ROBUST MAXIMIZATION OF ASYMPTOTIC GROWTH

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Abstract. This paper addresses the question of how to invest in an extremely robust growth-optimal way in a market where the instantaneous expected return of the underlying process is unknown. The optimal investment strategy is identified using a generalized version of the principle eigenfunction for an elliptic second-order differential operator which depends on the covariance structure of the underlying process used for investing. The aforementioned robust growth-optimal strategy can also be seen as a limit, as the terminal date does to infinity, of optimal arbitrages in the terminology of Fernholz and Karatzas [4].

0. Discussion

This paper addresses the question of how to optimally invest in a market when the financial planning horizon is long and the dynamics of the underlying assets are uncertain. For long time-horizons, it is reasonable to question whether fixed parameter estimation, especially for drift rates, remain valid. Therefore, determining a robust way to invest across potential model mis-identifications is desirable, if not indispensable.

Let X be a d-dimensional vector process modeling the underlying assets, discounted by some baseline wealth process. It is assumed that there exists a probability $\mathbb{Q}$ under which X has dynamics of the form $dX_t = \sigma(X_t) dW_t^{\mathbb{Q}}$, where $c := \sigma\sigma'$ represents the instantaneous covariance matrix, and $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$. The significance of the local martingale probability $\mathbb{Q}$ lies in that it acts as a "dominating" measure used to form a class of probabilities Π, out of which an unknown representative is supposed to capture the true dynamics of the process. The class Π is built by exactly all probabilities satisfying the following two conditions:

- Firstly, under $\mathbb{P} \in \Pi$ the process X stays in an open and connected subset $E \subseteq \mathbb{R}^d$. Qualitatively, if X represents either asset prices or (relative) capitalizations, this condition asserts that assets should not cease to exist over the time horizon.
- Secondly, each $\mathbb{P} \in \Pi$ is locally (i.e. for each $t \in \mathbb{R}_+$) absolutely continuous with respect to $\mathbb{Q}$. This last fact implies that the volatility process of X under each $\mathbb{P} \in \Pi$ is the same; even though model mis-identification is possible, the allowable models are not permitted to be wildly inconsistent with one another.

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Note that the family Π as described above does not necessarily induce any ergodic or stability property of the assets, although it certainly contains all such models; in particular, models where the assets display transient behavior are allowable. Furthermore, it is *not* assumed that $\mathbb{Q} \in \Pi$. Indeed, it is often the case that X "explodes" under $\mathbb{Q}$; more precisely, with ζ denoting the first exit time of X from E, $\mathbb{Q}[\zeta < \infty] > 0$ is allowed.

There are good reasons to let the class of models be defined in the above way. While the covariance structure given by the function c is easy to assess, the returns process of X under the "true" probability is statistically impossible to estimate in practice.[1]

Given that the underlying dynamics are only specified within a range of models $\mathbb{P} \in \Pi$, a natural question is to find a reasonable criterion for "optimal investment in X". Here, optimal investment is defined as a wealth process which ensures the largest possible asymptotic growth under *all* models. Given the class $\mathcal{V}$ of all possible positive stochastic integrals against X staring from some fixed initial capital, the asymptotic growth of $V \in \mathcal{V}$ under $\mathbb{P} \in \Pi$ is defined as the largest $\gamma \in \mathbb{R}_+$ such that $\lim_{t \uparrow \infty} \mathbb{P}[(1/t) \log V_t \geq \gamma] = 1$ holds. (An alternative definition of asymptotic growth via almost-sure limits is also considered in the paper.) With this definition of growth, the investor seeks to find a wealth process in $\mathcal{V}$ that achieves maximal growth uniformly over all possible models in Π, or at least in a large enough suitable subclass of Π that covers all "non-pathological" cases.

The solution to the above problem is given in terms of the generalized principal eigenvalue-eigenvector pair (λ^*, η^*) of the eigenvalue equation

$$\frac{1}{2}\sum_{i,j=1}^{k} c_{i,j}(x)\frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) = -\lambda \eta(x), \quad x \in E. \tag{0.1}$$

More precisely, the main result of Section 2 states that, when restricted to a large sub-class Π^* of Π, λ^* is the maximal growth and the process $V \in \mathcal{V}$ defined via $V_t = e^{\lambda^* t}\eta^*(X_t)$ achieves this maximal growth. There are, of course, technicalities on an analytical level arising from the use of the eigenvalue equation (0.1), since it is unreasonable in the present setting to assume either that c is uniformly positive definite on E or that E is bounded with smooth boundary. (Consider, for example, the case where X represents the prices of d assets. In this instance $E = (0, \infty)^d$, which is unbounded with corners. Furthermore, once the stock price goes to zero, it remains stuck there. Thus, the covariance matrix c degenerates along the boundary of E and hence cannot be both continuous and uniformly elliptic.) In order to allow for degenerate c and unbounded E with non-smooth boundary, but still retain some tractability in the problem, it is assumed that E can be

[1] Actually, under continuous-time observations, perfect estimation of c is possible. More realistically, high-frequency data give good estimators for c. In contrast, consider a one-dimensional model for an asst-price of the form $\mathrm{d}X_t/X_t = b\mathrm{d}t + .2\mathrm{d}W_t$, where $b \in \mathbb{R}$ — note that $\sigma = .2$ is considered a "typical" value for annualized volatility. Given observations $(X_t)_{t \in [0,T]}$, where $T > 0$, the best linear unbiased estimator for b is $\widehat{b}_T := (1/T)\log(X_T/X_0)$. Easy calculations show that in order for $|\widehat{b}_T - b| \leq .01$ to happen with probability at least 95%, one needs $T \approx 1600$ (in years). This simple exercise demonstrates the futility of attempting to estimate drifts.

"filled up" by bounded subregions with smooth boundary and that c is pointwise positive definite and continuous. Under this assumptions, [21, Chapter 4] gives a detailed account of eigenvalue equations the form (0.1).

Growth-optimal trading in the face of model uncertainty has been investigated by other authors. One strand of research considers the case where asset returns are assumed stationary and ergodic. In [2], asymptotically growth-optimal trading strategies based upon historical data are constructed. There have been a number of follow up papers on this topic — see [1], [12] and the references cited within. In contrast to the aforementioned approach, knowledge of the entire past is not required in this paper. In fact, the optimal strategy is only based on the current level of X and is, therefore, closely-related to the idea of functionally-generated portfolios studied in [7]. Furthermore, it is also not assumed here that X represents asset returns; in fact, the primary example is when X are relative capitalizations, and not asset returns. In this setting, stationarity of the relative capitalizations does not automatically transfer to stationarity of returns.

The concept of robust growth optimality is also related to that of robust utility optimization, the idea of which dates back to [9] and is considered in detail in [11], [8], [22] and [23] amongst others. Though this paper differs from those above by not considering penalty functions and by focusing on growth rather than general utility functions, the growth optimal strategy provides a "good" long term robust optimal strategy for general utility functions due to the exponential increase in terminal wealth as time progresses. Two recent papers which are close in spirit to this paper are [16] and [15]. [16] considers long-run robust utility maximization in the case of model uncertainty for power and logarithmic utility and [15] addresses the problem of finding wealth processes that minimize long-term downside risk. The precise manner in which the class of models is defined in these papers can only be identified up to a (stochastic) affine perturbation away from a fixed model.This paper differs from the above two in that, to the extent that underlying economic factors affect the asset dynamics, it is only through the drift of X. Furthermore, there is no a priori fixed model from which all other models are recovered via perturbations. This enables the class of models to be determined by qualitative properties, without additional technical restrictions. However, here, as well as in [15], there is a fundamental PDE, playing the role of an ergodic Bellman equation, which governs the robust trading strategies.

The problem of constructing robust growth-optimal strategies can be extended to the case where even the covariance matrix c is not known precisely, but rather assumed to belong to a class of admissible matrices $\mathcal{C}$. Such a situation has been studied in [5] in the setting of optimal arbitrage mentioned below. There is a natural definition of an "extremely" robust growth optimal trading strategy in terms of sub-solutions of (0.1) which are uniform over $\mathcal{C}$. A sketch of how such trading strategies are obtained is given at the end of Section 2.

A second goal of this paper is to relate robust growth optimal trading strategies to optimal arbitrages, as considered in [4]. Optimal arbitrages are trading strategies designed to optimally outperform the index almost surely over a given time horizon. In [4], it was shown that the existence of optimal arbitrages is equivalent to $\mathbb{Q}[\zeta < \infty] > 0$ (positive probability of explosion of the coordinate process under $\mathbb{Q}$), when E is the simplex in $\mathbb{R}^d$. In fact, optimal arbitrages are naturally expressed in terms of (conditional) tails of the distribution of ζ under $\mathbb{Q}$.

The robust growth optimal trading strategies considered here can be regarded as a long term limit of the optimal arbitrages; this is a topic taken up in Section 4. A better understanding of this connection requires exploring a particular probability $\mathbb{P}^*$, under which X has dynamics of the form $dX_t = (c(X_t)\nabla \log \eta^*(X_t))\, dt + \sigma(X_t) dW_t^{\mathbb{P}^*}$ for $t \in \mathbb{R}_+$, where $W^{\mathbb{P}^*}$ is a standard Brownian motion under $\mathbb{P}^*$. Loosely speaking, ergodicity of X under $\mathbb{P}^*$ implies the convergence of the optimal arbitrages to the robust growth-optimal wealth process as the horizon becomes large. This is part of the reason why Section 3 is devoted to investigating the properties of X under $\mathbb{P}^*$. An application of ergodic results for unbounded functions from [19], coupled with powerful probabilistic arguments, allows to show the aforementioned convergence of optimal arbitrages to the robust growth-optimal one. Furthermore, convergence of the probabilities $\mathbb{Q}[\cdot \mid \zeta > T]$ to $\mathbb{P}^*$ on $\mathcal{F}_t$ as $T \uparrow \infty$ in the total-variation norm is established. This extends results on diffusions conditioned to remain in a bounded region, first obtained in [20], to regions with non-smooth boundaries where the matrix c need not be uniformly positive definite, and where the the process X under $\mathbb{Q}$ need not be m-reversing for any measure m.

In the special one-dimensional case, considered in Section 5, simple tests for transience and recurrence of diffusions are readily available. This allows to provide tight conditions upon c in the case of a bounded interval, in which $\lambda^* = 0$ or $\lambda^* > 0$, and characterize both the nature of η^* and of $\mathbb{P}^*$. The main message is essentially the following: if X can explode to both endpoints under $\mathbb{Q}$ then everything works out nicely, in the sense that $\lambda^* > 0$ and X is positive recurrent under $\mathbb{P}^*$. The technical proofs of the results in Section 5, some of which rely heavily on singular Sturm-Liouville theory, are given in Section 7.

Finally, Section 6 provides examples that illustrate the results obtained in previous sections. In contrast to the case where c is uniformly positive definite on E, multi-dimensional examples where the function η^* does not vanish on the boundary of E, even if E is bounded, are given.

1. The Set-Up

Consider an open and connected set $E \subseteq \mathbb{R}^d$ and a function c mapping E to the space of $d \times d$ matrices. The following assumptions will be in force throughout:

Assumption 1.1. For each $x \in E$, $c(x)$ is a symmetric and strictly positive definite $d \times d$ matrix. For $1 \leq i, j \leq d$, $c_{ij}(x)$ is locally $C^{2,\alpha}$ on E for some $\alpha \in (0, 1]$. Furthermore, there exists a sequence $(E_n)_{n \in \mathbb{N}}$ of bounded open connected subsets of E such that each boundary ∂E_n is $C^{2,\alpha}$, $\bar{E}_n \subset E_{n+1}$ for $n \in \mathbb{N}$, and $E = \bigcup_{n=1}^{\infty} E_n$.

1.1. **The generalized martingale problem on E.** It will now be discussed how Assumption 1.1 implies the existence of a unique solution to the generalized martingale problem on E for the operator L which acts on $f \in C^2(E)$ via

$$(Lf)(x) = \frac{1}{2} \sum_{i,j=1}^{d} c_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x), \quad x \in E. \tag{1.1}$$

Let $\widehat{E} = E \cup \triangle$ be the one-point compactification of E; the point $\triangle$ is identified with ∂E if E is bounded and with ∂E plus the point at ∞ if E is unbounded. Let $C(\mathbb{R}_+, \widehat{E})$ be the space of continuous functions from $[0, \infty)$ to $\widehat{E}$. For $\omega \in C(\mathbb{R}_+, \widehat{E})$, define the exit times:

$$\begin{aligned} \zeta_n(\omega) &:= \inf \{ t \in \mathbb{R}_+ \mid \omega_t \notin E_n \}, \\ \zeta(\omega) &:= \lim_{n \uparrow \infty} \zeta_n(\omega). \end{aligned}$$

Then, define

$$\Omega = \left\{ \omega \in C(\mathbb{R}_+, \widehat{E}) \mid \omega_{\zeta + t} = \triangle \text{ for all } t \in \mathbb{R}_+ \text{ if } \zeta(\omega) < \infty \right\}.$$

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be the coordinate mapping process for $\omega \in \Omega$. Set $\mathcal{B} = (\mathcal{B}_t)_{t \in \mathbb{R}_+}$ to be the natural filtration of X. It follows that $\mathcal{B}_\infty := \bigvee_{t \in \mathbb{R}_+} \mathcal{B}_t$ is the Borel σ-algebra on Ω. Furthermore, $\mathcal{B}_\infty = \bigvee_{n \in \mathbb{N}} \mathcal{B}_{\zeta_n}$, since paths in Ω stay in $\triangle$ upon arrival.

A solution to the generalized martingale problem on E is a family of probability measures $(\mathbb{Q}_x)_{x \in \widehat{E}}$ such that $\mathbb{Q}_x[X_0 = x] = 1$ and

$$f(X_{t \wedge \zeta_n}) - \int_0^{t \wedge \zeta_n} (Lf)(X_s) ds$$

is a $(\Omega, (\mathcal{B}_t)_{t \in \mathbb{R}_+}, \mathbb{Q}_x)$-martingale for all $n \in \mathbb{N}$ and all $f \in C^2(E)$ with Lf given as in (1.1).

Assumption 1.1 ensures a solution to the generalized martingale problem, as the following proposition, taken from [21, Theorem 1.13.1], shows.

Proposition 1.2. *Under Assumption 1.1 there is a unique solution $(\mathbb{Q}_x)_{x \in \widehat{E}}$ to the generalized martingale problem on E. The family $(\mathbb{Q}_x)_{x \in \widehat{E}}$ possesses the strong Markov property.*

Set $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ to be the right-continuous enlargement of $(\mathcal{B}_t)_{t \in \mathbb{R}_+}$ and $\mathcal{F} = \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t = \mathcal{B}_\infty$. Assumption 1.1 implies that

$$f(X_{t \wedge \zeta_n}) - \int_0^{t \wedge \zeta_n} (Lf)(X_s) ds$$

is a $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{Q}_x)$-martingale for all $n = 1, 2, 3, ...$ and $f \in C^2(E)$ since f and Lf are bounded on each E_n. By setting $f(x) = x^i, i = 1, ..., d$ and $f(x) = x^i x^j, i, j = 1, ...d$ it follows that for each n and each $x \in \widehat{E}$, $X_{t \wedge \zeta_n}$ is a $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{Q}_x)$-martingale with quadratic covariation process $\int_0^\cdot \mathbb{I}_{\{t \leq \zeta_n\}} c(X_t) \mathrm{d}t$.

1.2. **Asymptotic growth.** For a fixed $x_0 \in E$, set $\mathbb{Q} = \mathbb{Q}_{x_0}$. Going forward, whenever there is no subscript associated to the probabilities it will be assumed they charge only the event $\{X_0 = x_0\}$.

Denote by Π the class of probabilities on $(\Omega, \mathcal{F})$ which are *locally absolutely continuous* with respect to $\mathbb{Q}$ (written $\mathbb{P} \ll_{\mathsf{loc}} \mathbb{Q}$) and for which the coordinate process X does not explode, i.e., $\mathbb{P} \in \Pi$ if and only if $\mathbb{P}|_{\mathcal{F}_t} \ll \mathbb{Q}|_{\mathcal{F}_t}$ for all $t \geq 0$ and $\mathbb{P}[\zeta < \infty] = 0$. For each $\mathbb{P} \in \Pi$, X is a $\left(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P}\right)$-semimartingale such that $\mathbb{P}\left[X \in C\left(\mathbb{R}_+, E\right)\right] = 1$. Therefore, X admits the representation

$$X = x_0 + \int_0^\cdot b_t^{\mathbb{P}} dt + \int_0^\cdot \sigma(X_t) dW_t^{\mathbb{P}},$$

where $W^{\mathbb{P}}$ is a standard d-dimensional Brownian motion on $\left(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P}\right)$, σ is the unique symmetric strictly positive definite square root of c and $b^{\mathbb{P}}$ is a d-dimensional $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-progressively-measurable process.

Let $(\xi_t)_{t \in \mathbb{R}_+}$ be an adapted process. For $\mathbb{P} \in \Pi$, define

$$\mathbb{P}\text{-}\liminf_{t \to \infty} \xi_t \;:=\; \operatorname*{ess\,sup}_{\mathbb{P}} \left\{ \zeta \text{ is } \mathcal{F}\text{-measurable} \;\mid\; \lim_{t \to \infty} \mathbb{P}[\xi_t \geq \zeta] = 1 \right\}.$$

If, in addition, $\mathbb{P}\left[\xi_t > 0\right] = 1$ for each $t \in \mathbb{R}_+$, let

$$g(\xi; \mathbb{P}) \;:=\; \sup \left\{ \gamma \in \mathbb{R} \mid \mathbb{P}\text{-}\liminf_{t \to \infty} \left(t^{-1} \log \xi_t\right) \geq \gamma, \quad \mathbb{P}\text{- a.s.} \right\}$$

be the *asymptotic growth of* ξ *under* $\mathbb{P}$. Since $\mathbb{P} \in \Pi$ and $\mathbb{Q}$ are not necessarily equivalent on $\mathcal{F}$, $g(\xi; \mathbb{P})$ indeed depends on $\mathbb{P} \in \Pi$. The following result, the proof of which is straightforward and hence omitted, provides an alternative representation for $g(\xi; \mathbb{P})$.

Lemma 1.3. *For a given* $\mathbb{P} \in \Pi$ *and adapted process* $(\xi_t)_{t \in \mathbb{R}_+}$ *such that* $\mathbb{P}[\xi_t > 0] = 1$ *for all* $t \in \mathbb{R}_+$,

$$g(\xi; \mathbb{P}) = \sup \left\{ \gamma \in \mathbb{R} \mid \lim_{t \to \infty} \mathbb{P}\left[t^{-1} \log \xi_t \geq \gamma\right] = 1 \right\}.$$

1.3. **The problem.** Let $\mathcal{V}$ denote the class of processes with $V_0 = 1$ that are nonnegative stochastic integrals with respect to X for all $\mathbb{P} \in \Pi$. The problem considered is to calculate

$$\sup_{V \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi} g(V; \mathbb{P}) \tag{1.2}$$

and to find $V^* \in \mathcal{V}$ that achieves this value, at least for all $\mathbb{P}$ in a large sub-class of Π. To this end, for a given $\lambda \in \mathbb{R}$ and L as in (1.1), define the cone of positive harmonic functions with respect to $L + \lambda$ as

$$H_\lambda \;:=\; \left\{ \eta \in C^2(E) \mid L\eta = -\lambda \eta \text{ and } \eta > 0 \right\} \tag{1.3}$$

Set

$$\lambda^* \;:=\; \sup \left\{ \lambda \in \mathbb{R} \mid H_\lambda \neq \emptyset \right\} \tag{1.4}$$

Since $H_0 \neq \emptyset$ (take $\eta \equiv 1$), it follows that $\lambda^* \geq 0$. If $H_{\lambda^*} \neq \emptyset$ then, by construction, there is an $\eta^* \in C^2(E)$ satisfying

$$L\eta^* = -\lambda^* \eta^* \tag{1.5}$$

and λ^* is the largest real for which such an η^* exists. Thus, λ^* is a generalized version of the principal eigenvalue for L on E. The following result, taken from [21, Theorem 4.3.2], states that, indeed, $H_{\lambda^*} \neq \emptyset$.

Proposition 1.4. *Let Assumption 1.1 hold. Then $\lambda^* < \infty$ and $H_{\lambda^*} \neq \emptyset$.*

Remark 1.5. In [21, Theorem 4.3.2], $\lambda^* = \inf\{\lambda \in \mathbb{R} \mid H_{-\lambda} \neq \emptyset\}$ and hence to connect the results therein with Proposition 1.4, λ must be multiplied by -1.

Remark 1.6. Proposition 1.4 makes no claim regarding the uniqueness of η^* corresponding to λ^*. For example, when $E = (0, \infty)$ and $c \equiv 1$, it holds that $\lambda^* = 0$; hence, η^* could be either x or 1. However, Proposition 1.7 below shows that typically η^* is unique up to a constant multiple, and Example 4.7 in Section 4 shows even when uniqueness fails, a particular η^* may be advantageous.

The following result, taken from [21, Theorems 4.3.3 and 4.3.4], provides a way of checking if a particular pair (η, λ) such that $\eta \in H_\lambda$ corresponds to the optimal pair (η^*, λ^*).

Proposition 1.7. *Let Assumption 1.1 hold. Let (η, λ) be such that $\eta \in H_\lambda$. Let $(\mathbb{P}_x^\eta)_{x \in \widehat{E}}$ be the solution to the generalized martingale problem on $\widehat{E}$ for the operator*

$$L^\eta = L + c\nabla \log \eta \cdot \nabla \tag{1.6}$$

Such a solution exists under Assumption 1.1. If the coordinate mapping process X is recurrent under $(\mathbb{P}_x^\eta)_{x \in E}$, then η is unique up to multiplication by a positive constant, $\eta^* = \eta$ and $\lambda^* = \lambda$.

Remark 1.8. It should be noted that Proposition 1.7 does not imply that if the coordinate mapping process X under $(\mathbb{P}_x^\eta)_{x \in E}$ is transient then $\eta \neq \eta^*$ and $\lambda \neq \lambda^*$. Indeed, in Example 4.7 from Section 4, $\lambda^* = 0$ even though $\mathbb{Q}_x[\zeta < \infty] > 0$ for all $x \in E$, and thus $\eta^* = 1$ does not yield a recurrent process.

2. The Min-Max Result

2.1. **The result.** The following theorem identifies λ^* with the value in (1.2):

Theorem 2.1. *Let Assumption 1.1 hold. Let η^* be the solution of (1.5) corresponding to λ^* with $\eta^*(x_0) = 1$, and define V^* via $V_t^* = e^{\lambda^* t}\eta^*(X_t)$ for all $t \in \mathbb{R}_+$. Define also*

$$\Pi^* := \left\{ \mathbb{P} \in \Pi \mid \mathbb{P}\text{-}\liminf_{t \to \infty} \left(t^{-1} \log \eta^*(X_t) \right) \geq 0 \ \ \mathbb{P}\text{- } a.s. \right\}$$

Then, $V^* \in \mathcal{V}$ and $g(V^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. Furthermore,

$$\lambda^* = \sup_{V \in \mathcal{V}} \inf_{\mathbb{P} \in \Pi^*} g(V; \mathbb{P}) = \inf_{\mathbb{P} \in \Pi^*} \sup_{V \in \mathcal{V}} g(V; \mathbb{P}). \tag{2.1}$$

Remark 2.2. While restricting attention to Π^* may seem artificial at first, two observations are in place. Firstly, Π^* only depends on the matrix c and region E, which are inputs to the problem. Secondly, Π^* does contain all the probabilities $\mathbb{P}$ such that X is eventually tight in E, and hence naturally corresponds to those $\mathbb{P}$ for which X is stable. To see the latter fact, let $\epsilon > 0$ and $K^\epsilon \subseteq E$ be compact such that $\sup_{t \geq t_0} \mathbb{P}\left[X_t \notin K^\epsilon\right] \leq \epsilon$ for some t_0. Set $\beta_\epsilon = \max_{x \in K^\epsilon} |\log \eta^*(x)|$ and note that for any $\delta > 0$ and $t > \max\{t_0, \beta^\epsilon/\delta\}$,

$$\mathbb{P}\left[t^{-1}\log\eta^*(X_t) < -\delta\right] \leq \mathbb{P}\left[\left|t^{-1}\log\eta^*(X_t)\right| > \delta; X_t \notin K^\epsilon\right] \leq \epsilon.$$

Thus, $\lim_{t\to\infty} \mathbb{P}\left[t^{-1}\log\eta^*(X_t) \geq -\delta\right] = 1$ for all $\delta > 0$; hence, $\mathbb{P} \in \Pi^*$.

Proof of Theorem 2.1. Set

$$\ell^*(x) = \log\eta^*(x), \quad \text{for } x \in E. \tag{2.2}$$

To see why $V^* \in \mathcal{V}$ note that Itô's formula gives, for each $n \in \mathbb{N}$, each $t \in \mathbb{R}_+$ and each $\mathbb{P} \in \Pi$,

$$\begin{aligned} V^*_{t\wedge\zeta_n} &= 1 + \int_0^{t\wedge\zeta_n} e^{\lambda^* s}\nabla\eta^*(X_s)'\mathrm{d}X_s \\ &= 1 + \int_0^{t\wedge\zeta_n} V^*_s\nabla\ell^*(X_s)'\mathrm{d}X_s \end{aligned} \tag{2.3}$$

where the prime symbol ($'$) denotes transposition. Since $\mathbb{P}[\zeta < \infty] = 0$ for all $\mathbb{P} \in \Pi$, it follows that the equalities in (2.3) hold for all $t \geq 0$. By the construction of Π^*, $\lim_{t\to\infty} \mathbb{P}\left[t^{-1}\log(V^*_t) \geq \gamma\right] = 1$ holds for all $\gamma < \lambda$ and all $\mathbb{P} \in \Pi^*$. Therefore, Lemma 1.3 implies $g(V^*; \mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi^*$. In particular, $\lambda^* \leq \sup_{V\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*} g(V; \mathbb{P})$.

Now, let λ^*_n, η^*_n and ℓ^*_n be the equivalents of λ^*, η^* and ℓ^* when E is replaced by E_n in (1.3), (1.4), (2.2) and (2.3). Assumption 1.1 gives that c is uniformly elliptic on E_n and hence $\eta^*_n \in C^{2,\alpha}\left(\bar{E}_n\right)$ and vanishes on ∂E_n [21, Theorem 3.5.5]. Furthermore, there exists a solution to the generalized martingale problem $\left(\mathbb{P}^*_{x,n}\right)_{x\in E_n}$ for the operator $L^{\eta^*_n}$ in (1.6) and the coordinate process X under $\left(\mathbb{P}^*_{x,n}\right)_{x\in E_n}$ is recurrent in E_n ([21, Theorem 4.2.4]). Set $\mathbb{P}^*_n = \mathbb{P}^*_{x_0,n}$. It follows that $\mathbb{P}^*_n[\zeta < \infty] = 0$ and $\lim_{t\to\infty} \mathbb{P}^*_n\left[t^{-1}\log\eta^*(X_t) = 0\right] = 1$ since there exists a $K_n > 0$ such that $1/K_n < \eta^* < K_n$ on E_n. Thus, $\mathbb{P}^*_n \in \Pi^*$ if $\mathbb{P}^*_n \ll_{\mathsf{loc}} \mathbb{Q}$. To show this, let $(\mathbb{Q}_{x,n})_{x\in\widehat{E}_n}$ be the solution to the generalized martingale problem for L on $\widehat{E}_n$. Let $\mathbb{Q}_n = \mathbb{Q}_{x_0,n}$. It follows from [21, Corollary 4.1.2] and the recurrence of X under $\mathbb{P}^*_n$ that for $t > 0$,

$$\left.\frac{d\mathbb{P}^*_n}{d\mathbb{Q}_n}\right|_{\mathcal{B}_t} = e^{\lambda^*_n t}\frac{\eta^*_n(X_t)}{\eta^*_n(x_0)}\mathbb{I}_{\{\zeta_n > t\}},$$

and thus $\mathbb{P}^*_n|_{\mathcal{B}_t} \ll \mathbb{Q}_n|_{\mathcal{B}_t}$. This immediately gives $\mathbb{P}^*_n|_{\mathcal{B}_{t\wedge\zeta_n}} \ll \mathbb{Q}_n|_{\mathcal{B}_{t\wedge\zeta_n}}$ for each n. But, $\mathbb{Q}_n|_{\mathcal{B}_{t\wedge\zeta_n}} = \mathbb{Q}|_{\mathcal{B}_{t\wedge\zeta_n}}$. Thus, if $B \in \mathcal{B}_t$ such that $\mathbb{Q}[B] = 0$ then $\mathbb{Q}[B \cap \{\zeta_n > t\}] = 0$. Since $B \cap \{\zeta_n > t\} \in \mathcal{B}_{t\wedge\zeta_n}$ it follows that $\mathbb{P}^*_n[B \cap \{\zeta_n > t\}] = 0$. But, $\mathbb{P}^*_n[\zeta_n > t] = 1$ for each t so $\mathbb{P}^*_n[B \cap \{\zeta_n > t\}] = 0$ implies $\mathbb{P}^*_n[B] = 0$. Therefore, $\mathbb{P}^*_n|_{\mathcal{B}_t} \ll \mathbb{Q}|_{\mathcal{B}_t}$ and hence on $\mathbb{P}^*_n|_{\mathcal{F}_t} \ll \mathbb{Q}|_{\mathcal{F}_t}$ as well, proving $\mathbb{P}^*_n \in \Pi$.

Let $V_n^* = \exp(\lambda_n^* t)\eta_n^*(X_t)$ be the numéraire portfolio under $\mathbb{P}_n^*$. Then, $g(V_n^*; \mathbb{P}_n^*) \leq \lambda_n^*$ is immediate since E_n is bounded and η_n^* goes to 0 on ∂E_n. That $g(V; \mathbb{P}_n^*) \leq g(V_n^*; \mathbb{P}_n^*)$ for all $V \in \mathcal{V}$ holds from the $\mathbb{P}_n^*$-supermartingale property of V/V_n^*. Therefore, $\sup_{V\in\mathcal{V}} g(V; \mathbb{P}_n^*) \leq \lambda_n^*$, and $\inf_{\mathbb{P}\in\Pi} \sup_{V\in\mathcal{V}} g(V;\mathbb{P}) \leq \lim_{n\to\infty} \lambda_n^*$. However, $\downarrow \lim_{n\to\infty} \lambda_n^* = \lambda^*$ holds in view of Assumption 1.1 [21, Theorem 4.4.1]. This gives $\inf_{\mathbb{P}\in\Pi^*} \sup_{V\in\mathcal{V}} g(V;\mathbb{P}) \leq \lambda^*$ and completes the argument. □

2.2. **An "almost sure" class of measures.** Define the following class of probability measures

$$\Pi_{a.s.}^* := \left\{\mathbb{P} \in \Pi \mid \liminf_{t\to\infty} \left(t^{-1}\log \eta^*(X_t)\right) \geq 0,\ \mathbb{P}\text{-a.s.}\right\}.$$

For $\mathbb{P} \in \Pi_{a.s.}^*$ define

$$g_{a.s.}(V;\mathbb{P}) := \sup\left\{\gamma \in \mathbb{R} \mid \liminf_{t\to\infty}\left(t^{-1}\log V_t\right) \geq \gamma,\ \mathbb{P}\text{-a.s.}\right\}.$$

The following result is the analog of Theorem 2.1 for the class of measures $\Pi_{a.s.}^*$ and for the growth rate $g_{a.s.}(V;\mathbb{P})$.

Proposition 2.3. *Let Assumption 1.1 hold. Then $g_{a.s.}(V;\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi_{a.s.}^*$ and*

$$\lambda^* = \sup_{V\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi_{a.s.}^*} g_{a.s.}(V;\mathbb{P}) = \inf_{\mathbb{P}\in\Pi_{a.s.}^*} \sup_{V\in\mathcal{V}} g_{a.s.}(V;\mathbb{P})$$

Remark 2.4. Unlike the situation with Π^*, where if the coordinate process X is eventually $\mathbb{P}$-tight then $\mathbb{P} \in \Pi^*$, giving a useful characterization of even a subset of $\Pi_{a.s.}^*$ independent of η^* is difficult. On the positive side, if $\mathbb{P}$ is such that X never exits E_n for some n then $\mathbb{P} \in \Pi_{a.s.}^*$. However, even if X is positive recurrent under $\mathbb{P}$, it cannot immediately be said that $\mathbb{P} \in \Pi_{a.s.}^*$.

Proof. The inequality $\lambda^* \leq \sup_{V\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi_{a.s}^*} g_{a.s.}(V;\mathbb{P})$ follows since by construction $g_{a.s.}(V^*,\mathbb{P}) \geq \lambda^*$ for all $\mathbb{P} \in \Pi_{a.s.}^*$. The inequality $\lambda^* \geq \inf_{\mathbb{P}\in\Pi_{a.s.}^*} \sup_{V\in\mathcal{V}} g_{a.s.}(V;\mathbb{P})$ follows by the same argument as in Theorem 2.1 since $\mathbb{P}_n^* \in \Pi_{a.s.}^*$ and $\sup_{V\in\mathcal{V}} g_{a.s.}(V;\mathbb{P}_n) \leq \lambda_n^*$. □

2.3. **Extreme robust growth optimality.** Even more robust treatment of asymptotic growth can be carried out in the case where the matrix c is identified only up to a class of matrices $\mathcal{C}$. Assume E and each $c \in \mathcal{C}$ satisfies Assumption 1.1. Write L^c for the operator in (1.1), let $\mathbb{Q}^c$ be the solution to the generalized martingale problem for L^c on E, and set $\lambda^{*,c}$ for λ^* from (1.4). For $\lambda \in \mathbb{R}$, define

$$\tilde{H}_\lambda = \left\{\eta \in C^2(E) \;\middle|\; \sup_{c\in\mathcal{C}} L^c\eta \leq -\lambda\eta \text{ and } \eta > 0\right\}$$

and

$$\tilde{\lambda}^* = \sup\left\{\lambda \in \mathbb{R} \mid \tilde{H}_\lambda \neq \emptyset\right\}$$

Note that, as before, $\tilde{\lambda}^* \geq 0$. Assume $\tilde{H}_{\tilde{\lambda}^*} \neq \emptyset$ and let $\tilde{\eta}^* \in \tilde{H}_{\tilde{\lambda}^*}$ normalized such that $\tilde{\eta}^*(x_0) = 1$. The analog of the class of probabilities Π is $\tilde{\Pi}$, consisting of all probabilities on $(\Omega, \mathcal{F})$ such that $\mathbb{P}[\zeta < \infty]$ and $\mathbb{P} \ll_{\mathsf{loc}} \mathbb{Q}^c$ holds for *some* $c \in \mathcal{C}$. Similarly, Π^* consists of all $\mathbb{P} \in \tilde{\Pi}$ such that $\mathbb{P}\text{-}\liminf_{t\to\infty}\left(t^{-1}\log\tilde{\eta}^*(X_t)\right) \geq 0$, $\mathbb{P}$- a.s. Note that as before, $\tilde{\Pi}^*$ contains all the measures $\mathbb{P}$ such that $(X_t)_{t\geq 0}$ forms a $\mathbb{P}$-tight family of random variables.

By Itô's formula it follows that the wealth process $V^* := 1 + \int_0^\cdot e^{\tilde{\lambda}^* t} \nabla \tilde{\eta}^*(X_t)' \mathrm{d}X_t$ is such that $V^*_t \geq e^{\tilde{\lambda}^* t} \tilde{\eta}^*(X_t)$ holds almost surely for any $\mathbb{P} \in \tilde{\Pi}$. If $\tilde{\lambda}^* = \inf_{c \in \mathcal{C}} \lambda^{*,c}$ then copying the proof in Theorem 2.1 yields

$$\tilde{\lambda}^* = \sup_{V \in \mathcal{V}} \inf_{\mathbb{P} \in \tilde{\Pi}^*} g(V; \mathbb{P}) = \inf_{\mathbb{P} \in \tilde{\Pi}^*} \sup_{V \in \mathcal{V}} g(V; \mathbb{P}).$$

It follows that there exist extreme robust growth optimal trading strategies across $\tilde{\Pi}^*$ if $\tilde{H}_{\tilde{\lambda}^*} \neq \emptyset$ and $\tilde{\lambda}^* = \inf_{c \in \mathcal{C}} \lambda^{*,c}$. Regarding the latter requirement, according to [21, Theorem 3.4.5], for each $c \in \mathcal{C}$, $\lambda^{*,c}$ admits the variational representation

$$\lambda^{*,c} = \sup_{u \in C^{2,\alpha}(E)} \inf_{x \in E} \left(-(L^c u)(x) - \frac{1}{2} \nabla u(x)' c(x) u(x) \right); \tag{2.4}$$

therefore,

$$\inf_{c \in \mathcal{C}} \lambda^{*,c} = \inf_{c \in \mathcal{C}} \sup_{u \in C^{2,\alpha}(E)} \inf_{x \in E} \left(-(L^c u)(x) - \frac{1}{2} \nabla u(x)' c(x) u(x) \right)$$

On the other hand, straightforward computations show that

$$\tilde{\lambda}^* = \sup_{u \in C^{2,\alpha}(E)} \inf_{c \in \mathcal{C}} \inf_{x \in E} \left(-(L^c u)(x) - \frac{1}{2} \nabla u(x)' c(x) u(x) \right)$$

Therefore, with $g(u, c) = \inf_{x \in E} \left(-(L^c u)(x) - \frac{1}{2} \nabla u(x)' c(x) u(x) \right)$, $\tilde{\lambda}^* = \inf_{c \in \mathcal{C}} \lambda^{*,c}$ will hold if

$$\inf_{c \in \mathcal{C}} \sup_{u \in C^{2,\alpha}(E)} g(u, c) = \sup_{u \in C^{2,\alpha}(E)} \inf_{c \in \mathcal{C}} g(u, c).$$

The above is a min-max problem, and appropriate topological and convexity assumptions will have to be made. No deeper analysis of this issue will be given, as it is not the focal point of the paper.

3. An Interesting Probability

Let $\eta^* \in H_{\lambda^*}$ and let $(\mathbb{P}^*_x)_{x \in \widehat{E}}$ be the solution to the generalized martingale problem on $\widehat{E}$ for the operator L^{η^*} given in (1.6). Set $\mathbb{P}^* \equiv \mathbb{P}^*_{x_0}$.

It is of great interest to know whether $\mathbb{P}^* \in \Pi^*$. To begin with, if this is indeed true and $g(V^*, \mathbb{P}^*) = \lambda^*$, the pair $(V^*, \mathbb{P}^*)$ constitutes a saddle point for the minimax problem described in (2.1). Indeed, in this case

$$g(V; \mathbb{P}^*) \leq g(V^*; \mathbb{P}^*) \leq g(V^*; \mathbb{P}), \text{ for all } V \in \mathcal{V} \text{ and } \mathbb{P} \in \Pi^*.$$

Furthermore, in Section 4 where connections between robust growth-optimal portfolios and optimal arbitrages are studied, the behavior of the coordinate process X under $\mathbb{P}^*$ becomes important. To this end, presented in the sequel are some results that explore the behavior of X under $\mathbb{P}^*$. In particular, all the results give sufficient conditions to ensure that $\mathbb{P}^* \in \Pi^*$.

Remark 3.1. Although only *sufficient* conditions ensuring that $\mathbb{P}^* \in \Pi^*$ are presented in this section, examples where $\mathbb{P}^* \notin \Pi^*$ have not been found. It is thus conjectured that $\mathbb{P}^* \in \Pi^*$ is true under Assumption 1.1, but it is an open question. For a potential counterexample, see Example 6.5 in Section 6. (In cases where H_{λ^*} is two-dimensional at least one of the resulting $\mathbb{P}^*$ is in Π^*.)

Recall from Remark 2.2 that eventual $\mathbb{P}^*$-tightness of $(X_t)_{t\in\mathbb{R}_+}$ implies that $\mathbb{P}^* \in \Pi^*$. The following result is useful because it shows under Assumption 1.1 that positive recurrence and eventual tightness of $(X_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$ are equivalent notions. Note that, in general, even in the one-dimensional bounded case, the behavior of $(X_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$ can vary from positive recurrence to transience as is shown in the examples in Section 6.1.

Proposition 3.2. *Let Assumption 1.1 hold. Then the following are equivalent:*

(1) *The coordinate mapping process X is positive recurrent under $(\mathbb{P}^*_x)_{x\in E}$.*
(2) *For some $x \in E$ and $t_0 \geq 0$, the family of random variables $(X_t)_{t\geq t_0}$ is $\mathbb{P}^*_x$-tight in E.*

Proof. Under Assumption 1.1, X is recurrent under $(\mathbb{P}^*_x)_{x\in E}$ if $\mathbb{P}^*_x[\zeta < \infty] = 0$ for all $x \in E$ and for any $x, y \in E$ and $\varepsilon > 0$ if $\tau_{B(y,\varepsilon)}$ is the first time the coordinate process enters into the closed ball of radius ε around y then $\mathbb{P}^*_x\left[\tau_{B(y,\varepsilon)} < \infty\right] = 1$. Furthermore, if X is recurrent then X is positive recurrent under $(\mathbb{P}^*_x)_{x\in E}$ if there exists a function $\tilde{\eta}^* > 0$ such that $\tilde{L}^*\eta = 0$ and $\tilde{\eta}^* \in \mathbb{L}^1(E, \mathsf{Leb})$ where $\tilde{L}^*$ is the formal adjoint to L^* [21, Section 4.9]. Under Assumption 1.1, $\tilde{L}^*$ is the differential operator acting on $f \in C^2(E)$ by

$$\tilde{L}^* f(x) = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}\left(c_{ij}(x) f(x)\right) - \sum_{i=1}^d \frac{\partial}{\partial x_i}\left(\left(c(x)\nabla \ell^*(x)\right)_i f(x)\right).$$

Assume that X is positive recurrent under $(\mathbb{P}^*_x)_{x\in E}$ and normalize $\tilde{\eta}^*$ so that $\int_E \tilde{\eta}^*(y)dy = 1$. By the ergodic theorem [21, Theorem 4.9.9] it follows that for any bounded measurable function $f : E \mapsto \mathbb{R}$

$$\lim_{t\uparrow\infty} \mathbb{E}^{\mathbb{P}^*}_x\left[f(X_t)\right] = \int_E f(y)\tilde{\eta}^*(y)dy \tag{3.1}$$

Since $\tilde{\eta}^*$ is a probability density, for any $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset E$ such that

$$\int_{K_\varepsilon^c} \tilde{\eta}^*(y)dy \leq \varepsilon.$$

Thus, taking $f_\varepsilon(x) = \mathbb{I}_{K_\varepsilon^c}(x)$ in (3.1), the continuity of X and $\mathbb{P}^*\left[\zeta < \infty\right] = 0$ imply that $(X_t)_{t\in\mathbb{R}_+}$ is eventually $\mathbb{P}^*_x$-tight for any $x \in E$.

As for the reverse implication, assume for some $x \in E$ and $t_0 \geq 0$ that $(X_t)_{t\geq t_0}$ is $\mathbb{P}^*_x$ - tight in E and for each ε let $K_\varepsilon \subset E$ be the compact set such that $\inf_{t\geq t_0} \mathbb{P}^*_x\left[X_t \in K_\varepsilon\right] \geq 1 - \varepsilon$.

Under Assumption 1.1 there are only three possibilities for the coordinate process X under $(\mathbb{P}^*_x)_{x\in \hat{E}}$ [21, Section 2.2.8]:

(1) X is transient: for all $x \in E$ and $n \in \mathbb{N}$, $\mathbb{P}^*_x\left[X \text{ is eventually in } E_n^c\right] = 1$;
(2) X is null recurrent: for any $\phi \in C^2(E)$ such that $\tilde{L}^*\phi = 0$, $\int_E \phi(y)dy = \infty$;
(3) X is positive recurrent.

Clearly, if $(X_t)_{t\geq t_0}$ is $\mathbb{P}^*_x$ - tight in E for some $x \in E$ then X cannot be transient. Furthermore, if X were null recurrent then for each $x \in E$ and any compact set $K \subset E$ it would follow that [21, Theorem 4.9.5]

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{P}^*_x\left[X_s \in K\right] ds = 0.$$

But, by the assumption of tightness

$$\begin{aligned}\liminf_{t\uparrow\infty}\frac{1}{t}\int_0^t \mathbb{P}^*_x\left[X_s \in K_\varepsilon\right] ds &\geq \liminf_{t\uparrow\infty}\frac{1}{t}\int_{t_0}^t \mathbb{P}^*_x\left[X_s \in K_\varepsilon\right] ds \\ &\geq \liminf_{t\uparrow\infty}(1-\varepsilon)\frac{t-t_0}{t} \\ &= (1-\varepsilon).\end{aligned}$$

Therefore, X cannot be null-recurrent. Thus, X is positive recurrent under $(\mathbb{P}^*_x)_{x\in E}$. □

The following result is useful when point-wise estimates for η^* are available.

Proposition 3.3. *Let Assumption 1.1 hold. If* $\lambda^* > 0$, $\mathbb{P}^*\left[\zeta < \infty\right] = 0$ *and*

$$\lim_{n\uparrow\infty}\inf_{x\in E_n^c}\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) \geq \lambda^*, \tag{3.2}$$

then $\mathbb{P}^* \in \Pi^*$.

Remark 3.4. If c is uniformly elliptic on E and E is bounded with a smooth boundary, λ^* corresponds to the principal eigenvalue for L acting on functions η which vanish on ∂E. Since $\left(e^{\lambda^* t}\eta^*(X_t)\right)^{-1}$ is a $\mathbb{P}^*$ supermartingale it follows that $\mathbb{P}^*\left[\zeta < \infty\right]$. Furthermore, Hopf's lemma asserts that $\nabla\eta^*$ does not vanish on ∂E so 3.2 holds as well; indeed, the quantity on the left hand side is unbounded from above.

Proof of Proposition 3.3. That $\mathbb{P}^* \ll_{\mathsf{loc}} \mathbb{Q}$ follows by the same line of reasoning as in the proof of Theorem 2.1. Recall that $\eta^*(x_0) = 1$. Now,

$$\frac{1}{t}\ell^*(X_t) = \frac{1}{t}\int_0^t\left(\frac{1}{2}\nabla\ell^*(X_s)'c(X_s)\nabla\ell^*(X_s) - \lambda^*\right)ds + \frac{1}{t}\int_0^t \nabla\ell^*(X_s)'\sigma(X_s)dW_s^{\mathbb{P}^*}, \tag{3.3}$$

where $W^{\mathbb{P}^*}$ is a Brownian motion under $\mathbb{P}^*$. Under Assumption 1.1, X is either positive recurrent, null recurrent or transient under $(\mathbb{P}^*_x)_{x\in E}$. From (3.2) it follows that in each of these three cases

$$\lim_{t\uparrow\infty}\int_0^t \nabla\ell^*(X_s)'c(X_s)\nabla\ell^*(X_s)ds = \infty, \quad \mathbb{P}^*\text{-a.s.}$$

Let $M = \int_0^\cdot \nabla\ell^*(X_s)'\sigma(X_s)\mathrm{d}W_s^{\mathbb{P}^*}$, so that $[M,M] = \int_0^\cdot \nabla\ell^*(X_s)'c(X_s)\nabla\ell^*(X_s)\mathrm{d}s$. By the Dambins, Dubins and Schwarz Theorem, [13, Theorem 3.4.6], there exists a standard Brownian motion (under $\mathbb{P}^*$) B such that $M = B_{[M,M]_\cdot}$. Therefore, one can write (3.3) as

$$\frac{1}{t}\ell^*(X_t) = -\lambda^* + \frac{[M,M]_t}{2t}\left(1 + 2\frac{B_{[M,M]_t}}{[M,M]_t}\right).$$

By the strong law of large numbers,

$$\lim_{t\uparrow\infty}\frac{B_{[M,M]_t}}{[M,M]_t} = 0 \quad \mathbb{P}^*\text{-a.s.},$$

which means that

$$\liminf_{t\uparrow\infty} \frac{1}{t}\ell^*(X_t) \geq -\lambda^* + \liminf_{t\uparrow\infty} \frac{[M,M]_t}{2t}, \quad \mathbb{P}^*\text{-a.s.} \tag{3.4}$$

If X is positive recurrent under $\mathbb{P}^*$ then $\mathbb{P}^* \in \Pi^*$ as shown in Proposition 3.2 and Remark 2.2. Else, note that because of (3.2) for any $\delta > 0$ and $n \in \mathbb{N}$ large enough,

$$\begin{aligned} -\lambda^* + \frac{[M,M]_t}{2t} &\geq -\delta \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s \in E_n^c\}} ds - \lambda^* \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}} ds \\ &\geq -\delta - \lambda^* \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}} ds. \end{aligned}$$

Now, if X is null-recurrent under $\mathbb{P}^*$ then from [21, Theorem 4.9.5] it follows that

$$\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}} ds = 0, \quad \mathbb{P}^*\text{-a.s}$$

proving, in view of (3.4), that $\mathbb{P}^* \in \Pi^*_{a.s.}$, and hence $\mathbb{P}^* \in \Pi^*$. Clearly,

$$\{X \text{ eventually in } E_n^c\} \subseteq \left\{\lim_{t\uparrow\infty} \frac{1}{t}\int_0^t \mathbb{I}_{\{X_s\in E_n\}} ds = 0\right\}.$$

Therefore, if X is transient it follows that $\mathbb{P}^* \in \Pi^*$. □

The next result gives a condition on whether $\mathbb{P}^* \in \Pi^*$ based on the tail-decay of the distribution of ζ under $\mathbb{Q}$.

Proposition 3.5. *Let Assumption 1.1 hold. If* $\mathbb{P}^*[\zeta < \infty] = 0$ *and*

$$\liminf_{t\uparrow\infty}\left(-\frac{1}{t}\log \mathbb{Q}[\zeta > t]\right) \geq \lambda^*, \tag{3.5}$$

then $\mathbb{P}^* \in \Pi^*$.

Proof. By Proposition 4.2 later in the text

$$\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right) = \lambda^* t + \log\left(\mathbb{Q}[\zeta > t]\right) - \log \eta^*(x_0).$$

Thus, (3.5) implies

$$\limsup_{t\uparrow\infty}\left(\frac{1}{t}\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right)\right) \leq 0. \tag{3.6}$$

Now, by Chebyshev's inequality, for each $\epsilon > 0$,

$$\begin{aligned} \frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t) \leq -\epsilon\right]\right) &= \frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{\eta^*(X_t)} \geq \exp(\epsilon t)\right]\right) \\ &\leq \frac{1}{t}\log\left(\exp(-\epsilon t)\,\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right) \\ &= -\epsilon + \frac{1}{t}\log\left(\mathbb{E}^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_t)}\right]\right). \end{aligned}$$

In conjunction with (3.6), this gives

$$\limsup_{t\uparrow\infty}\left(\frac{1}{t}\log\left(\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t)\leq -\epsilon\right]\right)\right)\leq -\epsilon,$$

which implies in particular that

$$\lim_{t\uparrow\infty}\mathbb{P}^*\left[\frac{1}{t}\log\eta^*(X_t)\leq -\epsilon\right]=0.$$

Since this is true for all $\epsilon>0$, it follows that $\mathbb{P}^*\in\Pi^*$. □

Remark 3.6. From [21, Theorem 4.4.4] (note that there, λ_c is used in place of $-\lambda^*$),

$$-\lambda^*=\lim_{n\uparrow\infty}\lim_{t\uparrow\infty}\frac{1}{t}\log\mathbb{Q}[\zeta_n>t].$$

Since $\mathbb{Q}[\zeta_n>t]\leq\mathbb{Q}[\zeta>t]$ it holds that

$$\lambda^*+\liminf_{t\uparrow\infty}\frac{1}{t}\log\mathbb{Q}[\zeta>t]\geq 0.$$

In particular, (3.5) is really equivalent to

$$\lim_{t\uparrow\infty}\left(\frac{1}{t}\log\mathbb{Q}\left[\zeta>t\right]\right)=\lambda^*.$$

4. Connections with Optimal Arbitrages

In [4], and quite close to the setting considered here, the authors treat the problem of *optimal arbitrage* under $\mathbb{P}\in\Pi$ on a given finite time horizon $[0,T]$, $T\in\mathbb{R}_+$. Using notation of the present paper, they show that there exist relative arbitrages over a time horizon $[0,T]$ if and only if $\mathbb{Q}[\zeta>T]<1$. With $U:\mathbb{R}_+\times E\mapsto[0,1]$ being defined via $U(T,x)=\mathbb{Q}_x[\zeta>T]$ for $(T,x)\in\mathbb{R}_+\times E$, the optimal arbitrage is given by $V^T=(V^T_t)_{t\in[0,T]}$, where

$$V^T_t=\frac{\mathbb{Q}[\zeta>T\mid\mathcal{F}_t]}{\mathbb{Q}[\zeta>T]}=\frac{U(T-t,X_t)}{U(T,x_0)},\ \text{for } t\in[0,T]. \tag{4.1}$$

Remark 4.1. In [4, section 10 and onwards], the problem of optimal arbitrages is essentially treated in the special case of the setting here where

$$E=\left\{x\in\mathbb{R}^d\ \middle|\ \min_{i=1,\dots,d}x_i>0,\ \text{and}\ \sum_{i=1}^d x_i<1\right\},$$

i.e., E is the interior of the simplex on $\mathbb{R}^d$. The interpretation is that the coordinate process X are *relative* capitalizations of stocks, and the corresponding optimal arbitrages are in fact *relative* arbitrages with respect to the market portfolio. In principle, the treatment of [4] does not really utilize the special structure of the simplex; therefore, the general case is considered.

Observe that the optimal arbitrage V^T in (4.1) is normalized so that $V^T_0=1$. In [4], the normalization is such that the *terminal* value of the optimal relative arbitrage is unit; in that case, $U(T,x_0)$ is the minimal capital required at time zero to ensure a unit of capital at time T.

It is natural to study the asymptotic behavior of these optimal arbitrages as the time-horizon becomes arbitrarily large. It is shown below that, under suitable assumptions, the sequence of wealth processes $(V^T)_{T\in\mathbb{R}_+}$ (parameterized via their maturity) converges to the robust asymptotically growth optimal wealth process. The following result, which relates the tail probabilities of ζ under $\mathbb{Q}$ and robust growth-optimal strategies, provides a tool in proving this convergence.

Proposition 4.2. *Let Assumption 1.1 hold and let $\eta^* \in H_{\lambda^*}$ be such that $\mathbb{P}_x^*[\zeta < \infty] = 0$ holds for all $x \in E$. Then,*

$$\mathbb{Q}_x\left[\zeta > T\right] = \eta^*(x)\mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{V_T^*}\right] \text{ holds for all } T \in \mathbb{R}_+ \text{ and } x \in E. \tag{4.2}$$

Proof. Given that $V_T^* = \exp(\lambda^* T)\eta^*(X_T)$, this follows immediately from [21, Theorem 4.1.1]. □

From the above result, it follows that if $\lambda^* > 0$ and $\mathbb{P}_x^*\left[\zeta < \infty\right] = 0$ for each $x \in E$, relative arbitrages occur if and only if the local $\mathbb{P}_x^*$-martingale $1/V^*$ is a strict local $\mathbb{P}_x^*$-martingale in the terminology of [3]. If $1/V^*$ is a $\mathbb{P}_x^*$-martingale then, even though relative arbitrages do not exist, it is still possible to construct robust growth optimal trading strategies, as seen in Example 6.7.

Equation (4.2) may be re-written as

$$e^{\lambda^* T}\mathbb{Q}_x\left[\zeta > T\right] = \eta^*(x)\mathbb{E}_x^{\mathbb{P}^*}\left[\frac{1}{\eta^*(X_T)}\right]. \tag{4.3}$$

Thus, to study the asymptotic behavior of V_t^T as $T \uparrow \infty$ in (4.1) it is necessary to study the long time (as $T \uparrow \infty$) behavior of $\mathbb{E}_x^{\mathbb{P}^*}\left[(\eta^*(X_T))^{-1}\right]$. Assume that X is positive recurrent, or equivalently, eventually tight, under $(\mathbb{P}_x^*)_{x\in E}$ with invariant probability measure μ. Under Assumption 1.1, [19, Theorem 1.2 (iii), Equations (3.29),(3.30)] extends the ergodic result in (3.1) to functions f which are integrable with respect to μ. Thus, for all positive measurable functions $f : E \mapsto \mathbb{R}$

$$\lim_{T\uparrow\infty} \mathbb{E}_x^{\mathbb{P}^*}\left[f(X_T)\right] = \int_E f d\mu \tag{4.4}$$

and this limit is the same for all $x \in E$. This yields the following proposition:

Proposition 4.3. *Let Assumption 1.1 hold. Suppose that $\eta^* \in H_{\lambda^*}$ is such that*

$$\lim_{n\uparrow\infty} \sup_{x\in E_n^c} \eta^*(x) = 0 \tag{4.5}$$

Then, $\mathbb{P}_x^*[\zeta < \infty] = 0$ for all $x \in E$, and the following are equivalent:

(1) $\lim_{T\uparrow\infty} e^{\lambda^* T}\mathbb{Q}_x\left[\zeta > T\right] = \kappa\eta^*(x)$ *for all $x \in E$ where $\kappa > 0$ does not depend upon x.*
(2) $\limsup_{T\uparrow\infty} e^{\lambda^* T}\mathbb{Q}_x\left[\zeta > T\right] < \infty$ *for some $x \in E$.*
(3) *X is positive recurrent under $(\mathbb{P}_x^*)_{x\in E}$ and $\int_E(\eta^*)^{-1}d\mu < \infty$ where μ is the invariant measure for X.*

Remark 4.4. Note that (3) implies (1) even if (4.5) does not hold. Note also that by Example 4.7 below that some condition like (4.5) is necessary for (1), (2) and (3) to be equivalent.

Proof of Proposition 4.3. Let $x \in E$. Note that $\left(e^{\lambda^* t}\eta^*(X_t)\right)^{-1}$ is a $\mathbb{P}^*_x$ super-martingale. By (4.5) if $\mathbb{P}^*_x[\zeta < \infty] > 0$ then the super-martingale property would be violated. Thus, an explosion cannot occur.

Regarding the equivalences, (1) $\Rightarrow$ (2) is trivial. As for (2) $\Rightarrow$ (3), if (2) holds then by (4.3) it follows that there is some $T_0 \geq 0$ such that

$$\sup_{T \geq T_0} \mathbb{E}^{\mathbb{P}^*}_x \left[\frac{1}{\eta^*(X_T)}\right] < \infty$$

Therefore, (4.5) yields that $(X_T)_{T \geq T_0}$ form a $\mathbb{P}^*_x$ tight family of random variables for each $x \in E$. By Proposition 3.2 it follows that X is positive recurrent under $(\mathbb{P}^*_x)_{x\in E}$; hence, (4.4) gives

$$\int_E \frac{1}{\eta^*} d\mu = \lim_{T\uparrow\infty} \mathbb{E}^{\mathbb{P}^*}_x \left[\frac{1}{\eta^*(X_T)}\right] \leq \limsup_{T\uparrow\infty} \mathbb{E}^{\mathbb{P}^*}_x \left[\frac{1}{\eta^*(X_T)}\right] < \infty,$$

proving (3). Implication (3) $\Rightarrow$ (1) follows by applying (4.4) to $1/\eta^*$ and using (4.3). $\square$

The following is the main result of the section.

Theorem 4.5. *Suppose that $\eta^* \in H_{\lambda^*}$ is such that $\mathbb{P}^*[\zeta < \infty] = 0$ and that condition (1) in Proposition 4.3 holds. Fix $\mathbb{P} \in \Pi$. Then, for any fixed $t \in \mathbb{R}_+$,*

$$\mathbb{P}\text{-}\lim_{T\to\infty} \sup_{\tau\in[0,t]} \left|V^T_\tau - V^*_\tau\right| = 0. \tag{4.6}$$

Additionally, for each $T \in \mathbb{R}_+$, let $(\vartheta^T_t)_{t\in[0,T]}$ be a predictable process such that

$$V^T = 1 + \int_0^\cdot V^T_t \left(\vartheta^T_t\right)' \mathrm{d}X_t. \tag{4.7}$$

With $\vartheta^* = \nabla \ell^*(X)$ it follows that for any fixed $t \in \mathbb{R}_+$

$$\mathbb{P}\text{-}\lim_{T\to\infty} \int_0^t \left(\vartheta^T_\tau - \vartheta^*_\tau\right)' c(X_\tau) \left(\vartheta^T_\tau - \vartheta^*_\tau\right) \mathrm{d}\tau = 0. \tag{4.8}$$

Proof. Fix $t \in \mathbb{R}_+$. Equation (4.1), coupled with Proposition 4.3, imply that $\mathbb{P}\text{-}\lim_{T\to\infty} V^T_t = V^*_t$. Let $Z^T = (Z^T_\tau)_{\tau\in[0,t]}$ be defined via $Z^T_\tau := V^T/V^*$. As V^* is the numéraire portfolio under $\mathbb{P}^*$, Z^T is a nonnegative $\mathbb{P}^*$-supermartingale on $[0,t]$ for all $T \in (t,\infty)$. Then, [14, Theorem 2.5] implies that $\mathbb{P}^*\text{-}\lim_{T\to\infty} \sup_{\tau\in[0,t]} \left|Z^T_\tau - 1\right| = 0$. Using the fact that $\mathbb{P}^*\left[\inf_{\tau\in[0,t]} V^*_\tau > 0\right] = 1$, it follows that $\mathbb{P}^*\text{-}\lim_{T\to\infty} \sup_{\tau\in[0,t]} \left|V^T_\tau - V^*_\tau\right| = 0$. Now, with $R^T = (R^T_\tau)_{\tau\in[0,t]}$ defined via

$$R^T = \int_0^\cdot \left(\vartheta^T_s - \vartheta^*_s\right)' \left(\mathrm{d}X_s - c(X_s)\nabla\ell^*(X_s)\mathrm{d}s\right),$$

it holds that $Z^T = 1 + \int_0^\cdot Z^T_s \mathrm{d}R_s$. Invoking [14, Theorem 2.5] again yields $\mathbb{P}^*\text{-}\lim_{T\to\infty}[R^T, R^T]_t = 0$ for all $t \in \mathbb{R}_+$. As

$$[R^T, R^T]_t = \int_0^t \left(\vartheta^T_\tau - \vartheta^*_\tau\right)' c(X_\tau) \left(\vartheta^T_\tau - \vartheta^*_\tau\right) \mathrm{d}\tau,$$

(4.8) follows, with $\mathbb{P}^*$ replacing $\mathbb{P}$ there.

Up to now, the validity of both (4.6) and (4.8), for the special case $\mathbb{P} = \mathbb{P}^* \in \Pi$ has been shown. For a general $\mathbb{P} \in \Pi$, the result follows by noting that $\mathbb{P}^*$ and $\mathbb{Q}$ are equivalent on each $\mathcal{F}_{\zeta_n}$, $n \in \mathbb{N}$, and that $\lim_{n\to\infty} \mathbb{P}[\zeta_n > t] = 1$. □

Remark 4.6. The result of Theorem 4.5 is expected to hold is much more generality than its assumptions suggest. It is conjectured that the results hold under Assumption 1.1, but it is an open question. See Example 6.4 in Section 6 for a potential counterexample. The next example shows that it can even hold when $\lambda^* = 0$.

Example 4.7. Let $E = (0, \infty)$ and $c(x) = 1$ for $x \in E$. It is straightforward to check that

$$U(T, x) = \mathbb{Q}_x[\zeta > T] = 2\Phi\left(x/\sqrt{T}\right) - 1, \text{ for } (T, x) \in \mathbb{R}_+ \times E,$$

where Φ is the cumulative distribution function of the standard normal law. With $x_0 = 1$, it follows that

$$V_t^T = \frac{2\Phi\left(X_t/\sqrt{T-t}\right) - 1}{2\Phi\left(1/\sqrt{T}\right) - 1}, \text{ for } t \in [0, T].$$

From this explicit formula it is straightforward that $\mathbb{P}\text{-}\lim_{T\to\infty} \sup_{\tau\in[0,t]} \left|V_\tau^T - X_\tau\right| = 0$ holds whenever $t \in \mathbb{R}_+$. Observe that $V^* = X$ exactly for the choice $\eta^*(x) = x$ corresponding to $\lambda^* = 0$, and $\mathbb{P}^*$ being the probability that makes X behave as a 3-dimensional Bessel process. Remember that in this example the dimensionality of the set of principal eigenfunctions is two — the other one is $\eta \equiv 1$. It is interesting to note that the sequence (V^T) "chooses" to converge to the optimal strategy of the optimal probability $\mathbb{P}^*$ that satisfies $\mathbb{P}^* \in \Pi$.

As in [6, Section 5.1], for $T \in \mathbb{R}_+$ and $x \in E$ define the measure $\mathbb{P}_x^{\star,T}$ on $\mathcal{F}_T$ via

$$\mathbb{P}_x^{\star,T}[A] = \mathbb{Q}_x[A \mid \zeta > T], \quad \text{for } A \in \mathcal{F}_T.$$

It is shown therein that for each $t \in [0, T]$ and $x \in E$

$$\left.\frac{d\mathbb{P}_x^{\star,T}}{d\mathbb{Q}_x}\right|_{\mathcal{F}_t} = \frac{U(T-t, X_t)}{U(T, x)} \mathbb{I}_{\{\zeta > t\}}.$$

Furthermore, under the assumption $U \in C^{1,2}((0,T)\times E)$, the coordinate process X under $\left(\mathbb{P}_x^{\star,T}\right)_{x\in E}$ has dynamics on $[0, T]$ of

$$\begin{aligned} dX_\tau &= c(X_\tau)\frac{\nabla_x U(T-\tau, X_\tau)}{U(T-\tau, X_\tau)}d\tau + \sigma(X_\tau)dW_\tau \\ &= c(X_\tau)\vartheta_\tau^T d\tau + \sigma(X_\tau)dW_\tau \end{aligned}$$

using the notation of (4.7) in Theorem 4.5. Assuming $\mathbb{P}_x^*[\zeta < \infty] = 0$, it follows that $\mathbb{P}_x^{\star,T}$ and $\mathbb{P}_x^*$ are equivalent on $\mathcal{F}_t$ for $t \in [0, T]$ with

$$\left.\frac{d\mathbb{P}_x^{\star,T}}{d\mathbb{P}_x^*}\right|_{\mathcal{F}_t} = \mathcal{E}\left(\int_0^\cdot \left(\vartheta_\tau^T - \vartheta_\tau^*\right)' \sigma(X_\tau)dW_\tau\right)_t. \tag{4.9}$$

Thus, the results of Theorem 4.5 immediately imply the following:

Proposition 4.8. *Suppose the hypotheses of Theorem 4.5 hold. Then, for any $t \in \mathbb{R}_+$, $\mathbb{P}_x^{\star,T}$ converges in variation norm to $\mathbb{P}_x^*$ on $\mathcal{F}_t$ as $T \uparrow \infty$.*

Proof. The process on the right hand side of (4.9) is the process $Z^T = V^T/V^*$ in the proof of Theorem 4.5. Since for each $A \in \mathcal{F}_t$

$$\left|\mathbb{P}_x^{\star,T}(A) - \mathbb{P}_x^*(A)\right| \leq \mathbb{E}_x^{\mathbb{P}^*}\left[\left|Z_t^T - 1\right|\right],$$

the result follows from [14, Theorem 2.5 (i)]. □

Remark 4.9. In [20], a similar result to Proposition 4.8 is obtained, though not in the setting of convergence of relative arbitrages. Namely, it is assumed that

$$\lim_{T\uparrow\infty} \frac{\nabla_x U(T,x)}{U(T,x)} = \nabla \ell^*(x), \quad \text{for } x \in E, \tag{4.10}$$

where the convergence takes place exponentially fast with rate λ^* and is uniform on compact subsets of E. Under this assumption, the measures $\mathbb{P}_x^{\star,T}$ are shown to weakly converge as $T \uparrow \infty$ to $\mathbb{P}_x^*$ on $\mathcal{F}_t$ for each $t \in \mathbb{R}_+$.

In the case where E is bounded with smooth boundary and c is uniformly elliptic over E, (4.10) holds if there exists a function $H : E \mapsto \mathbb{R}$ such that, for each $i = 1, \dots d$,

$$\sum_{j=1}^d c_{ij}(x)\frac{\partial}{\partial_{x_j}}H(x) = -\frac{1}{2}\sum_{j=1}^d \frac{\partial}{\partial_{x_j}}c_{ij}(x).$$

In vector notion, this gradient condition takes the form $\nabla H = c^{-1}f$, where f is the *Fichera drift* associated to $\mathbb{Q}$. Under this hypothesis, the measure $m(dx) = \exp(2H(x))\,dx$ is reversing for the transition probability function $\mathbb{Q}(t,x,\cdot)$ and the convergence result in (4.10) follows by representing $U(T,x) = \mathbb{Q}_x[\zeta > T]$ as an eigenfunction expansion where the underlying space is $L^2(E,m)$.

The message of Proposition 4.8 is that analytic convergence assumptions of the type in (4.10), which are difficult to prove in the general setup of Assumption 1.1, can be replaced by the probabilistic convergence assumptions in Proposition 4.3.

5. A Thorough Treatment of the One-Dimensional Case

This section considers the case $d = 1$, where $E = (\alpha, \beta)$ is a bounded interval. If $E = \mathbb{R}$, then $\lambda^* = 0$ holds by Proposition 1.7, because the coordinate process under $\mathbb{Q}$ is recurrent. If E is a half interval, it is possible for:

- $\lambda^* = 0$, even though there is explosion under $\mathbb{Q}$ — see Example 4.7.
- $\lambda^* > 0$, even though there is no explosion under $\mathbb{Q}$ — see Example 6.6 with $d = 1$.

and hence making a general statement connecting $\lambda^* > 0$ with explosion or non-explosion under $\mathbb{Q}$ is difficult. Thus, to enlighten the connections with relative arbitrages the following will assumed throughout the section:

Assumption 5.1. Assumption 1.1 holds for $E = (\alpha, \beta)$ with $-\infty < \alpha < \beta < \infty$.

Under the validity of Assumption 5.1, results are provided that almost completely cover all the cases that can occur. The proofs of these results are lengthy and technical, and will be given in Section 7.

The first proposition establishes point-wise tests for c which yield $\lambda^* > 0$ or $\lambda^* = 0$. However, in the case $\lambda^* > 0$, nothing is claimed regarding η^* or $\mathbb{P}^*$. The second proposition gives integral tests which yield $\lambda^* > 0$ or $\lambda^* = 0$. Condition (5.5) is equivalent to the coordinate process X under $(\mathbb{Q}_x)_{x \in [\alpha,\beta]}$ exploding to both α, β with positive probability. Additionally, condition (5.5) not only yields $\lambda^* > 0$ but also that $\mathbb{P}^* \in \Pi^*_{a.s.}$ (and hence $\mathbb{P}^* \in \Pi^*$).

Recall the following facts regarding explosion, transience, recurrence and positive recurrence in the one dimensional case under Assumption 5.1:

- Since E is bounded the coordinate process X under $(\mathbb{Q}_x)_{x \in [\alpha,\beta]}$ is transient. Furthermore it explodes to α and/or β with positive probability if for some $a \in (\alpha, \beta)$:
$$\int_\alpha^a \frac{x-\alpha}{c(x)} dx < \infty \qquad \text{and/or} \qquad \int_a^\beta \frac{\beta - x}{c(x)} dx < \infty.$$
- The coordinate process X under $(\mathbb{P}^*_x)_{x \in (\alpha,\beta)}$ is recurrent if
$$\int_\alpha^a \frac{1}{(\eta^*(x))^2} dx = \infty \quad \text{and} \quad \int_a^\beta \frac{1}{(\eta^*(x))^2} dx = \infty. \tag{5.1}$$
If either of the integrals in (5.1) are finite then the coordinate process X is transient towards the endpoint with finite integral.
- The coordinate process X under $(\mathbb{P}^*_x)_{x \in (\alpha,\beta)}$ is positive recurrent if
$$\int_\alpha^\beta \frac{(\eta^*(x))^2}{c(x)} dx < \infty. \tag{5.2}$$

Proposition 5.2 (Pointwise result)**.** *Let Assumption 5.1 hold. If*
$$\sup_{x \in (\alpha,\beta)} \frac{(x-\alpha)^2 (\beta - x)^2}{c(x)} < \infty, \tag{5.3}$$
then $\lambda^* > 0$. *If*
$$\lim_{x \downarrow \alpha} \frac{(x-\alpha)^2}{c(x)} = \infty \quad \text{or} \quad \lim_{x \uparrow \beta} \frac{(\beta - x)^2}{c(x)} = \infty, \tag{5.4}$$
then $\lambda^* = 0$.

Proposition 5.3 (Integral result)**.** *Let Assumption 5.1 hold. If*
$$\int_\alpha^\beta \frac{(x-\alpha)(\beta-x)}{c(x)} dx < \infty, \tag{5.5}$$
then:

(1) $\lambda^* > 0$.

(2) $\lim_{x\downarrow\alpha} \eta^*(x) = 0 = \lim_{x\uparrow\beta} \eta^*(x)$.
(3) *The coordinate process* X *under* $(\mathbb{P}^*_x)_{x\in(\alpha,\beta)}$ *is positive recurrent and so* $\mathbb{P}^* \in \Pi^*$.
(4) $\mathbb{P}^* \in \Pi^*_{a.s}$.

If for some $a \in (\alpha, \beta)$

$$\int_\alpha^a \frac{(x-\alpha)^2}{c(x)} dx = \infty \quad \text{or} \quad \int_a^\beta \frac{(\beta-x)^2}{c(x)} dx = \infty, \tag{5.6}$$

then $\lambda^* = 0$.

6. Examples

6.1. **One-dimensional examples.** The following examples display a variety of outcomes regarding η^* and $\mathbb{P}^*$. Proofs of all the statements follow from Propositions 5.2, 5.3 and/or from the tests for recurrence, null recurrence or positive recurrence under $\mathbb{P}^*$ given in equations (5.1) and (5.2) in conjunction with Proposition 1.7.

Example 6.1. Let $E = (0,1)$ and $c(x) = x(1-x)$. Then:

- (5.5) holds and so the results of Proposition 5.3 follow.
- $\eta^*(x) = x(1-x)$, $\lambda^* = 1$.
- (4.5) holds as well as condition (3) in Proposition 4.3. Thus, the results of Theorem 4.5 and Proposition 4.8 follow.

Example 6.2. Let $E = (0,1)$ and $c(x) = x^2(1-x)^2$. Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\eta^*(x) = \sqrt{x(1-x)}$, $\lambda^* = 1/8$.
- The coordinate process X is null recurrent under $(\mathbb{P}^*_x)_{x\in E}$; however, $\mathbb{P}^* \in \Pi^*_{a.s.}$.

Note that there is a multidimensional generalization of this in Example 6.7.

Example 6.3. Let $E = (0,1)$ and $c(x) = x^3(1-x)^3$. Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\lambda^* = 0$ by either Proposition 5.2 or 5.3.
- η^* can be any affine function $\alpha + \beta x$ such that $\eta^* > 0$ on $(0,1)$. For any such η^*, $\mathbb{P}^* \in \Pi^*_{a.s.}$.

Example 6.4. Let $E = (0, \hat{x})$, where

$$\hat{x} := \min\left\{ x > 0 \;\middle|\; \int_0^x \log(-\log(y))\, dy = 0 \right\} \approx 0.75.$$

Furthermore, let $c : E \mapsto \mathbb{R}_+$ be defined via

$$c(x) = -2x\log(x) \int_0^x \log(-\log(y))\, dy, \quad \text{for } x \in E.$$

Then:

- (5.5) holds and so the results of Proposition 5.3 follow.

- $\eta^*(x) = \int_0^x \log(-\log(y))\,dy$, $\lambda^* = 1$.
- $(\eta^*)^{-1}$ is not integrable with respect to the invariant measure for $\mathbb{P}^*$.

Example 6.5. Let $E = (0,\infty)$ and

$$c(x) = \frac{4\left(x^{3/2}\int_0^x \cos\left(y^{-1/2}\right)dy + 4x^2 - x^{5/2}\right)}{2 - \sin\left(x^{-1/2}\right)}, \quad \text{for } x \in E.$$

Then:

- $\mathbb{Q}[\zeta < \infty] = 0$.
- $\eta^*(x) = \int_0^x \cos\left(y^{-1/2}\right)dy + 4\sqrt{x} - x$, $\lambda^* = 1$.
- The coordinate process X under $(\mathbb{P}^*_x)_{x\in E}$ is null-recurrent. Whether or not $\mathbb{P}^* \in \Pi^*_{a.s.}$ or Π^* is entirely dependent upon the behavior near 0 and ∞ of (see Proposition 3.3)

$$\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) - \lambda^*.$$

- No conclusion can be drawn based on the results of the paper, since

$$\limsup_{x\downarrow 0}\left(\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) - \lambda^*\right) = 0 \text{ and } \liminf_{x\downarrow 0}\left(\frac{1}{2}\nabla\ell^*(x)'c(x)\nabla\ell^*(x) - \lambda^*\right) = -\frac{2}{3}.$$

6.2. **Multi-dimensional examples.** The following examples show that the optimal η^* need not vanish on the boundary of E even when E is bounded, and that asymptotic growth is possible even when $\mathbb{Q}[\zeta < \infty] = 0$.

Example 6.6 (Correlated geometric Brownian Motion). Let $E = (0,\infty)^d$, and define the matrix c via

$$c_{ij}(x) = x_i x_j A_{ij}, \qquad 1 \le i,j \le d.$$

where A is a symmetric, strictly positive definite $d\times d$ matrix. Define the vectors $\hat{A},\hat{B} \in \mathbb{R}^d$ by

$$\hat{A}_i = A_{ii} \quad (1 \le i \le d), \qquad \hat{B} = \frac{1}{2}A^{-1}\hat{A}.$$

Then

$$\eta^*(x) = \prod_{i=1}^d x_i^{\hat{B}_i}, \qquad \lambda^* = \frac{1}{8}\hat{A}'A^{-1}\hat{A}, \tag{6.1}$$

and $\mathbb{P}^* \in \Pi^*_{a.s.}$.

To see the validity of the above claims, set η, λ as the respective right hand sides of (6.1). A straightforward calculation shows that $L\eta = -\lambda\eta$ and hence that $\lambda^* \ge \lambda$. Set $(\mathbb{P}^\eta_x)_{x\in\widehat{E}}$ as the solution to the generalized martingale problem for L^η as in (1.6) and $\mathbb{P}^\eta = \mathbb{P}^\eta_{x_0}$. The coordinate process X under $\mathbb{P}^\eta$ is given by $X_t = \exp(aW_t)$ where a is the unique positive definite square root of A and W a Brownian motion under $\mathbb{P}^\eta$. Thus, under $\mathbb{P}^\eta$,

$$\frac{1}{t}\log\eta(X_t) = \frac{1}{t}\hat{B}'aW_t.$$

The strong law of large number for Brownian motion gives that $\mathbb{P}^\eta \in \Pi^*_{a.s.}$. Theorem 2.1 then yields $\lambda^* \leq \sup_{V\in\mathcal{V}} g(V;\mathbb{P}^\eta) \leq \lambda$, and hence $\lambda^* = \lambda$, $\eta^* = \eta$ and $\mathbb{P}^* = \mathbb{P}^\eta$.

Example 6.7 (Relative capitalizations of a correlated geometric Brownian Motion). For $d \geq 2$, let

$$E = \left\{ x \in \mathbb{R}^{d-1} \mid \min_{i=1,\dots,d-1} x_i > 0; \sum_{i=1}^{d-1} x_i < 1 \right\}.$$

For the matrix A of Example 6.6 define the $d-1$ dimensional square matrix $\mathcal{A}$ by

$$\mathcal{A}_{ij} = A_{ij} - A_{id} - A_{jd} + A_{dd} \qquad 1 \leq i, j \leq d-1,$$

and the matrix c via

$$c_{ij}(x) = x_i x_j \left(\mathcal{A}_{ij} - (\mathcal{A}x)_i - (\mathcal{A}x)_j + x'\mathcal{A}x \right), \qquad 1 \leq i, j \leq d-1.$$

Set the $d-1$ dimensional vectors

$$\hat{\mathcal{A}}_i = \mathcal{A}_{ii} \quad (1 \leq i \leq d-1), \qquad \hat{\mathcal{B}} = \frac{1}{2}\mathcal{A}^{-1}\hat{\mathcal{A}}.$$

Then,

$$\eta^*(x) = \left(\prod_{i=1}^{d-1} x_i^{\hat{\mathcal{B}}_i} \right) \left(1 - \sum_{i=1}^{d-1} x_i \right)^{1-\sum_{i=1}^{d-1}\hat{\mathcal{B}}_i}, \qquad \lambda^* = \frac{1}{8}\hat{\mathcal{A}}'\mathcal{A}^{-1}\hat{\mathcal{A}}, \tag{6.2}$$

and $\mathbb{P}^* \in \Pi^*_{a.s.}$. Furthermore, the coordinate process under $\mathbb{P}^*$ on the simplex has the same dynamics as the coordinate process under $\mathbb{P}^*$ in Example 6.6 moved to the simplex.

To prove the validity of the claims, rewrite $\tilde{\mathbb{P}}^*$ for the probability measure $\mathbb{P}^*$ of Example 6.6. Set η, λ as the right hand sides of (6.2). Set $(\mathbb{P}^\eta_x)_{x\in\widehat{E}}$ as the solution to the generalized martingale problem for L^η is in (1.6) and $\mathbb{P}^\eta = \mathbb{P}^\eta_{x_0}$. A long calculation using Itô's formula shows that $L\eta = -\lambda\eta$ and the equivalence between the dynamics on E under $\mathbb{P}^\eta$ and the dynamics under of $\tilde{\mathbb{P}}^*$ after making the transformation $Y = X/(\mathbf{1}_d'X)$ where $\mathbf{1}_d$ is the vector of all 1's in $\mathbb{R}^d$ and noting that $Y_d = 1 - \mathbf{1}_{d-1}'X$. Under $\tilde{\mathbb{P}}$, $X = \exp(aW)$. Thus, under $\mathbb{P}^\eta$,

$$\log \eta(Y_t) = \hat{\beta}(*)'aW_t - \log\left(\mathbf{1}_d' e^{aW_t}\right),$$

where

$$\hat{\beta}(*)_i = \hat{\beta}_i \quad 1 \leq i \leq d-1, \qquad \hat{\beta}(*)_d = 1 - \sum_{j=1}^{d-1}\hat{\beta}_j.$$

Thus, it follows that $\lim_{t\uparrow\infty} \frac{1}{t}\log\eta(Y_t) = 0 \quad \mathbb{P}^\eta$ a.s and hence $\mathbb{P}^\eta \in \Pi^*_{a.s}$. The same argument as in Example 6.6 yields the optimality of η, λ and $\mathbb{P}^\eta$.

An interesting numerical example. Using the same notation as in Examples 6.6 and 6.7, consider for $d = 3$ the matrix A and associated vectors $\hat{B}$, $\hat{\mathcal{B}}$ given by

$$A = \begin{pmatrix} 5/3 & 3 & 0 \\ 3 & 7 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{B} = \begin{pmatrix} -7/4 \\ 5/4 \\ 1/2 \end{pmatrix}, \qquad \hat{\mathcal{B}} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

The eigenvalues of A are 1 and $13/3\left(1 \pm \sqrt{145/169}\right)$ and hence A is positive definite. The η^* from (6.1) and (6.2) respectively are

$$\eta^*(x,y,x) = \sqrt[4]{\frac{y^5 z^2}{x^7}}, \qquad \text{for } (x,y,z) \in (0,\infty)^3,$$
$$\eta^*(x,y) = \frac{y(1-x-y)}{x}, \qquad \text{for } x > 0,\ y > 0,\ x+y < 1.$$

Therefore, η^* goes to ∞ along the boundary of E in each case, even when the region is bounded.

7. Proofs of the One-Dimensional Results of Section 5

7.1. **Some helpful lemmata.** The proofs of Proposition 5.2 and Proposition 5.3 rely upon the following three auxiliary results.

Lemma 7.1. *Let Assumption 5.1 hold. Let $\eta \in C^2(\alpha,\beta)$ be strictly positive and concave, and set*

$$\delta(\eta) = \inf_{x\in(\alpha,\beta)} \frac{-c(x)\ddot{\eta}(x)}{2\eta(x)}. \tag{7.1}$$

Then,

(1) *If* (5.4) *holds and* $\inf_{x\in(\alpha,\beta)} \eta(x) > 0$ *then* $\delta(\eta) = 0$.
(2) *If* (5.6) *holds, then* $\delta(\eta) = 0$, *even if* $\lim_{x\downarrow\alpha} \eta(x) = 0 = \lim_{x\uparrow\beta} \eta(x)$.

Proof. It suffices to treat the case near α as the proof near β is the same. Let $x_0 \in (\alpha,\beta)$. Let η be any positive, strictly concave function function on (α,β) normalized so that $\eta(x_0) = 1$. Note that this will not change the value of $\delta(\eta)$ from (7.1). Using integration by parts, for $\alpha < x < x_0$

$$\eta(x) = 1 - (x_0 - x)\dot{\eta}(x_0) - \int_x^{x_0} (y-x)(-\ddot{\eta}(y))dy,$$

and hence

$$\int_\alpha^{x_0} \mathbb{I}_{\{y\geq x\}}(y-x)(-\ddot{\eta}(y))dy \leq 1 + (\beta-\alpha)\left|\dot{\eta}(x_0)\right|.$$

Fatou's lemma and the concavity of η yield

$$\int_\alpha^{x_0} (y-\alpha)(-\ddot{\eta}(y))dy \leq 1 + (\beta-\alpha)\left|\dot{\eta}(x_0)\right| \tag{7.2}$$

First, assume for some $\epsilon > 0$ and $x_1 \in (\alpha, x_0)$ that $\eta(x) \geq \varepsilon$ on on (α, x_1). If (5.4) holds then by taking ϵ small enough, it also holds that on (α, x_1) that $\epsilon(\alpha - x)^2 \geq c(x)$. Thus, if $\delta(\eta) > 0$,

$$\begin{aligned}\int_\alpha^{x_0} (x-\alpha)(-\ddot{\eta}(x))dx &\geq 2\delta(\eta)\int_\alpha^{x_0} \frac{(x-\alpha)\eta(x)}{c(x)}dx \\ &= 2\delta(\eta)\int_\alpha^{x_1} \frac{(\alpha-x)^2\eta(x)}{c(x)(\alpha-x)}dx \\ &\geq 2\delta(\eta)\int_\alpha^{x_1} \frac{1}{\alpha-x}dx \;=\; \infty.\end{aligned}$$

However, this violates (7.2). Thus, $\delta(\eta) = 0$. Now, assume that $\lim_{x\downarrow\alpha}\eta(x) = 0 = \lim_{x\uparrow\beta}\eta(x)$ but also that (5.6) holds near α for c. In this case, the concavity of η yields

$$\eta(x) = \eta\left(\frac{x-\alpha}{x_0-\alpha}x_0 + \frac{x_0-x}{x_0-\alpha}\alpha\right) \geq \frac{x-\alpha}{x_0-\alpha}.$$

Thus, if $\delta(\eta) > 0$, (5.6) gives

$$\begin{aligned}\int_\alpha^{x_0} (x-\alpha)(-\ddot{\eta}(x))dx &\geq 2\delta(\eta)\int_\alpha^{x_0} \frac{(x-\alpha)\eta(x)}{c(x)}dx \\ &\geq \frac{2\delta(\eta)}{x_0-\alpha}\int_\alpha^{x_1} \frac{(x-\alpha)^2}{c(x)}dx \;=\; \infty,\end{aligned}$$

which again contradicts (7.2). Thus, $\delta(\eta) = 0$. □

Lemma 7.2. *Let Assumption 5.1 hold. If $\eta \in C^2(\alpha, \beta)$ is strictly positive, concave and such that $\lim_{x\downarrow\alpha}\eta(x) = 0 = \lim_{x\uparrow\beta}\eta(x)$, then*

$$\liminf_{x\downarrow\alpha} \frac{-(x-\alpha)^2\ddot{\eta}(x)}{\eta(x)} \leq 1 \quad \textit{and} \quad \liminf_{x\uparrow\beta} \frac{-(\beta-x)^2\ddot{\eta}(x)}{\eta(x)} \leq 1.$$

Proof. It suffices to treat the case near α, the conclusion near β follows by the same reasoning. Suppose there exists an $N > 1$ and $x_0 \in (\alpha, \beta)$ such that for $x \in (\alpha, x_0)$

$$\frac{-(x-\alpha)^2\ddot{\eta}(x)}{\eta(x)} \geq N. \tag{7.3}$$

Since

$$-\frac{\ddot{\eta}(x)}{\eta(x)} = -\left(\frac{\dot{\eta}(x)}{\eta(x)}\right)^{\cdot} - \frac{\dot{\eta}(x)^2}{\eta(x)^2} \leq -\left(\frac{\dot{\eta}(x)}{\eta(x)}\right)^{\cdot},$$

(7.3) implies for each $x \in (\alpha, x_0)$,

$$\frac{\dot{\eta}(x)}{\eta(x)} - \frac{\dot{\eta}(x_0)}{\eta(x_0)} = \int_x^{x_0} -\left(\frac{\dot{\eta}(y)}{\eta(y)}\right)^{\cdot} dy \geq \int_x^{x_0} \frac{N}{(y-\alpha)^2}dy = \frac{N}{x-\alpha} - \frac{N}{x_0-\alpha}.$$

Integrating again this means for each $x \in (\alpha, x_0)$

$$\int_x^{x_0}\left(\frac{\dot{\eta}(y)}{\eta(y)} - \frac{\dot{\eta}(x_0)}{\eta(x_0)}\right)dy \geq \int_x^{x_0}\left(\frac{N}{y-\alpha} - \frac{N}{x_0-\alpha}\right)dy,$$

or that

$$\log\left(\frac{\eta(x_0)}{\eta(x)}\right) \geq N\log\left(\frac{x_0-\alpha}{x-\alpha}\right) + \frac{\dot{\eta}(x_0)}{\eta(x_0)}(x_0-x) - N\frac{x_0-x}{x_0-\alpha}.$$

Multiplying the inequality by -1, exponentiating, multiplying by $\eta(x_0)$ and dividing by $(y-\alpha)^N$ yields

$$\frac{\eta(x)}{(x-\alpha)^N} \leq \eta(x_0)(x_0-\alpha)^{-N} \exp\left(N\left(\frac{x_0-x}{x_0-\alpha}\right) - \frac{\dot{\eta}(x_0)}{\eta(x_0)}(x_0-x)\right). \tag{7.4}$$

Since η is concave and $\lim_{x\downarrow\alpha}\eta(x)=0$, it follows by defining $\eta(\alpha)=0$ that η is still concave and

$$\eta(x) = \eta\left(\frac{x-\alpha}{x_0-\alpha}x_0 + \frac{x_0-x}{x_0-\alpha}\alpha\right) \geq (x-\alpha)\frac{\eta(x_0)}{x_0-\alpha}.$$

In conjunction with (7.4), this yields

$$\frac{1}{(x-\alpha)^{N-1}} \leq (x_0-\alpha)^{-(N-1)} \exp\left(N\left(\frac{x_0-x}{x_0-\alpha}\right) - \frac{\dot{\eta}(x_0)}{\eta(x_0)}(x_0-x)\right).$$

Since $N>1$, as $x\downarrow\alpha$ the left hand side of the above equation goes to ∞ while the right hand side remains finite, yielding a contradiction. Thus, for each $N>1$ and $x_0\in(\alpha,\beta)$, there exists $\hat{x}\in(\alpha,x_0)$ such that $-(\hat{x}-\alpha)^2\ddot{\eta}(\hat{x})/\eta(\hat{x}) < N$. Therefore, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n\downarrow\alpha$ such that

$$\sup_{n\in\mathbb{N}} \frac{-(x_n-\alpha)^2\ddot{\eta}(x_n)}{\eta(x_n)} \leq N,$$

which yields the desired result upon sending $N\downarrow 1$. $\square$

Lemma 7.3. *Let Assumption 5.1 hold. Let $\lambda>0$ and $\eta\in H_\lambda$ be such that*

$$\lim_{x\downarrow\alpha}\eta(x) = 0 = \lim_{x\uparrow\beta}\eta(x) \tag{7.5}$$

and

$$\int_\alpha^\beta \frac{\eta^2(x)}{c(x)}dx < \infty. \tag{7.6}$$

Then, $\lambda^*=\lambda$ and $\eta^*=\eta$. The coordinate process X under $(\mathbb{P}^*_x)_{x\in(\alpha,\beta)}$ is positive recurrent. Furthermore, $\mathbb{P}^*\in\Pi^*_{a.s.}$.

Proof. If X is recurrent under $(\mathbb{P}^*_x)_{x\in E}$ then from Proposition 1.7, $\lambda^*=\lambda$ and $\eta^*=\eta$. Furthermore, by (7.6) positive recurrence will follow with the invariant measure $\widetilde{\eta}$ that has density proportional to η^2/c with respect to Lebesgue measure, appropriately normalized so $\widetilde{\eta}$ is a probability measure.

To check recurrence it will be shown that (5.1) holds near α, the proof near β is the same. Note that, since $\eta\in H_\lambda$ and (7.5) holds, there exists a unique $x_0\in(\alpha,\beta)$ such that $\dot{\eta}(x_0)=0$. For $\alpha<x<x_0$,

$$\int_x^{x_0}\frac{2\lambda\eta(y)^2}{c(y)}dy = -\int_x^{x_0}\eta(y)\ddot{\eta}(y)dy = \eta(x)\dot{\eta}(x) + \int_x^{x_0}\dot{\eta}(y)^2dy.$$

Thus, as $x\downarrow\alpha$ since η is positive and concave it must hold that $\eta(x)\dot{\eta}(x)>0$ and hence by (7.6) it follows that $\int_\alpha^{x_0}\dot{\eta}(y)^2dy<\infty$. Therefore, by the concavity of η and (7.5),

$$0 \leq \liminf_{x\downarrow\alpha}\eta(x)\dot{\eta}(x) \leq \lim_{x\downarrow\alpha}\int_0^x \dot{\eta}(y)^2dy = 0. \tag{7.7}$$

This implies that for any $\varepsilon > 0$ there is x_ε near α such that for $x \in (\alpha, x_\varepsilon)$, $\eta^2(x) \leq 2\varepsilon(x-\alpha)$, or that

$$\int_\alpha^{x_\varepsilon} \frac{1}{\eta(y)^2} dy \geq \frac{1}{2\varepsilon} \int_\alpha^{x_\varepsilon} \frac{1}{y-\alpha} dy = \infty,$$

and recurrence follows. It remains to prove that $\mathbb{P}^* \in \Pi^*_{a.s.}$. To this end, it follows from equations (3.3) and (3.4) in the proof of Proposition 3.3 that $\mathbb{P}^* \in \Pi^*_{a.s.}$ if

$$\liminf_{t\uparrow\infty} \frac{1}{t} \int_0^t \left(\frac{1}{2} c(X_s) \left(\frac{\dot\eta(X_s)}{\eta(X_s)} \right)^2 - \lambda \right) ds \geq 0 \quad \mathbb{P}^*\text{-a.s.}$$

By the ergodic theorem [21, Theorem 4.9.5] and the monotone convergence theorem it follows that

$$\liminf_{t\uparrow\infty} \frac{1}{t} \int_0^t \left(\frac{1}{2} c(X_s) \left(\frac{\dot\eta(X_s)}{\eta(X_s)} \right)^2 - \lambda \right) ds \geq \int_\alpha^\beta \left(\frac{1}{2} c(y) \left(\frac{\dot\eta(y)}{\eta(y)} \right)^2 - \lambda \right) \frac{\eta(y)^2}{c(y)} dy, \ \mathbb{P}^*\text{-a.s.}$$

Continuing, $\eta \in H_\lambda$ implies

$$\int_\alpha^\beta \left(\frac{1}{2} c(y) \left(\frac{\dot\eta(y)}{\eta(y)} \right)^2 - \lambda \right) \frac{\eta(y)^2}{c(y)} dy = \lim_{x\downarrow\alpha} \eta(x)\dot\eta(x) - \lim_{x\uparrow\beta} \eta(x)\dot\eta(x) = 0,$$

where the last equality follows from (7.7) since the same equality holds near β. Thus, $\mathbb{P}^* \in \Pi^*_{a.s.}$. □

7.2. **Proof of Proposition 5.2.** By [21, Theorem 3.4.5] (note that λ_c from [21, Theorem 3.4.5] is equal to $-\lambda^*$ here), λ^* admits the following variational representation:

$$\lambda^* = \sup_{\substack{\eta \in C^2(\alpha,\beta) \\ \eta > 0}} \inf_{x\in(\alpha,\beta)} \frac{-c(x)\ddot\eta(x)}{2\eta(x)} = \sup_{\substack{\eta \in C^2(\alpha,\beta) \\ \eta > 0}} \delta(\eta) \tag{7.8}$$

for $\delta(\eta)$ as in (7.1). Let $\eta(x) = \sqrt{(x-\alpha)(\beta-x)}$ for $x \in (\alpha,\beta)$. Then,

$$\delta(\eta) = \inf_{x\in(\alpha,\beta)} \frac{(\beta-\alpha)^2 c(x)}{8(x-\alpha)^2(\beta-x)^2}.$$

Thus, if (5.3) holds then $\delta(\eta) > 0$ and hence $\lambda^* > 0$.

Now, assume (5.4) holds for $x \downarrow \alpha$. The proof for $x \uparrow \beta$ is the same. Clearly $\lambda^* \geq 0$. To check if $\lambda^* > 0$, the positivity of η and c implies it suffices to consider functions η which are strictly concave on (α,β). Since (5.4) holds for $x \downarrow \alpha$, by Lemma 7.1 it suffices to consider functions η which also go to 0 at α, β. For such functions, Lemma 7.2 implies there exists a sequence $x_n \downarrow \alpha$ so that

$$\lim_{n\uparrow\infty} \frac{-(x_n-\alpha)^2\ddot\eta(x_n)}{\eta(x_n)} \equiv K \leq 1.$$

Since (5.4) holds, for any $\varepsilon > 0$ there is some N large enough so that $n \geq N$ implies both

$$\frac{-(x_n-\alpha)^2\ddot\eta(x_n)}{\eta(x_n)} \leq K + \varepsilon \quad \text{and} \quad \frac{(x_n-\alpha)^2}{c(x_n)} \geq \frac{1}{\varepsilon}.$$

It then follows that

$$\frac{-c(x_n)\ddot\eta(x_n)}{2\eta(x_n)} = \frac{-(x_n-\alpha)^2\ddot\eta(x_n)}{\eta(x_n)} \frac{c(x_n)}{2(x_n-\alpha)^2} \leq \frac{\varepsilon}{2}(K+\varepsilon),$$

so that $\delta(\eta) \leq (\varepsilon/2)(K+\varepsilon)$. Taking $\varepsilon \downarrow 0$ proves $\lambda^* = 0$. $\square$

7.3. **Proof of Proposition 5.3.** The proof of how (5.6) implies $\lambda^* = 0$ is handled first. By (7.8), it suffices to consider strictly concave functions η. However, since (5.6) holds, Lemma 7.1 applies and hence $\delta(\eta) = 0$ for all such η. Thus $\lambda^* = 0$.

Regarding the assertions when (5.5) holds, in light of Lemma 7.3 it suffices to show that (5.5) yields the existence of a $\lambda > 0$, $\eta \in H_\lambda$ such that conditions (7.5) and (7.6) are satisfied. To this end, define the σ-finite measure m via $m(dx) = c(x)^{-1}dx$. Note that condition (7.6) now reads $\eta \in L^2((\alpha,\beta),m)$. The desired pair (λ,η) are the principle eigenvalue and eigenfunction for the operator $(L,\mathcal{D}(L))$ where $(L\eta)(x) = -(1/2)c(x)\ddot{\eta}(x)$ for $x \in (\alpha,\beta)$ and the domain $\mathcal{D}(L)$ consists of functions which vanish at α,β and is constructed so that $(L,\mathcal{D}(L))$ is self adjoint in $L^2((\alpha,\beta),m)$. $\mathcal{D}(L)$ is highly dependent upon the behavior of m near α and β. The study of the spectral properties of such operators falls under the name *Sturm-Liouville theory.* For a detailed exposition on the topics covered/results given below, see [18] and [24].

The case when $m\left((\alpha,\beta)\right) < \infty$ is called the *regular case*. Here $\mathcal{D}(L)$ is given by

$$\mathcal{D}(L) = \left\{\eta \in L^2((\alpha,\beta),m) \mid \dot{\eta} \in AC(\alpha,\beta), \eta(\alpha) = \eta(\beta) = 0, c\ddot{\eta} \in L^2((\alpha,\beta),m)\right\}. \tag{7.9}$$

and the existence of a $\lambda > 0$, $\eta \in H_\lambda \cap \mathcal{D}(L)$ is given by [18, Theorem 2.7.4] and [24, Theorem 10.12.1].

Now, suppose that (5.5) holds, but for some $a \in (\alpha,\beta)$ either $m\left((\alpha,a)\right) = \infty$ or $m\left((a,\beta)\right) = \infty$, or both. These cases are called the *singular cases*. In each of these three cases there exists a domain $\mathcal{D}(L) \subset L^2((\alpha,\beta),m)$, similar to that in (7.9), such that $(L,\mathcal{D}(L))$ is self adjoint. For explicit formulas for the domains, see [24, Chapters 7 and 10].

According to [24, Theorem 10.12.1 (8)], if the spectrum of $(L,\mathcal{D}(L))$ is discrete and bounded from below then in fact there exists a $\lambda > 0$ and $\eta \in H_\lambda \cap \mathcal{D}(L)$ such that (7.5) holds. (this last fact follows by construction of $\mathcal{D}(L)$ but also because otherwise $\eta \notin L^2((\alpha,\beta),m)$).

To prove the spectrum is discrete and bounded from below, it suffices to treat the case of one regular and one singular endpoint. This follows using the *spectral decomposition method* on which a detailed description may be found in [10]. Without loss of generality, consider the case when α is regular and β is singular. Under the transformation $z = f(x) = \int_\alpha^x (1/c(y))\,dy$, (α,β) is taken to be $(0,\infty)$. Set $\varphi(z) = \eta(x)$ and $g(z) = f^{-1}(z)$. Note that $\eta \in L^2((\alpha,\beta),m)$ is equivalent to $\varphi \in L^2((0,\infty),\mathrm{Leb}) \equiv L^2(0,\infty)$. Furthermore, the operator $(M,\mathcal{D}(M))$ defined by

$$(M\varphi)(z) = -\frac{1}{2}\left(\frac{1}{\dot{g}(z)}\dot{\varphi}(z)\right)^{\cdot}, \qquad \mathcal{D}(M) = \{\varphi \mid \varphi(z) = \eta(x), \eta \in \mathcal{D}(L)\}$$

is self-adjoint in $L^2(0,\infty)$. Let $N > 0$ and

$$Q_N = \left\{v \in C_0\left((N,\infty),\mathbb{C}\right) \mid v \in AC_{loc}(0,\infty), \dot{v} \in L^2(0,\infty)\right\},$$

where C_0 means that v is continuous and compactly supported in (N, ∞). For $v \in Q_N$, set

$$I(v, N) = \frac{1}{2} \int_N^\infty \frac{|\dot{v}(z)|^2}{\dot{g}(z)} dz.$$

According to [17, Lemma 4.2], $(M, \mathcal{D}(M))$ has a discrete spectrum bounded from below if and only if for each $\theta > 0$ there exists an $N > 0$ such that

$$I(v, N) \geq \theta \int_N^\infty v(z)^2 dz$$

for each real valued $v \in Q_N$. To show this, fix $\theta > 0$. For any $N > 0$ and $v \in Q_N$,

$$v(z) = -\int_z^\infty \dot{v}(\tau) d\tau.$$

Since $\tau = f(g(\tau))$, it follows that $\dot{g}(\tau) = c(g(\tau)) > 0$. By Hölder's inequality, for real valued $v \in Q_N$,

$$v(z)^2 \leq \left(\int_z^\infty \frac{\dot{v}(\tau)^2}{\dot{g}(\tau)} d\tau \right) \left(\int_z^\infty \dot{g}(\tau) d\tau \right) \leq 2I(v, N) \left(\beta - g(z) \right).$$

Therefore,

$$\begin{aligned} \theta \int_N^\infty v(z)^2 dz &\leq 2\theta I(v, N) \int_N^\infty (\beta - g(z)) dz \\ &= 2\theta I(v, N) \int_{g(N)}^\beta \frac{\beta - x}{c(x)} dx, \end{aligned}$$

where the last equality follows from the substitution $x = g(z)$ or $z = f(x)$. Since $\lim_{z \uparrow \infty} g(x) = \beta$, by (5.5)

$$2\theta \int_{g(N)}^\beta \frac{\beta - x}{c(x)} dx \leq 1,$$

for N large enough, yielding the desired result. $\square$

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