{R_t^{\pi_{b^*}^*}, L_t^{\pi_{b^*}^*}\}$ is uniquely determined by the following stochastic differential equation

$$\begin{cases} dR_t^{\pi_{b^*}^*} = (a^*(R_t^{\pi_{b^*}^*})\mu + rR_t^{\pi_{b^*}^*})dt + \sqrt{(a^*(R_t^{\pi_{b^*}^*}))^2\sigma^2 + \sigma_p^2 \cdot (R_t^{\pi_{b^*}^*})^2} dW_t \\ \quad - dL_t^{\pi_{b^*}^*}, \\ m \leq R_t^{\pi_{b^*}^*} \leq b^*, \\ \int_0^\infty I_{\{t: R_t^{\pi_{b^*}^*} < b^*\}}(t) dL_t^{\pi_{b^*}^*} = 0. \end{cases} \quad (3.8)$$

The solvency of the company is $1 - \varepsilon$.

(iii) For any $x \leq b_0$,

$$\frac{F_{b^*}(x)}{F_{b_0}(x)} = \frac{h'(b^*)}{h'(b_0)} < 1. \quad (3.9)$$

Economic and financial explanation of theorem 3.1 is as follows:

(1) For a given level of risk and time horizon, if probability of bankruptcy is less than the level of risk, the optimal control problem of (2.4) and (2.5) is the traditional one, the company has higher solvency, so it will have good reputation. The solvency constraints here do not work. This is a trivial case. In view of Theorem 4.2 below, the model treated in [20] can be reduced to this trivial case.

(2) If probability of bankruptcy is large than the level of risk, the traditional optimal policy will not meet the standard of security and solvency, the company needs to find a sub-optimal policy $\pi_{b^*}^*$ to improve its solvency. The sub-optimal reserve process $R_t^{\pi_{b^*}^*}$ is a diffusion process reflected at b^* , the process $L_t^{\pi_{b^*}^*}$ is the process which ensures the reflection. The sub-optimal action is to pay out everything in excess of b^* as dividend and pay no dividend when the reserve is below b^* , and $a^*(x)$ is the sub-optimal feedback control function.

(3) On the one hand, the inequality (3.9) states that $\pi_{b^*}^*$ will reduce the company's profit, on the other hand, in view of (3.6) and $\mathbb{P}[\tau_{b^*}^{\pi_{b^*}^*} \leq T] = \varepsilon$ as well as lemma 6.7 below, the cost of improving solvency is minimal. Therefore the policy $\pi_{b^*}^*$ is the best equilibrium action between making profit and improving solvency.

Effect of the risk level ε and minimum reserve requirement m on the optimal reaction and dividend payout level of the insurance company is given as follows:

(4) We see from the figure 4 below(based on PDE(6.2)satisfied by solvency probability) that the dividend payout level b^* is an increasing function of minimum reserve requirement m . Using comparison theorem for one-dimensional Itô process we know that the reserve process $R_t^{\pi_{b^*}^*}$ of the insurance company is also an increasing function of b^* . Therefore, since the sub-optimal feedback control function $a^*(x)$ is increasing with respect to x , by theorem3.1 we conclude that the optimal retention ratio $a^*(R_t^{\pi_{b^*}^*})$ increases with m , that is, increasing minimum reserve requirement will improve the optimal retention ratio. However, this increasing action must result in lower profit because the optimal return function $V(x, b^*)$ is a decreasing of b^* (see Lemma 6.7). So the process $L_t^{\pi_{b^*}^*}$ is a decreasing function of m too.

(5) We see from the figure 3 below that the dividend payout level b^* is a decreasing function of the risk ε . So, by the same argument as in (4) above, the optimal retention ratio $a^*(R_t^{\pi_{b^*}^*})$ decreases with ε , the process $L_t^{\pi_{b^*}^*}$ increases with ε .

(6) We also see from the figure 6 below that, for given the risk ε , the dividend payout level b is an increasing function of underwriting risk σ^2 , so it decreases the company's profit.

Remark 3.1. Because the [20] had no continuity of probability of bankruptcy and actual b^* , the authors of [20] did not obtain the best equilibrium policy $\pi_{b^*}^*$.

Remark 3.2. By (6.2) one knows that the equation $\psi(T, m, b^*) = 1 - \phi(T, m, b^*) = \varepsilon$ can set a risk-based capital standard (m, b^*) to ensure the capital requirement of can cover the total given risk ε , then establish the optimal return function, the optimal retention rate and dividend payout level via Theorem 3.1.

Remark 3.3. By using the same approach as in [14] we can show that the b^* is an increasing function of σ_p^2 , so the company has possibility of making larger gain from the reinvestments. We omit the analysis here. We focus on the effect of investments risk on probability of bankruptcy for the topic of this paper in next section.

4. Analysis on risk of a large insurance company

The first result of this section is the following, which states that the company has to find optimal policy to improve its solvency.

Theorem 4.1. For $b \geq m > 0$, let $\{R_t^{\pi_b^*}, L_t^{\pi_b^*}\}$ be defined by the following SDE(see Lions and Sznitman [21])

$$\left\{ \begin{array}{l} dR_t^{\pi_b^*} = (a^*(R_t^{\pi_b^*})\mu + rR_t^{\pi_b^*})dt + \sqrt{(a^*(R_t^{\pi_b^*}))^2\sigma^2 + \sigma_p^2 \cdot (R_t^{\pi_b^*})^2} dW_t \\ \quad \quad \quad - dL_t^{\pi_b^*}, \\ m \leq R_t^{\pi_b^*} \leq b, \\ \int_0^\infty I_{\{t: R_t^{\pi_b^*} < b\}}(t) dL_t^{\pi_b^*} = 0, \\ R_0^{\pi_b^*} = b. \end{array} \right. \quad (4.1)$$

Then

$$\mathbb{P}\{\tau_b^{\pi_b^*} \leq T\} \geq \varepsilon(b, T) \equiv \frac{4[1 - \Phi(\frac{b-m}{\sqrt{kT}})]^2}{\exp\{\frac{(\lambda\mu+r)^2T}{\sigma_p^2}\}} > 0, \quad (4.2)$$

where $\tau_b^{\pi_b^*} = \inf\{t : R_t^{\pi_b^*} \leq m\}$, $k = (\lambda^2\sigma^2 + \sigma_p^2)m^2$, $\lambda = \frac{\mu}{\sigma^2(1-\alpha^*)}$.

Proof. Since $a^*(x)$ is a bounded Lipschitz continuous function, the following SDE

$$dR_t^{(1)} = (a^*(R_t^{(1)})\mu + rR_t^{(1)})dt + \sqrt{a^{*2}(R_t^{(1)})\sigma^2 + \sigma_p^2 R_t^{(1)2}} dW_t, R_0^{(1)} = b$$

has a unique solution $R_t^{(1)}$. Using comparison theorem for one-dimensional Itô process, we have

$$\mathbb{P}\{R_t^{(1)} \geq R_t^{\pi_b^*}\} = 1. \quad (4.3)$$

Let \mathbb{Q} be a measure on \mathcal{F}_T defined by

$$d\mathbb{Q}(\omega) = M_T(\omega)d\mathbb{P}(\omega), \quad (4.4)$$

where

$$\begin{aligned} M_t = & \exp\left\{-\int_0^t \frac{a^*(R_s^{(1)})\mu + rR_s^{(1)}}{\sqrt{a^{*2}(R_s^{(1)})\sigma^2 + \sigma_p^2 R_s^{(1)2}}} dW_s \right. \\ & \left. - \frac{1}{2} \int_0^t \frac{(a^*(R_s^{(1)})\mu + rR_s^{(1)})^2}{a^{*2}(R_s^{(1)})\sigma^2 + \sigma_p^2 R_s^{(1)2}} ds\right\}. \end{aligned}$$

Since $\{M_t\}$ is a martingale w.r.t. \mathcal{F}_t , we have $\mathbb{E}[M_T] = 1$. Using Girsanov theorem, we know that \mathbb{Q} is a probability measure on \mathcal{F}_T and the process $\{R_t^{(1)}\}$ satisfies the following SDE

$$dR_t^{(1)} = \sqrt{a^{*2}(R_t^{(1)})\sigma^2 + \sigma_p^2 R_t^{(1)2}} d\tilde{W}_t, R_0^{(1)} = b,$$

where \tilde{W}_t is a Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$.

In view of (4.3), $R_t^{(1)} \geq R_t^{\pi_b^*} \geq m > 0$ for any $t \geq 0$, so we can define $\rho(t)$ by

$$\dot{\rho}(t) = \frac{1}{a^{*2}(R_t^{(1)})\sigma^2 + \sigma_p^2 R_t^{(1)2}}$$

and define $\hat{R}_t^{(1)}$ by $R_{\rho(t)}^{(1)}$. Then $\rho(t)$ is a strictly increasing function and

$$\hat{R}_t^{(1)} = b + \hat{W}_t,$$

where \hat{W}_t is a standard Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$. Moreover, for $t \geq 0$

$$\begin{aligned} \dot{\rho}(t) &= \frac{1}{a^{*2}(R_t^{(1)})\sigma^2 + \sigma_p^2 R_t^{(1)2}} \\ &\leq \frac{1}{(\lambda^2 \sigma^2 + \sigma_p^2)m^2} \\ &:= \frac{1}{\kappa} > 0, \end{aligned}$$

so $\rho(t) \leq \frac{1}{\kappa}t$ and $\rho^{-1}(t) \geq \kappa t$. As a result

$$\begin{aligned}
\mathbb{Q}[\inf\{t : R_t^{(1)} \leq m\} \leq T] &= \mathbb{Q}[\inf\{t : \hat{R}_{\rho^{-1}(t)}^{(1)} \leq m\} \leq T] \\
&= \mathbb{Q}[\inf\{\rho(t) : b + \hat{W}_t \leq m\} \leq T] \\
&= \mathbb{Q}[\inf\{t : \hat{W}_t \leq m - b\} \leq \rho^{-1}(T)] \\
&\geq \mathbb{Q}[\inf\{t : \hat{W}_t \leq m - b\} \leq \kappa T] \\
&= 2[1 - \Phi(\frac{b-m}{\sqrt{\kappa T}})] > 0, \tag{4.5}
\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal distribution function. By virtue of (4.4),

$$\begin{aligned}
\mathbb{Q}[\inf\{t : R_t^{(1)} \leq m\} \leq T] &= \int_{\omega} \mathbb{1}_{[\inf\{t : R_t^{(1)} \leq m\} \leq T]} d\mathbb{Q}(\omega) \\
&= \int_{\omega} \mathbb{1}_{[\inf\{t : R_t^{(1)} \leq m\} \leq T]} M_T d\mathbb{P}(\omega) \\
&= \mathbb{E}^{\mathbb{P}}[M_T \mathbb{1}_{[\inf\{t : R_t^{(1)} \leq m\} \leq T]}] \\
&\leq \mathbb{E}^{\mathbb{P}}[M_T^2]^{\frac{1}{2}} \mathbb{P}[\inf\{t : R_t^{(1)} \leq m\} \leq T]^{\frac{1}{2}}. \tag{4.6}
\end{aligned}$$

Substituting (4.5) and

$$\mathbb{E}^{\mathbb{P}}[M_T^2] \leq \exp\left\{\frac{(\lambda\mu + r)^2 T}{\sigma_p^2}\right\},$$

into (4.6), we get

$$\begin{aligned}
\mathbb{P}[\inf\{t : R_t^{(1)} \leq m\} \leq T] &\geq \frac{\mathbb{Q}[\inf\{t : R_t^{(1)} \leq m\} \leq T]^2}{\mathbb{E}^{\mathbb{P}}[M_T^2]} \\
&\geq \frac{4[1 - \Phi(\frac{b-m}{\sqrt{\kappa T}})]^2}{\exp\left\{\frac{(\lambda\mu+r)^2 T}{\sigma_p^2}\right\}} > 0.
\end{aligned}$$

Thus by (4.3)

$$\begin{aligned}
\mathbb{P}[\tau_b^{\pi_b^*} \leq T] &\geq \mathbb{P}[\inf\{t : R_t^{(1)} \leq m\} \leq T] \tag{4.7} \\
&\geq \varepsilon(b, \sigma^2, \sigma_p^2, T) \equiv \frac{4[1 - \Phi(\frac{b-m}{\sqrt{\kappa T}})]^2}{\exp\left\{\frac{(\lambda\mu+r)^2 T}{\sigma_p^2}\right\}} > 0.
\end{aligned}$$

□

The economic interpretation of theorem 4.1 is the following.

(1) The lower boundary $\varepsilon(b, \sigma^2, \sigma_p^2, T)$ of bankrupt probability for the company is an increasing function of σ_p^2 , thus the reinvestments will make the company have larger risk.

(2) The lower boundary $\varepsilon(b, \sigma^2, \sigma_p^2, T)$ of bankrupt probability for the company is an increasing function of m , so the minimum reserve requirement m will increase the risk of the company goes to bankruptcy.

(3) The lower boundary $\varepsilon(b, \sigma^2, \sigma_p^2, T)$ of bankrupt probability for the company is a decreasing function of b , so the optimal dividend payout barrier should keep reasonable high so that the company gets good solvency.

(4) The company does have larger risk before the contract between insurer and policy holders goes into effect (i.e., $0 < T$ is less than the time of the contract issue) because the lower boundary $\varepsilon(b, \sigma^2, \sigma_p^2, T)$ is positive for any $T > 0$, the company has to find an optimal policy to improve the ability of the insurer to fulfill its obligation to policy holders.

Now we prove the second result of this section.

Theorem 4.2. *Let $m = 0$ in Theorem 4.1. Then for any T and b $\mathbb{P}[\tau_b^{\pi_b^*} \leq T] = 0$.*

Proof. Let $\tau_b^{\pi_b^*} = \inf\{t : R_t^{\pi_b^*} = 0, R_0^{\pi_b^*} = b\}$, $\tau_n = \inf\{t : R_t^{\pi_b^*} = 2^{-2n}x_0\}$, $A = \{\tau_b^{\pi_b^*} \leq T\}$ and $B_n = \{\tau_n \leq T\}$. Then for any $n > 0$ $A \subset B_n$. As a result,

$$\mathbb{P}[A] = \mathbb{P}[A \cap B_n] \leq \mathbb{P}[A|B_n].$$

Noting that $\{R_t^{\pi_b^*}\}$ is a Markov process, we have

$$\begin{aligned} \mathbb{P}[A|B_n] &= \mathbb{P}[\inf_{0 \leq t \leq T} R_t^{\pi_b^*} \leq 0 | \tau_n \leq T] \\ &\leq \mathbb{P}^{2^{-2n}x_0}[\inf_{0 \leq t \leq T} R_t^{\pi_b^*} \leq 0] \\ &\leq \mathbb{P}^{2^{-2n}x_0}[\inf_{0 \leq t \leq T} R_t^{\pi_b^*} \leq 2^{-3n}x_0 \text{ or } \sup_{0 \leq t \leq T} R_t^{\pi_b^*} \geq 2^{-n}x_0] \\ &= 1 - \mathbb{P}^{2^{-2n}x_0}[\inf_{0 \leq t \leq T} R_t^{\pi_b^*} \geq 2^{-3n}x_0 \text{ and } \sup_{0 \leq t \leq T} R_t^{\pi_b^*} \leq 2^{-n}x_0] \\ &\equiv 1 - \mathbb{P}(D). \end{aligned}$$

Using definition of $a^*(x)$, on the set D

$$\begin{aligned} R_t^{\pi_b^*} &= 2^{-2n} x_0 \exp\left[\left(\lambda\mu + r - \frac{1}{2}(\lambda^2\sigma^2 + \sigma_p^2)\right)t + \sqrt{\lambda^2\sigma^2 + \sigma_p^2} W_t\right] \\ &:= 2^{-2n} x_0 \exp[X_t], \end{aligned}$$

where X_t is a Brownian motion with drift. So

$$\begin{aligned} f(n) &:= \mathbb{P}^{2^{-2n}x_0} \left[\inf_{0 \leq t \leq T} R_t^{\pi_b^*} \geq 2^{-3n} x_0 \text{ and } \sup_{0 \leq t \leq T} R_t^{\pi_b^*} \leq 2^{-n} x_0 \right] \\ &= \mathbb{P} \left[\inf_{0 \leq t \leq T} X_t \geq -n \ln 2 \text{ and } \sup_{0 \leq t \leq T} X_t \leq n \ln 2 \right] \rightarrow 1 \end{aligned}$$

as $n \rightarrow \infty$. Thus $\mathbb{P}[\tau_b^{\pi_b^*} \leq T] = 0$ follows from $\mathbb{P}[\tau_b^{\pi_b^*} \leq T] \leq 1 - f(n)$. \square

The interpretation of Theorem 4.2 is that when $m = 0$ the company of the model will never go to bankruptcy. Indeed, this is an ideal model and does not exist in reality. Thus the assumption $m > 0$ in this paper is reasonable and more closer to real world.

5. Numerical examples

In this section we consider some numerical samples to demonstrate the bankrupt probability is a decreasing function of dividend payout level b or initial reserve x based on PDE (6.2) below. The dividend payout level $b(\varepsilon, m, T)$ decreases with ε , and increases with m, σ^2 and T via the equation $\psi(T, b, m, x) = \varepsilon$ (see (6.2)).

Example 5.1. *Let $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1$ in PDE (6.2) below, the figures 1 and 2 of the bankrupt probability $1 - \phi(T, x)$ state that solvency will improve with dividend payout level b or initial reserve x , but the company's profit will reduce (see Lemma 6.7 below).*

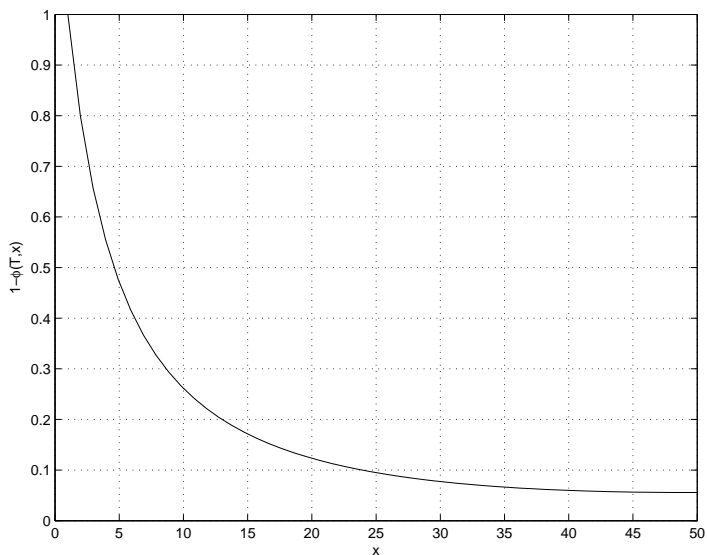


FIGURE 1. Bankrupt probability $1 - \phi(T, x)$ as a function of x (Parameters: $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1, b = 50$)

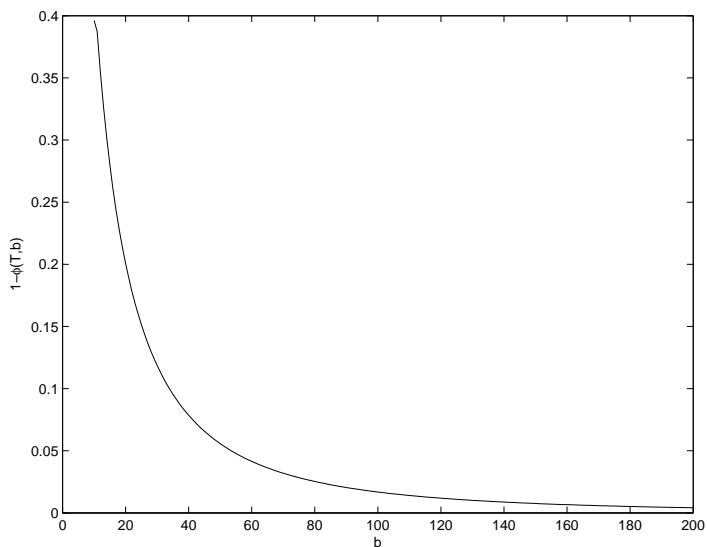


FIGURE 2. Bankrupt probability $1 - \phi(T, b)$ as a function of b (Parameters: $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1$)

Example 5.2. Let $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1$ and solve $b(\varepsilon)$ by $1 - \phi(T, b) = \varepsilon$, we get the figure 3. It shows that the risk ε greatly impacts on dividend payout level b . The dividend payout level b decreases with the risk ε , so the risk ε increases the company's profit.

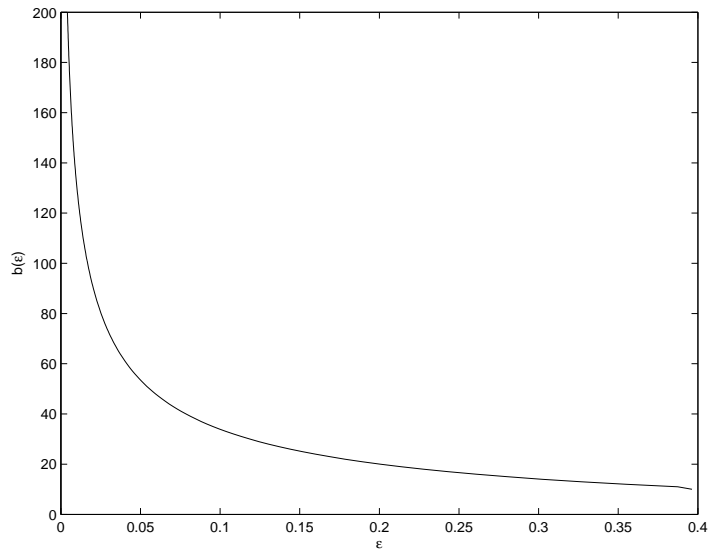


FIGURE 3. Dividend payout level $b(\varepsilon)$ as a function of ε (Parameters: $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1, m = 1$)

Example 5.3. Let $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1$ and solve $b(\varepsilon)$ by $1 - \phi(T, b) = \varepsilon$, we get the figure 4 below. The two curves in this figure show that the minimum reserve requirement m increases dividend payout level b , but decreases the company's profit.

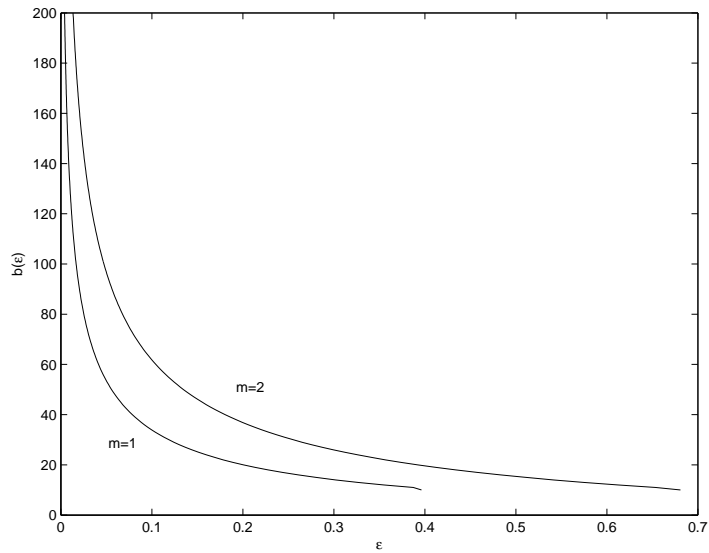


FIGURE 4. Dividend payout level $b(\varepsilon)$ as a function of ε (Parameters: $\sigma^2 = \mu = 1, \sigma_p^2 = 2, T = 1$).

Example 5.4. Let $\sigma^2 = \mu = 1, \sigma_p^2 = 2, m = 1$ and solve $b(\varepsilon)$ by $1 - \phi(T, b) = \varepsilon$, we get the figure 5 below. It portrays that the dividend payout level b is an increasing function of time horizon T , so it decreases the company's profit.

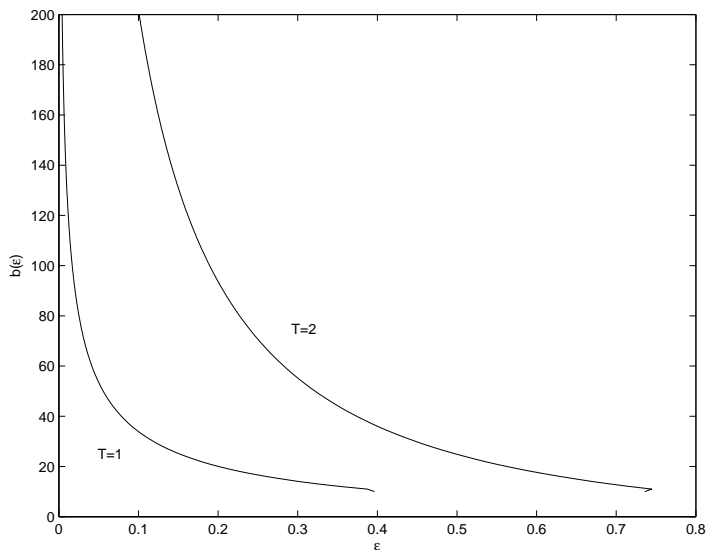


FIGURE 5. Dividend payout level $b(\varepsilon)$ as a function of ε (Parameters: $\sigma^2 = \mu = 1, \sigma_p^2 = 2, m = 1$).

Example 5.5. Let $\mu = 1, \sigma_p^2 = 2, m = 1$ and solve $b(\varepsilon)$ by $1 - \phi(T, b) = \varepsilon$, we get the figure 6 below. It portrays that the dividend payout level b is an increasing function of underwriting risk σ^2 , so it decreases the company's profit.

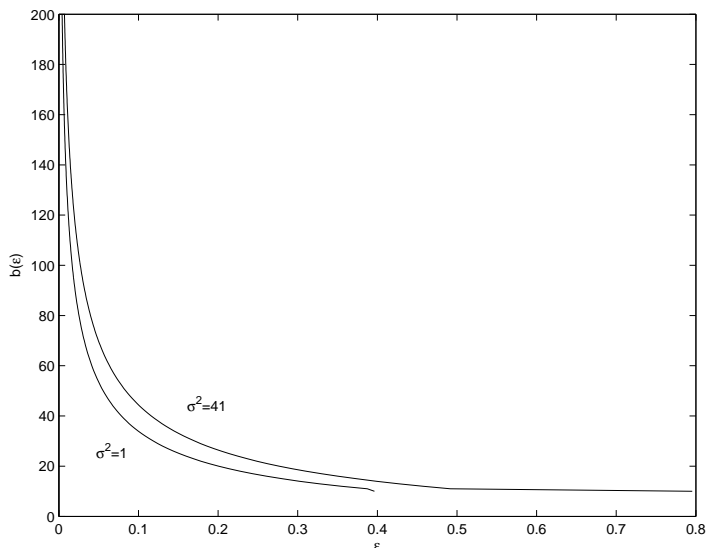


FIGURE 6. Dividend payout level $b(\varepsilon)$ as a function of ε (Parameters: $\mu = 1, \sigma_p^2 = 2, m = 1$).

6. Properties on bankrupt probability and $V(x, b)$

In this section, to prove Theorem 3.1, we list some lemmas on properties of bankrupt probability and $V(x, b)$ which will be used late. The rigorous proofs of these lemmas will be given in the appendix below.

Lemma 6.1. *The probability of bankruptcy $\mathbb{P}[\tau_b^b \leq T]$ is a decreasing function of b , where $\tau_b^b := \tau_b^{\pi_b^*}$.*

Lemma 6.2.

$$\lim_{b \rightarrow \infty} \mathbb{P}[\tau_b^b \leq T] = 0. \quad (6.1)$$

Lemma 6.3. *Let $\phi(t, x) \in C^1(0, \infty) \cap C^2(m, b)$ and satisfy the following partial differential equation*

$$\begin{cases} \phi_t(t, x) = \frac{1}{2}[a^{*2}(x)\sigma^2 + \sigma_p^2 x^2]\phi_{xx}(t, x) + [a^*(x)\mu + rx]\phi_x(t, x), \\ \phi(0, x) = 1, \text{ for } m < x \leq b, \\ \phi(t, m) = 0, \phi_x(t, b) = 0, \text{ for } t > 0. \end{cases} \quad (6.2)$$

Then $\phi(T, x) = 1 - \psi^b(T, x)$, i.e., $\phi^b(T, x)$ is probability that the company will survive on time interval $[0, T]$, the function $\psi^b(t, x)$ is defined by

$$\psi^b(t, x) := \mathbb{P}[\tau_x^b \leq t],$$

where $\tau_b^x := \tau_x^{\pi_b^*}$, i.e., probability of bankruptcy for the process $\{R_t^{\pi_b^*, x}\}_{t \geq 0}$ with the initial asset x and a dividend barrier b is employed before time t . where $a^*(\cdot)$ is defined by (3.3).

Let $\sigma(x) := \frac{1}{2}[a^{*2}(x)\sigma^2 + \sigma_p^2 x^2]$ and $\mu(x) := a^*(x)\mu + rx$. Then the equation (6.2) becomes

$$\phi_t(t, x) = \sigma^2(x)\phi_{xx}(t, x) + \mu(x)\phi_x(t, x). \quad (6.3)$$

By properties of $a^*(\cdot)$, it is easy to show that $\sigma(x)$ and $\mu(x)$ are continuous in $[m, b]$. So there exists a unique solution (6.2) and the solution is in $C^1(0, \infty) \cap C^2(m, b)$. Moreover, $\sigma'(x)$ and $\mu'(x)$ are bounded on (m, x_0) and (x_0, b) respectively.

Lemma 6.4. *Let $\phi^b(t, x)$ be a solution of the equation(6.2). Then the $\phi^b(T, b)$ is a continuous function of b on $[b_0, \infty)$.*

Lemma 6.5. *Let $F_b(x)$ be defined by (3.2) and b_0 be given by part (ii) of Lemma 3.1. Then*

$$\mathcal{L}F_b(x) \leq 0, \text{ for all } x \geq 0, \quad (6.4)$$

where

$$\mathcal{L} = \frac{1}{2}(a^2\sigma^2 + \sigma_p^2 x^2)\frac{d^2}{dx^2} + (a\mu + rx)\frac{d}{dx} - c.$$

Lemma 6.6. (i) *For any $b \leq b_0$ we have $V(x, b) = V(x, b_0) = V(x) = F_{b_0}(x) = J(x, \pi_{b_0}^*)$. Moreover, the optimal policy is $\pi_{b_0}^* = \{a^*(R_t^{\pi_{b_0}^*}), L_t^{\pi_{b_0}^*}\}$, where $(R_t^{\pi_{b_0}^*}, L_t^{\pi_{b_0}^*})$ is uniquely determined by the SDE (3.4).*

(ii) *For any $b \geq b_0$ we have $V(x, b) = F_b(x) = J(x, \pi_b^*)$. The optimal policy $\pi_b^* = \{a^*(R_t^{\pi_b^*}), L_t^{\pi_b^*}\}$, where $(R_t^{\pi_b^*}, L_t^{\pi_b^*})$ is uniquely determined by the SDE(3.8).*

The lemma 6.6 mainly deals with relationships among $F_b(x)$, $V(x, b)$ and $V(x)$ defined by (2.3).

Lemma 6.7. *For any $b \geq b_0$ and $x \geq m$,*

$$\frac{d}{db}V(x, b) < 0. \quad (6.5)$$

Moreover, if $b_1, b_2 \geq b_0$ and $x \leq \min\{b_1, b_2\}$, then

$$\frac{V(x, b_1)}{V(x, b_2)} = \frac{h'(b_2)}{h'(b_1)}. \quad (6.6)$$

7. Appendix

In this section we will give the proofs of theorem and lemmas we concerned with throughout this paper.

Proof of theorem 3.1. If $\mathbb{P}[\tau_{b_0}^{b_0} \leq T] \leq \varepsilon$, then the conclusion is obvious because it is just the optimal control problem without constraints.

Assume that $\mathbb{P}[\tau_{b_0}^{b_0} \leq T] > \varepsilon$. By Lemma 6.1 and Lemma 6.2, there exists a unique $b^* (\geq b_0)$ such that

$$\begin{aligned} b^* &= \min\{b : \mathbb{P}[\tau_b^b \leq T] = \varepsilon\} = \min\{b : b \in \mathfrak{B}\}, \quad (7.1) \\ \mathbb{P}[\tau_b^b \leq T] &> \varepsilon, \quad \forall b \leq b^*, \\ \mathbb{P}[\tau_b^b \leq T] &\leq \varepsilon, \quad \forall b \geq b^*. \end{aligned}$$

By Lemma 6.7, we know that $V(x, b)$ is decreasing w.r.t. b , so b^* satisfies (3.5). Using Lemma 6.4, we get $b^* \in \mathfrak{B}$ and $\mathbb{P}[\tau_{b^*}^{b^*} \leq T] = \varepsilon$. Moreover, by Lemma 6.6 and (7.1), we have

$$F_{b^*}(x) = V(x, b^*) = J(x, \pi_{b^*}^*) = V(x).$$

So the optimal policy associated with the optimal return function $V(x)$ is $\{a^*(R_t^{\pi_{b^*}^*}), L_t^{\pi_{b^*}^*}\}$, where, $(R_t^{\pi_{b^*}^*}, L_t^{\pi_{b^*}^*})$ is determined uniquely by (3.8). The inequality (3.9) is a direct consequence of (6.6). \square

Proof of lemma 6.1. The proof of this lemma is the same as that of Theorem 3.1 in the [20], we omit it here. \square

Proof of lemma 6.2. Using the same argument as in the proof of theorem 3.1 in the [20], we have for some $n > 3$ and large $b \geq \max\{1, m^n\}$

$$\mathbb{P}[\tau_{\sqrt[n]{b}}^b \leq T] \geq \mathbb{P}[\tau_b^b \leq T]. \quad (7.2)$$

Let $R_t^{(2)}$ be the unique solution of the following SDE

$$\begin{aligned} dR_t^{(2)} &= (a^*(R_t^{(2)})\mu + rR_t^{(2)})dt + \sqrt{a^{*2}(R_t^{(2)})\sigma^2 + \sigma_p^2 R_t^{(2)2}} dW_t, \\ R_0^{(2)} &= \sqrt[n]{b}. \end{aligned} \quad (7.3)$$

Then by comparison theorem on SDE (see Ikeda and Watanabe [17](1981))

$$\{\tau_{\sqrt[n]{b}}^{\pi_b^*} \leq T\} \subseteq \{\exists t \leq T \text{ such that } R_t^{(2)} = m \text{ or } R_t^{(2)} = b\}.$$

As a result,

$$\begin{aligned} \mathbb{P}\{\tau_{\sqrt[n]{b}}^{\pi_b^*} \leq T\} &\leq \mathbb{P}\{\exists t \leq T \text{ such that } R_t^{(2)} = m \text{ or } R_t^{(2)} = b\} \\ &\leq \mathbb{P}\{\sup_{0 \leq t \leq T} R_t^{(2)} \geq b\} + \mathbb{P}\{\inf_{0 \leq t \leq T} R_t^{(2)} \leq m\}. \end{aligned} \quad (7.4)$$

Firstly, we estimate $\mathbb{P}\{\sup_{0 \leq t \leq T} R_t^{(2)} \geq b\}$.

Using Hölder inequality and $a^*(x) \leq 1$, it follows from SDE (7.3) that

$$\begin{aligned} \sup_{0 \leq t \leq T} (R_t^{(2)})^2 &\leq [3(\sqrt[n]{b})^2 + 6\mu^2 T^2] + 6r^2 T \int_0^T \sup_{0 \leq s \leq t} (R_s^{(2)})^2 ds \\ &\quad + 3 \sup_{0 \leq t \leq T} \left(\int_0^t \sqrt{a^{*2}(R_s^{(2)})\sigma^2 + \sigma_p^2 R_s^{(2)2}} dW_s \right)^2. \end{aligned} \quad (7.5)$$

Taking mathematical expectation at both sides of (7.5) and using B-D-G inequality, we derive

$$\begin{aligned} \mathbb{E}\{\sup_{0 \leq t \leq T} (R_t^{(2)})^2\} &\leq [3(\sqrt[n]{b})^2 + 6\mu^2 T^2] + 6r^2 T \int_0^T \mathbb{E}\{\sup_{0 \leq s \leq t} (R_s^{(2)})^2\} ds \\ &\quad + 12\mathbb{E}\left\{\int_0^T (a^{*2}(R_s^{(2)})\sigma^2 + \sigma_p^2 R_s^{(2)2}) dt\right\} \\ &\leq [3(\sqrt[n]{b})^2 + 6\mu^2 T^2 + 12\sigma^2 T^2] \\ &\quad + 6(r^2 T + 2\sigma_p^2) \int_0^T \mathbb{E}\{\sup_{0 \leq s \leq t} (R_s^{(2)})^2\} ds. \end{aligned} \quad (7.6)$$

Solving (7.6), we get

$$\mathbb{E}\{\sup_{0 \leq t \leq T} (R_t^{(2)})^2\} \leq [(3\sqrt[n]{b})^2 + 6\mu^2 T^2 + 12\sigma^2 T^2] \exp\{6(r^2 T + 2\sigma_p^2)T\}. \quad (7.7)$$

Combining Markov inequality and the inequality (7.7), we conclude that

$$\begin{aligned} \mathbb{P}\{\sup_{0 \leq t \leq T} R_t^{(2)} \geq b\} &\leq \frac{\mathbb{E}\{\sup_{0 \leq t \leq T} (R_t^{(2)})^2\}}{b^2} \\ &\leq \frac{(3(\sqrt[n]{b})^2 + 6\mu^2 T^2 + 12\sigma^2 T^2) \exp\{6(r^2 T + 2\sigma_p^2)T\}}{b^2}. \end{aligned} \quad (7.8)$$

Secondly, we estimate $\mathbb{P}\{\inf_{0 \leq t \leq T} R_t^{(2)} \leq m\}$.

Let M_1 be a martingale defined by

$$M_1(t) = \int_0^t I_{\{s: R_s^{(2)} \neq 0\}} \sqrt{\sigma^2 \left(\frac{a^*(R_s^{(2)})}{R_s^{(2)}}\right)^2 + \sigma_p^2} dW_s.$$

Then we can rewrite the SDE (7.3) as follows,

$$R_t^{(2)} = (\sqrt[n]{b} + \int_0^t (a^*(R_s^{(2)})\mu + rR_s^{(2)})ds) + \int_s^t R_s^{(2)} dM_1(s).$$

In view of Proposition 2.3 of Chapter 9 in [25],

$$R_t^{(2)} = \mathcal{E}(M_1)_t \left(\sqrt[n]{b} + \int_0^t \frac{(\mu a^*(R_s^{(2)}) + rR_s^{(2)})}{\mathcal{E}(M_1)_s} ds \right),$$

where $\mathcal{E}(M_1)_t = \exp\{M_1(t) - \frac{1}{2} \langle M_1 \rangle (t)\}$ is an exponential martingale, $\langle M_1 \rangle$ is the bracket of M_1 . So the fact $\inf\{f(t)g(t)\} \geq \inf\{f(t)\} \inf\{g(t)\}$ for any $f(t) \geq 0$ and $g(t) \geq 0$ implies that

$$\inf_{0 \leq t \leq T} \{R_t^{(2)}\} \geq \sqrt[n]{b} \inf_{0 \leq t \leq T} \{\mathcal{E}(M_1)_t\}.$$

As a result

$$\mathbb{P}\{\inf_{0 \leq t \leq T} R_t^{(2)} \leq m\} \leq \mathbb{P}\{\inf_{0 \leq t \leq T} \mathcal{E}(M_1)_t \leq \frac{m}{\sqrt[n]{b}}\}.$$

Since $\langle M_1 \rangle_T \leq (\lambda^2 \sigma^2 + \sigma_p^2)T < +\infty$, we have

$$\begin{aligned} \lim_{b \rightarrow \infty} \mathbb{P}\{\inf_{0 \leq t \leq T} R_t^{(2)} \leq m\} &\leq \mathbb{P}\{\inf_{0 \leq t \leq T} \mathcal{E}(M_1)_t = 0\} \\ &\leq \mathbb{P}\{\sup_{0 \leq t \leq T} |M_1(t)| = +\infty\}. \end{aligned} \quad (7.9)$$

By B-D-G inequalities, we get

$$\mathbb{E}\{\sup_{0 \leq t \leq T} |M_1(t)|^2\} \leq 4(\lambda^2 \sigma^2 + \sigma_p^2)T < +\infty,$$

which implies that

$$\mathbb{P}\{\sup_{0 \leq t \leq T} |M_1(t)| = +\infty\} = 0.$$

Thus by (7.9)

$$\lim_{b \rightarrow \infty} \mathbb{P}\{\inf_{0 \leq t \leq T} \{R_t^{(2)}\} \leq m\} = 0. \quad (7.10)$$

So the inequalities (7.2), (7.4), (7.8) and (7.10) yield that

$$\lim_{b \rightarrow \infty} \mathbb{P}\{\tau_b^{\pi_b^*} \leq T\} = 0.$$

□

Remark 7.1. *The proof of theorem 3.2 in [20] seems wrong, so we can use the way proving Lemma 6.2 to correct it. Theorem 3.2 in the [20] is indeed a direct consequence of Lemma 6.2.*

Proof of lemma 6.3. Let $(R_t^b, L_b(t))$ denote $(R_t^{\pi_b^*}, L_t^{\pi_b^*})$ defined by SDE (4.1). Since $(R_t^b, L_b(t))$ is continuous process, by the generalized Itô formula, we have

$$\begin{aligned} \phi(T - (t \wedge \tau_x^b), R_{t \wedge \tau_x^b}^b) &= \phi(T, x) \\ &+ \int_0^{t \wedge \tau_x^b} \left\{ \frac{1}{2} [a^{*2}(R_s^b) \sigma^2 + \sigma_p^2 \cdot (R_s^b)^2] \sigma^2 \phi_{xx}(T - s, R_s^b) \right. \\ &+ [a^*(R_s^b) \mu + r R_s^b] \phi_x(T - s, R_s^b) \\ &- \phi_t(T - s, R_s^b) \Big\} ds - \int_0^{t \wedge \tau_x^b} \phi_x(T - s, R_s^b) dL_b(s) \\ &+ \int_0^{t \wedge \tau_x^b} a(R_s^b) \sigma \phi_x(T - s, R_s^b) dW_s. \end{aligned} \quad (7.11)$$

Letting $t = T$ and taking mathematical expectation at both sides of (7.11) yields that

$$\begin{aligned} \phi(T, x) &= \mathbb{E}[\phi(T - (T \wedge \tau_x^b), R_{T \wedge \tau_x^b}^b)] \\ &= \mathbb{E}[\phi(0, R_T^b) 1_{T < \tau_x^b}] + \mathbb{E}[\phi(T - \tau_x^b, m) 1_{T \geq \tau_x^b}] \\ &= \mathbb{E}[1_{T < \tau_x^b}] = \mathbb{P}[\tau_x^b > T] = 1 - \psi(T, x). \end{aligned}$$

□

Now we use PDE method to prove lemma 6.4.

Proof of lemma 6.4. Let $x = by$, $0 \leq y \leq 1$ and $\theta^b(t, y) = \phi^b(t, (b-m)y+m)$.

Then the equation (6.2) becomes

$$\begin{cases} \theta_t^b(t, y) = [\sigma[(b-m)y+m]/(b-m)^2]\theta_{yy}^b(t, y) \\ + [\mu[(b-m)y+m]/(b-m)]\theta_y^b(t, y), \\ \theta^b(0, y) = 1, \text{ for } 0 \leq y \leq 1, \\ \theta^b(t, 0) = 0, \theta_y^b(t, 1) = 0, \text{ for } t > 0. \end{cases} \quad (7.12)$$

In view of (7.12), the proof of Lemma 6.4 reduces to proving $\lim_{b_2 \rightarrow b_1} \theta^{b_2}(t, 1) = \theta^{b_1}(t, 1)$ for fixed $b_1 > b_0$. Let $w(t, y) = \theta^{b_2}(t, y) - \theta^{b_1}(t, y)$. Since $\theta^b(t, y)$ is continuous at $y = 1$ for any $b > b_0$, we only need to show that

$$\int_0^t \int_0^1 w^2(s, y) ds dy \rightarrow 0, \text{ as } b_2 \rightarrow b_1. \quad (7.13)$$

Let $\sigma^b(y) = \sigma[(b-m)y+m]/(b-m)^2$, $\mu^b(y) = \mu[(b-m)y+m]/(b-m)$.

Then the (7.12) translates into

$$\begin{cases} w_t(t, y) = \sigma^{b_2}(y)w_{yy}(t, y) + \mu^{b_2}(y)w_y(t, y) \\ + \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\}\theta_{yy}^{b_1}(t, y) \\ + \{\mu^{b_2}(y) - \mu^{b_1}(y)\}\theta_y^{b_1}(t, y), \\ w(0, y) = 0, \text{ for } 0 < y \leq 1, \\ w(t, y) = 0, y = 0, w_y(t, 1) = 0, \text{ for } t > 0. \end{cases} \quad (7.14)$$

Multiplying both sides of the first equation in (7.14) by $w(t, z)$, and then integrating both sides of the resulting equation on $[0, t] \times [0, 1]$, we get

$$\begin{aligned} & \int_0^t \int_0^1 w(s, y)w_t(s, y) dy ds \\ &= \int_0^t \int_0^1 [\sigma^{b_2}(y)]w(s, y)w_{yy}(s, y) dy ds \\ &+ \int_0^t \int_0^1 [\mu^{b_2}(y)]w(s, y)w_y(s, y) dy ds \\ &+ \int_0^t \int_0^1 [\sigma^{b_2}(y) - \sigma^{b_1}(y)]w(s, y)\theta_{yy}^{b_1}(t, y) dy ds \\ &+ \int_0^t \int_0^1 w(s, y)[\mu^{b_2}(y) - \mu^{b_1}(y)]w(s, y)\theta_y^{b_1}(t, y) dy ds \\ &\equiv E_1 + E_2 + E_3 + E_4. \end{aligned} \quad (7.15)$$

Now we look at terms at both sides of (7.15).

Firstly, we have

$$\int_0^t \int_0^1 w(s, y)w_t(s, y) dy ds = \int_0^1 \frac{1}{2}w^2(t, y) dy. \quad (7.16)$$

Secondly, we deal with terms E_i , $i = 1, \dots, 4$ as follows.

It is easy to see from the expression of $a^*(\cdot)$ that there exist positive constants D_1 , D_2 and D_3 such that $[\mu(b_2y)/b_2]^2 \leq D_1$ and $\sigma_2^b(y) \geq D_2 > 0$ for $y \geq 0$, and $\sigma_2^b(y)' \leq D_3$ for $y \in (0, \frac{x_0-m}{b-m}) \cup (\frac{x_0-m}{b-m}, 1]$. As a result, for any $\lambda_1 > 0$ and $\lambda_2 > 0$

$$\begin{aligned}
E_1 &= \int_0^t \int_0^1 [\sigma^{b_2}(y)] w(s, y) w_{yy}(s, y) dy ds \\
&= - \int_0^t \int_0^1 [\sigma^{b_2}(y)] w_y^2(s, y) dy ds \\
&\quad - \int_0^t \left[\int_0^{(x_0-m)/(b-m)} + \int_{(x_0-m)/(b-m)}^1 \right] [(\sigma^{b_2}(y))' w_y(s, y) w(s, y)] dy ds \\
&\leq -D_2 \int_0^t \int_0^1 w_y^2(s, y) dy ds \\
&\quad + D_3 \int_0^t \int_0^1 \lambda_1 w_y^2(s, y) + \frac{1}{4\lambda_1} w^2(s, y) dy ds
\end{aligned} \tag{7.17}$$

and

$$\begin{aligned}
E_2 &= \int_0^t \int_0^1 [\mu^{b_2}(y)] w(s, y) w_y(s, y) dy ds \\
&\leq \lambda_2 \int_0^t \int_0^1 w_y^2(s, y) dy ds \\
&\quad + \frac{D_1}{4\lambda_2} \int_0^t \int_0^1 w^2(s, y) dy ds.
\end{aligned} \tag{7.18}$$

In order to estimate E_3 , we decompose E_3 as follows:

$$\begin{aligned}
E_3 &= \int_0^t \int_0^1 \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\} w(s, y) \theta_{yy}^{b_1}(s, y) dy ds \\
&= - \int_0^t \int_0^1 \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\} w_y(s, y) \theta_y^{b_1}(s, y) dy ds \\
&\quad - \int_0^t \int_0^{(x_0-m)/(b_2-m)} \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\}' w(s, y) \theta_y^{b_1}(s, y) dy ds \\
&\quad - \int_0^t \int_{(x_0-m)/(b_2-m)}^{(x_0-m)/(b_1-m)} \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\}' w(s, y) \theta_y^{b_1}(s, y) dy ds \\
&\quad - \int_0^t \int_{(x_0-m)/(b_1-m)}^1 \{\sigma^{b_2}(y) - \sigma^{b_1}(y)\}' w(s, y) \theta_y^{b_1}(s, y) dy ds \\
&= E_{31} + E_{32} + E_{33} + E_{34}.
\end{aligned} \tag{7.19}$$

So the estimating E_3 is reduced to estimating E_{3i} , $i = 1, \dots, 4$.

The fact $[\sigma(by)/b^2]$, $[\sigma(by)/b^2]'$ and $[\mu(by)/b]$ are Lipschitz continuous on $(0, \frac{x_0-m}{b_2-m})$, $(\frac{x_0-m}{b_2-m}, \frac{x_0-m}{b_1-m})$ and $(\frac{x_0-m}{b_1-m}, 1)$, that is, there exists $L > 0$ such that

$$\begin{aligned} |[\sigma^{b_2}(y)] - [\sigma^{b_1}(y)]| &\leq L|b_2 - b_1|, \\ |[\sigma^{b_2}(y)]' - [\sigma^{b_1}(y)]'| &\leq L|b_2 - b_1|, \\ |[\mu^{b_2}(y)] - [\mu^{b_1}(y)]| &\leq L|b_2 - b_1|, \end{aligned}$$

and Young's inequality yield that for any $\lambda_3 > 0$ and $\lambda_4 > 0$

$$\begin{aligned} E_{31} &= - \int_0^t \int_0^1 [\sigma^{b_2}(y) - \sigma^{b_1}(y)] w_y(s, y) \theta_y^{b_1}(s, y) dy ds \\ &\leq \frac{L^2(b_2 - b_1)^2}{4\lambda_3} \int_0^t \int_0^1 [\theta_y^{b_1}(s, y)]^2 dy ds \\ &\quad + \lambda_3 \int_0^t \int_0^1 w_y^2(s, y) dy ds, \end{aligned} \tag{7.20}$$

$$\begin{aligned} E_{32} + E_{34} &= - \int_0^t \left[\int_0^{(x_0-m)/(b_2-m)} + \int_{(x_0-m)/(b_1-m)}^1 \right] \{ \sigma^{b_2}(y) \\ &\quad - \sigma^{b_1}(y) \}' w(s, y) \theta_y^{b_1}(s, y) dy ds \\ &\leq \frac{L^2(b_2 - b_1)^2}{4\lambda_4} \int_0^t \int_0^1 [\theta_y^{b_1}(s, y)]^2 dy ds \\ &\quad + \lambda_4 \int_0^t \int_0^1 w^2(s, y) dy ds. \end{aligned} \tag{7.21}$$

The remaining part of estimating E_3 is to deal with E_{33} .

By the boundary conditions

$$\begin{aligned}
0 &= \int_0^t \int_0^1 \theta_t^b(s, y) \theta^b(s, y) \\
&\quad - \sigma^b(y) \theta_{yy}^b(s, y) \theta^b(s, y) - \mu^b(y) \theta_y^b(s, y) \theta^b(s, y) dy ds \\
&= \frac{1}{2} \int_0^1 [\theta^b(s, y)]^2 dy + \int_0^t \int_0^1 \sigma^b(y) [\theta_y^b(s, y)]^2 dy ds \\
&\quad + \int_0^t \left[\int_0^{(x_0-m)/(b-m)} + \int_{(x_0-m)/(b-m)}^1 \right] \sigma^b(y)' [\theta_y^b(s, y)] [\theta^b(s, y)] dy ds \\
&= \frac{1}{2} \int_0^1 [\theta^b(s, y)]^2 dy + \int_0^t \int_0^1 \sigma^b(y) [\theta_y^b(s, y)]^2 dy ds \\
&\quad + \int_0^t \left[\int_0^{(x_0-m)/(b-m)} + \int_{(x_0-m)/(b-m)}^1 \right] (\sigma^b(y)' \\
&\quad - \mu^b(y)) [\theta_y^b(s, y)] [\theta^b(s, y)] dy ds \\
&\geq \lambda_5 \int_0^t \int_0^1 [\theta_y^b(s, y)]^2 dy ds - \frac{\lambda_5}{2} \int_0^t \int_0^1 [\theta_y^b(s, y)]^2 dy ds \\
&\quad - \frac{\lambda_6}{2\lambda_5} \int_0^t \int_0^1 [\theta^b(s, y)]^2 dy ds \\
&\geq \frac{\lambda_5}{2} \int_0^t \int_0^1 [\theta_y^b(s, y)]^2 dy ds - \frac{\lambda_6}{2\lambda_5},
\end{aligned}$$

from which we know that

$$\int_0^t \int_0^1 [\theta_y^b(s, y)]^2 dy ds \leq \frac{\lambda_6}{\lambda_5^2},$$

where $\lambda_5 > 0$ is the lower boundary of $\sigma^b(y)$ and λ_6 is the upper boundary of $|\sigma^b(y)/b^2|' - |\mu^b(y)/b|$ on $(0, \frac{x_0-m}{b-m}) \cup (\frac{x_0-m}{b-m}, 1]$.

Therefore we conclude that $\int_0^t \int_0^1 [\theta_y^b(s, y)]^2 dy ds$ is bounded. So by using $w(s, y) \leq 2$, we have

$$\lim_{b_2 \rightarrow b_1} |E_{33}| = 0. \tag{7.22}$$

Thus the equalities (7.20), (7.21) and (7.22) yield that there exists a positive function $B_1^{b_1}(b_2)$ with $\lim_{b_2 \rightarrow b_1} B_1^{b_1}(b_2) = 0$ such that for $0 \leq t \leq T$

$$\begin{aligned}
E_3 &= \int_0^t \int_0^1 \{\sigma(b_2 y)/b_2^2 - \sigma(b_1 y)/b_1^2\} w(s, y) \theta_{yy}^{b_1}(t, y) dy ds \\
&= E_{31} + E_{32} + E_{33} + E_{34} \\
&\leq B^{b_1}(b_2) + (\lambda_3 + \lambda_4) \int_0^t \int_0^1 w_y^2(s, y) + w^2(s, y) dy ds.
\end{aligned} \tag{7.23}$$

By the same way as that of (7.18)

$$\begin{aligned}
E_4 &= \int_0^t \int_0^1 \{\mu(b_2 y)/b_2 - \mu(b_1 y)/b_1\} w(s, y) \theta_y^{b_1}(t, y) dy ds \\
&\leq \frac{L^2(b_2 - b_1)^2}{4\lambda_7} \int_0^t \int_0^1 [\theta_y^{b_1}(s, y)]^2 dy ds \\
&\quad + \lambda_7 \int_0^t \int_0^1 w^2(s, y) dy ds.
\end{aligned} \tag{7.24}$$

Let

$$B_2^{b_1}(b_2) = \frac{L^2(b_2 - b_1)^2}{4\lambda_7} \int_0^t \int_0^1 [\theta_y^{b_1}(s, y)]^2 dy ds.$$

Then

$$\lim_{b_2 \rightarrow b_1} B_2^{b_1}(b_2) = 0,$$

which, together with (7.24), implies that

$$E_4 \leq B_2^{b_1}(b_2) + \lambda_7 \int_0^t \int_0^1 w^2(s, y) dy ds. \tag{7.25}$$

Choosing λ_1 , λ_2 and λ_3 small enough such that $\lambda_1 + D_3\lambda_2 + \lambda_3 < D_2$, we can conclude from (7.15), (7.17), (7.18), (7.23) and (7.25) that there exist constants C_1 and C_2 such that

$$\int_0^1 w^2(t, y) dy \leq C_1 \int_0^t \int_0^1 w^2(s, y) dy ds + C_2 [B_1^{b_1}(b_2) + B_2^{b_1}(b_2)].$$

Using the Gronwall inequality, we get

$$\int_0^t \int_0^1 w^2(s, y) dy ds \leq C_2 [B_1^{b_1}(b_2) + B_2^{b_1}(b_2)] \exp\{C_1 t\}.$$

So

$$\lim_{b_2 \rightarrow b_1} \int_0^t \int_0^1 [\theta^{b_2}(s, y) - \theta^{b_1}(s, y)]^2 dy ds = 0.$$

Thus we complete the proof. \square

Proof of lemma 6.5. If $x < m$ then by (3.2), $F_b(x) = 0$. It suffices to prove (6.4) for $m \leq x$. If $m \leq x \leq b$, then

$$\mathcal{L}F_b(x) = \frac{\mathcal{L}h(x)}{h'(b)},$$

here $h(x)$ is a solution of (3.1), so $\mathcal{L}F_b(x) \leq 0$ follows from lemma 3.1. If $x > b$ then by using $F_b''(b) \geq 0$ for $b \geq b_0$

$$\begin{aligned} \mathcal{L}F_b(x) &= \frac{1}{2}(a^2\sigma^2 + \sigma_p^2x^2)F_b''(x) + (a\mu + rx)F_b'(x) - cF_b(x) \\ &\leq (\mu + rx) - c(x - b + F_b(b)) \\ &\leq (\mu + rb) - cF_b(b) \\ &= \mathcal{L}F_b(b) - \frac{1}{2}(a^2\sigma^2 + \sigma_p^2x^2)F_b''(b) \\ &\leq 0 \end{aligned}$$

Thus the proof follows. \square

Proof of lemma 6.6. The proof basically follows the same arguments as in the proof of theorem 5.2 in He and Liang [18] and so we omit it. \square

Proof of lemma 6.7. The lemma is a direct consequence of lemma 6.5 and lemma 6.6. \square

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