OUTPERFORMING THE MARKET PORTFOLIO WITH A GIVEN PROBABILITY

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Abstract. Our goal is to resolve a problem proposed by Karatzas and Fernholz (2008): Characterizing the minimum amount of initial capital that would guarantee the investor to beat the market portfolio with a certain probability as a function of the market configuration and time to maturity. We show that this value function is the largest subsolution of a non-linear PDE. As in Karatzas and Fernholz (2008), we do not assume the existence of an equivalent local martingale measure but merely the existence of a local martingale deflator.

1. Introduction

In this paper we consider the quantile hedging problem when the underlying market does not have an equivalent martingale measure. Instead, we assume that there exists a *local martingale deflator* (a strict local martingale which when multiplied with the asset prices yields a positive local martingale). We characterize the value function as the largest non-negative subsolution of a fully non-linear partial differential equation. This resolves the open problem proposed in the final section of [9]; also see page 37 of [31].

Our framework falls under the umbrella of the stochastic portfolio theory of Fernholz and Karatzas, see e.g. [12], [14], [13]; and the benchmark approach of Platen [29]. In this framework, the linear partial differential equation that the superhedging price satisfies does not have a unique solution; see e.g. [9], [13], [10], and [30]. Similar phenomenon occurs when the asset prices have *bubbles*: an equivalent local martingale measure exists, but the asset prices under this measure are strict local martingales; see e.g. [6], [18], [22], [21], [7], and [4]. In a related series of papers [1], [32], [27], [19], [26], [8], and [3] addressed the issue of bubbles in the context of stochastic volatility models. In particular, [3] gave necessary and sufficient conditions for linear partial differential equations appearing in the context of stochastic volatility models to have a unique solution.

Key words and phrases. Strict local martingale deflators, optimal arbitrage, quantile hedging, nonuniqueness of solutions of non-linear PDEs.

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In contrast, we show that the quantile hedging problem, which is equivalent to an optimal control problem, solves a fully non-linear PDE. As in the linear case, these PDEs may not have a unique solution, and, therefore, an alternative characterization for the value function needs to be provided. Recently, [24], [2], and [11] also considered stochastic control problems in this framework. The first reference solves the classical utility maximization problem, the second one solves the optimal stopping problem, whereas the third one determines the optimal arbitrage under model uncertainty, which is equivalent to solving a zero-sum stochastic game.

The structure of the paper is simple: In Section 2, we formulate the problem. In this section we also discuss the implications of assuming the existence of a local martingale deflator. In Section 3, we generalize the results of [15] on quantile hedging, in particular the Neyman-Pearson Lemma. We also prove other properties of the value function such as convexity. Section 4 is where we give the PDE characterization of the value function.

2. The model

We consider a financial market with a bond $B(\cdot) = 1$ and d stocks $X = (X_1, \cdots, X_d)$ which satisfy

$$dX_i(t) \quad = X_i(t) \left(b_i(X(t))dt + \textstyle\sum_{k=1}^{d} s_{ik}(X(t))dW_k(t) \right), i = 1; \cdots d, \quad X(0) = x = (x_1, \cdots, x_d). \tag{2.1}$$

Following the set up in [9, Section 8], we make the following assumption.

Assumption 2.1. *Let $b_i : (0,\infty)^d \to \mathbb{R}$ and $s_{ik} : (0,\infty)^d \to \mathbb{R}$ be continuous functions and $b(\cdot) = (b_1(\cdot), \cdots, b_d(\cdot))'$ and $s(\cdot) = (s_{ij}(\cdot))_{1\le i,j\le d}$, which we assume to be invertible for all $x \in (0,\infty)^d$. We also assume that* (2.1) *has a weak solution that is unique in distribution for every initial value. Another assumption we will impose is that*

$$\sum_{i=1}^{d} \int_0^T \left(|b_i(X(t))| + a_{ii}(X(t)) + \theta_i^2(X(t)) \right) < \infty, \tag{2.2}$$

where $\theta(\cdot) := s^{-1}(\cdot)b(\cdot)$, $a_{ij}(\cdot) := \sum_{k=1}^{d} s_{ik}(\cdot)s_{jk}(\cdot)$.

We will denote by $\mathbb{F}$ the augmentation of the natural filtration of $X(\cdot)$. Thanks to Assumption 2.1, every local martingale of $\mathbb{F}$ has the martingale representation property with respect to $W(\cdot)$ (which is adapted with respect to $\mathbb{F}$), the solution of (2.1) takes values in the positive orthant, and the exponential local martingale

$$Z(t) := \exp\left\{ -\int_0^t \theta(X(s))' dW(s) - \frac{1}{2}\int_0^t |\theta(X(s))|^2 ds \right\}, \quad 0 \le t < \infty, \tag{2.3}$$

the so-called *deflator* is well defined. We do not exclude the possibility that $Z(\cdot)$ is a strict local martingale.

Let $\mathcal{H}$ be the set of $\mathbb{F}$-progressively measurable processes $\pi : [0,T) \times \Omega \to \mathbb{R}^d$, which satisfies

$$\int_0^T \left(|\pi'(t)\mu(X(t))| + \pi'(t)\alpha(X(t))\pi(t)\right) dt < \infty, \quad \text{a.s.},$$

in which $\mu = (\mu_1, \cdots, \mu_d)$ and $\sigma = (\sigma_{ij})_{1 \le i,j \le d}$ with $\mu_i(x) = b_i(x)x_i$, $\sigma_{ik}(x) = s_{ik}(x)x_i$, and $\alpha(x) = \sigma(x)\sigma'(x)$.

At time t, investor invests $\pi_i(t)$ proportion of his wealth in the i^{th} stock. The proportion $1 - \sum_{i=1}^d \pi_i(t)$ gets invested in the bond. For each $\pi \in \mathcal{H}$ and initial wealth $y \ge 0$ the associated wealth process will be denoted by $Y^{y,\pi}(\cdot)$. This process solves

$$dY^{y,\pi}(t) = Y^{y,\pi}(t) \sum_{i=1}^d \pi_i(t) \frac{dX_i(t)}{X_i(t)}, \quad Y^{y,\pi}(0) = y.$$

It can be easily seen that $Z(\cdot)Y^{y,\pi}(\cdot)$ is a positive local martingale for any $\pi \in \mathcal{H}$. Let $g : (0,\infty)^d \to \mathbb{R}_+$ be a measurable function satisfying

$$\mathbb{E}[Z(T)g(X(T))] < \infty, \tag{2.4}$$

and define

$$V(T,x,1) := \inf\{y > 0 : \exists\, \pi(\cdot) \in \mathcal{H} \text{ s.t. } Y^{y,\pi}(T) \ge g(X(T))\}.$$

Thanks to Assumption 2.1, we have that $V(t,x,1) = \mathbb{E}[Z(T)g(X(T))]$. Note that if g has linear growth, then (2.4) is satisfied since the process ZX is a positive supermartingale.

2.1. **A Digression: What does the existence of a local martingale deflator entail?** Although, we do not assume the existence of equivalent local martingale measures, we assume the existence of a local martingale deflator. This is equivalent to the *No-Unbounded-Profit-with-Bounded-Risk* (NUPBR) condition; see [24, Theorem 4.12]. NUPBR is defined as follows: A sequence (π^n) of admissible portfolios is said to generate a UPBR if $\lim_{m\to\infty} \sup_n \mathbb{P}[Y_T^{1,\pi^n} > m] > 0$. If no such sequence exists, then we say that NUPBR holds; see [24, Proposition 4.2]. In fact, the the so-called *No-Free-Lunch-with-Vanishing-Risk* is equivalent to NUPBR plus the classical *no-arbitrage* assumption. So, in our setting (since we assumed the existence of local martingale deflators), although arbitrages exist they remain on the level of "cheap thrills", which was coined by [28]. (Note that the results of Karatzas and Kardaras also imply that one does not need NFLVR for the portfolio optimization problem of an individual to be well-defined. One merely needs the NUPBR condition to hold.) The failure of no-arbitrage means that the money market is not an optimal investment and is dominated by other investments. So a short position in the money market and long position in the dominating assets leads one to arbitrage. However, one can not scale the arbitrage and make an arbitrary profit because of the admissibility constraint, which requires the wealth to be positive. This is what is contained in NUPBR, which holds in our setting. Also, see [25], where these issues are further discussed.

3. On quantile hedging

In this section, we will try to determine

$$V(T,x,p) = \inf\{y>0 \,|\, \exists\, \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{y,\pi}(T) \geq g(X(T))\} \geq p\}, \tag{3.1}$$

for $p \in [0,1]$. Observe that

$$\widetilde{V}(T,x,p) = \frac{V(T,x,p)}{g(x)} = \inf\{r>0 \,|\, \exists\, \pi \in \mathcal{H} \text{ s.t. } \mathbb{P}\{Y^{rg(x),\pi}(T) \geq g(X(T))\} \geq p\}.$$

When $g(x) = \sum_{i=1}^d x_i$, observe that $\widetilde{V}(T,x,1)$ is equal to equation (6.1) of [9], the smallest relative amount to beat the market capitalization $\sum_{i=1}^d X_i(T)$.

Remark 3.1. *Clearly,*

$$0 = V(T,x,0) \leq V(T,x,p) \nearrow V(T,x,1) \leq g(x), \qquad \text{as } p \to 1. \tag{3.2}$$

Analogous to [15], we will present a probabilistic characterization of $V(T,x,p)$. First, we will generalize the Neyman-Pearson lemma (see e.g. [16, Theorem A.28]) in the next result.

Lemma 3.1. *Suppose that Assumption* 2.1 *holds and g satisfies* (2.4). *Let $A \in \mathcal{F}_T$ satisfy*

$$\mathbb{P}(A) \geq p. \tag{3.3}$$

Then

$$V(T,x,p) \leq \mathbb{E}[Z(T)g(X(T))1_A] \tag{3.4}$$

Furthermore, if $A \in \mathcal{F}_T$ satisfies (3.3) *with equality and*

$$\operatorname{ess\,sup}_A\{Z(T)g(X(T))\} \leq \operatorname{ess\,inf}_{A^c}\{Z(T)g(X(T))\}, \tag{3.5}$$

then A satisfies (3.4) *with equality.*

Proof. Assumption 2.1 implies that the market is complete. As a result, $g(X(T))1_A \in \mathcal{F}_T$ is replicable with initial capital $\mathbb{E}[Z(T)g(X(T))1_A]$; see e.g. Section 10.1 of [13]. If $\mathbb{P}(A) \geq p$, it follows from (3.1) that $V(T,x,p) \leq \mathbb{E}[Z(T)g(X(T))1_A]$.

Now, take an arbitrary pair (y_0, π_0) of initial capital and admissible portfolio that replicates $g(X(T))$ with probability greater than or equal to p, i.e.

$$\mathbb{P}\{B\} \geq p, \text{ where } B \triangleq \{Y^{y_0,\pi_0}(T) \geq g(X(T))\}.$$

Let $A \in \mathcal{F}_T$ satisfy (3.3) with equality and (3.5). To prove equality in (3.4), it's enough to show that

$$y_0 \geq \mathbb{E}[Z(T)g(X(T))1_A]$$

Observing that $\mathbb{P}(A^c \cap B) = \mathbb{P}(A \cup B) - \mathbb{P}(A) \geq \mathbb{P}(A \cup B) - \mathbb{P}(B) = \mathbb{P}(B^c \cap A)$ and using (3.5), we obtain that

$$\begin{aligned} y_0 \; &\geq \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)] = \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_B] + \mathbb{E}[Z(T)Y^{y_0,\pi_0}(T)1_{B^c}] \\ &\geq \mathbb{E}[Z(T)g(X(T))1_B] = \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A^c\cap B}] \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{P}(A^c \cap B)\operatorname{ess\,inf}_{A^c\cap B}\{Z(T)g(X(T))\} \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{P}(A \cap B^c)\operatorname{ess\,sup}_{A\cap B^c}\{Z(T)g(X(T))\} \\ &\geq \mathbb{E}[Z(T)g(X(T))1_{A\cap B}] + \mathbb{E}[Z(T)g(X(T))1_{A\cap B^c}] \\ &= \mathbb{E}[Z(T)g(X(T))1_A]. \end{aligned}$$

□

Let $F(\cdot)$ be the cumulative distribution function of $Z(T)g(X(T))$ and for any $a \in \mathbb{R}_+$ define

$$A_a := \{\omega : Z(T)g(X(T)) < a\}, \;\partial A_a := \{\omega : Z(T)g(X(T)) = a\},$$

and let $\bar{A}_a$ denote $A_a \cup \partial A_a$. Taking $A = \bar{A}_a$ in Lemma 3.1, it follows that

$$V(T, x, F(a)) = \mathbb{E}[Z(T)g(X(T))1_{\bar{A}_a}]. \tag{3.6}$$

On the other hand, taking $A = A_a$, we obtain that

$$V(T, x, F(a-)) = \mathbb{E}[Z(T)g(X(T))1_{A_a}]. \tag{3.7}$$

The last two equalities imply the following relationship

$$\begin{aligned} V(T, x, F(a)) \; &= V(t, x, F(a-)) + a\mathbb{P}\{\partial A_a\} \\ &= V(t, x, F(a-)) + a(F(a) - F(a-)). \end{aligned} \tag{3.8}$$

Next, we will determine $V(T, x, p)$ for $p \in (F(a-), F(a))$ when $F(a-) < F(a)$.

Proposition 3.1. *Suppose Assumption* 2.1 *holds. Fix an* $(x, p) \in (0, \infty)^d \times [0, 1]$.

(1) *There exists* $A \in \mathcal{F}_T$ *satisfying* (3.3) *with equality and* (3.5). *As a result,* (3.4) *holds with equality.*
(2) *If* $F^{-1}(p) := \{s \in \mathbb{R}_+ : F(s) = p\} = \emptyset$, *then letting* $a := \inf\{s \in \mathbb{R}_+ : F(s) > p\}$ *we have*

$$\begin{aligned} V(T, x, p) \; &= V(T, x, F(a-)) + a(p - F(a-)). \\ &= V(T, x, F(a)) - a(F(a) - p) \end{aligned} \tag{3.9}$$

Proof. **(1)** If there exists an a such that either $F(a) = p$ or $F(a-) = p$, $A = A_a$ or $A = \bar{A}_a$, thanks to (3.6) and (3.7). In the rest of the proof we will assume that $F^{-1}(p) = \emptyset$.

Let $\widetilde{W}$ be a Brownian motion with respect to $\mathbb{F}$ and define $B_b = \{\omega : \frac{\widetilde{W}(T)}{\sqrt{T}} < b\}$. Let us define $f(\cdot)$ by $f(b) = \mathbb{P}\{\partial A_a \cap B_b\}$. The function f satisfies $\lim_{b\to-\infty} f(b) = 0$ and $\lim_{b\to\infty} f(b) = \mathbb{P}(\partial A_a)$. Moreover, the function $f(\cdot)$ is continuous and nondecreasing. Right continuity can be shown as follows: For $\varepsilon > 0$

$$0 \le f(b+\varepsilon) - f(b) = \mathbb{P}(\partial A_a \cap B_{b+\varepsilon}) - \mathbb{P}(\partial A_a \cap B_b) \le \mathbb{P}(B_{b+\varepsilon} \cap B_b^c).$$

The right continuity follows from observing that the last expression goes to zero as $\varepsilon \to 0$. One can show left continuity of $f(\cdot)$ in a similar fashion.

Since $0 < p - \mathbb{P}(A_a) < \mathbb{P}(\partial A_a)$, thanks to the above properties of f there exists a $b^* \in \mathbb{R}$ satisfying $f(b^*) = p - \mathbb{P}(A_a)$.

Define $A := A_a \cup (\partial A_a \cap B_{b^*})$. Observe that $\mathbb{P}(A) = \mathbb{P}(A_a) + \mathbb{P}(\partial A_a \cap B_{b^*}) = p$. A also satisfies (3.5).

(2) This follows immediately from (1):

$$\begin{aligned} V(T,x,p) &= \mathbb{E}[Z(T)g(X(T))1_A] \\ &= \mathbb{E}[Z(T)g(X(T))1_{A_a}] + \mathbb{E}[Z(T)g(X(T))1_{\partial A_a \cap B_{b^*}}] \\ &= V(T,x,F(a-)) + a\mathbb{P}(\partial A_a \cap B_{b^*}) \\ &= V(t,x,F(a-)) + a(p - F(a-)). \end{aligned}$$

□

Remark 3.2. *Note that when Z is a martingale, using Neyman-Pearson Lemma,* [15] *showed that*

$$V(T,x,p) = \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi] = \mathbb{E}[Z(T)g(X(T))\varphi^*], \tag{3.10}$$

where

$$\mathcal{M} = \left\{ \varphi : \Omega \to [0,1] \middle| \mathcal{F}_T \text{ measurable}, \mathbb{E}[\varphi] \ge p \right\}. \tag{3.11}$$

The randomized test function φ^* is not necessarily an indicator function. Using Lemma 3.1 and the fine structure of the filtration $\mathcal{F}_T$, in Proposition 3.1, we provide another optimizer of (3.10) *that is an indicator function.*

Proposition 3.2. *Suppose Assumption* 2.1 *holds. Then, $V(T,x,\cdot)$ is convex, and thus continuously increasing from $V(T,x,0) = 0$ to $V(T,x,1)$. Hence, $V(T,x,p) \le pV(T,x,1) \le pg(x)$ for all $p \in (0,1)$.*

Proof. It is enough to show,

$$\frac{V(T,x,p_1) + V(T,x,p_2)}{2} \ge V\left(T, x, \frac{p_1+p_2}{2}\right), \quad \text{for all } 0 \le p_1 < p_2 \le 1. \tag{3.12}$$

Denote $\tilde{p} \triangleq \frac{p_1+p_2}{2}$. It follows from Proposition 3.1 that there exist $A_1 \subset \tilde{A} \subset A_2$ with $\mathbb{P}(A_1) = p_1 < \mathbb{P}(\tilde{A}) = \tilde{p} < \mathbb{P}(A_2) = p_2$ satisfying (3.5),

$$V(T, x, p_i) = \mathbb{E}[Z(T)g(X(T))1_{A_i}],\ i = 1, 2,$$

and

$$V(T, x, \tilde{p}) = \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}].$$

By (3.5),

$$\begin{aligned}\operatorname{ess\,inf}\{Z(T)g(X(T))1_{A_2\cap\tilde{A}^c}\} &\geq \operatorname{ess\,sup}\{Z(T)g(X(T))1_{\tilde{A}}\} \\ &\geq \operatorname{ess\,sup}\{Z(T)g(X(T))1_{\tilde{A}\cap A_1^c}\},\end{aligned}$$

which implies that

$$\mathbb{E}[Z(T)g(X(T))1_{A_2\cap\tilde{A}^c}] \geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}\cap A_1^c}].$$

As a result,

$$\mathbb{E}[Z(T)g(X(T))1_{A_2}] - \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}] \geq \mathbb{E}[Z(T)g(X(T))1_{\tilde{A}}] - \mathbb{E}[Z(T)g(X(T))1_{A_1}],$$

which is equivalent to (3.12). □

Example 3.1. *Consider a market with a single stock, whose dynamics follow a three-dimensional Bessel process, i.e.*

$$dX(t) = \frac{1}{X(t)}dt + dW(t) \quad X_0 = x > 0,$$

and let $g(x) = x$*. In this case*

$$Z(t) = \frac{x}{X(t)},$$

which is the classical example for a strict local martingale; see [23]*. On the other hand,* $Z(t)X(t) = x$ *is a martingale. Thanks to Proposition 3.1 there exits a set* $A \in \mathcal{F}_T$ *with* $\mathbb{P}(A) = p$ *such that*

$$V(T, x, p) = \mathbb{E}[Z(T)X(T)1_A] = px.$$

In [15], the following result was proved when Z is a martingale. Here, we generalize this result to the case where Z is only a local martingale.

Proposition 3.3. *Under Assumption 2.1*

$$V(T, x, p) = \inf_{\varphi\in\mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi]. \tag{3.13}$$

Proof. Thanks to Proposition 3.1 there exists a set $A \in \mathcal{F}_T$ such that $V(T,x,p) = \mathbb{E}[Z(T)g(X(T))1_A]$. Since $1_A \in \mathcal{M}$, clearly

$$V(T,x,p) \geq \inf_{\varphi \in \mathcal{M}} \mathbb{E}[Z(T)g(X(T))\varphi].$$

For the other direction, we will show that for any $\varphi \in \mathcal{M}$ and a given set $A \in \mathcal{F}_T$ with $\mathbb{P}(A) = p$ satisfying (3.5)

$$\mathbb{E}[Z(T)g(X(T))1_A] \leq \mathbb{E}[Z(T)g(X(T))\varphi].$$

Letting $M = \operatorname{ess\,sup}_A\{Z(T)g(X(T))\}$, we can write

$$\begin{aligned}
&\mathbb{E}[Z(T)g(X(T))\varphi] - \mathbb{E}[Z(T)g(X(T))1_A] \\
&\quad = \mathbb{E}[Z(T)g(X(T))\varphi 1_A] + \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A] \\
&\quad = \mathbb{E}[Z(T)g(X(T))\varphi 1_{A^c}] - \mathbb{E}[Z(T)g(X(T))1_A(1-\varphi)] \\
&\quad \geq M\mathbb{E}[\varphi 1_{A^c}] - M\mathbb{E}[1_A(1-\varphi)] \quad \text{(by (3.5))} \\
&\quad \geq 0.
\end{aligned}$$

□

3.1. **A Digression: Representation of V as a Stochastic Control Problem.** Let us denote by $P^p_\alpha(\cdot)$ the solution of

$$dP(t) = P(t)(1-P(t))\alpha'(t)dW(t), \;\; P(0) = p \in [0,1], \tag{3.14}$$

where $\alpha(\cdot)$ is an $\mathbb{F}-$progressively measurable $\mathbb{R}^d$-valued process such that $\int_0^T |\alpha(s)|^2 ds < \infty$ $\mathbb{P}$-a.s. We will denote the class of such processes by $\mathcal{A}$. The next result obtains an alternative representation for V in terms of P.

Proposition 3.4. *Under Assumption* 2.1*,*

$$V(T,x,p) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}[Z(T)g(X(T))P^p_\alpha(T)] < \infty. \tag{3.15}$$

Proof. The finiteness follows from (2.4). It can be shown using Proposition 3.3 that

$$V(T,x,p) = \inf_{\varphi \in \widetilde{\mathcal{M}}} \mathbb{E}[Z(T)g(X(T))\varphi],$$

where

$$\widetilde{\mathcal{M}} = \left\{ \varphi : \Omega \to [0,1] \middle| \mathcal{F}_T \text{ measurable}, \mathbb{E}[\varphi] = p \right\}.$$

Therefore, it's enough to show that $\widetilde{\mathcal{M}}$ satisfies,

$$\widetilde{\mathcal{M}} = \left\{ P^p_\alpha(T) \middle| \alpha \in \mathcal{A} \right\}.$$

The inclusion

$$\widetilde{\mathcal{M}} \supset \left\{ P^p_\alpha(T) \middle| \alpha \in \mathcal{A} \right\},$$

is clear. To show the other inclusion we will use the Martingale representation theorem: For any $\varphi \in \mathcal{F}_T$ there exists an $\mathbb{F}-$progressively measurable $\mathbb{R}^d$-valued process $\psi(\cdot)$ satisfying

$$\varphi = p + \int_0^T \psi'(t)dW(t).$$

Then we see that $\mathbb{E}[\varphi|\mathcal{F}_t]$ solves (3.14) with $\alpha(\cdot)$

$$\alpha(t) = 1_{\{\mathbb{E}[\varphi|\mathcal{F}_t]\in(0,1)\}} \cdot \frac{\psi(t)}{\mathbb{E}[\varphi|\mathcal{F}_t](1-\mathbb{E}[\varphi|\mathcal{F}_t])}.$$

□

4. The PDE characterization

4.1. **Notation.** We denote by $X^{t,x}(\cdot)$ the solution of (2.1) starting from x at time t and by $Z^{t,x,z}(\cdot)$ the solution of

$$dZ(s) = -Z(s)\theta(X^{t,x}(s))'dW(s),\ Z(t) = z. \tag{4.1}$$

We then introduce the value function

$$U(t,x,p) := \inf_{\varphi\in\mathcal{M}} \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))\varphi], \tag{4.2}$$

where $\mathcal{M}$ is defined in (3.11). Note that the original value function V can be written in terms of U as

$$V(T,x,p) = U(0,x,p).$$

We also consider the Legendre-Fenchel dual of U with respect to the p variable

$$w(t,x,q) := \sup_{p\in[0,1]} \{pq - U(t,x,p)\}, \tag{4.3}$$

and the function

$$\tilde{w}(t,x,q) := \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] = \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))^+], \tag{4.4}$$

where the process $Q(\cdot)$ satisfies the dynamics

$$\frac{dQ(s)}{Q(s)} = |\theta(X^{t,x}(s))|^2 ds + \theta(X^{t,x}(s))'dW(s),\ Q_{t,x,q}(t) = q.$$

Finally, for any $(t,x) \in [0,T] \times (0,\infty)^d$, we denote by $F(\cdot)$ the cumulative distribution function of $Z^{t,x,1}(T)g(X^{t,x}(T))$.

4.2. **Associated partial differential equation.** We will first present some properties of $\tilde{w}$, and then relate w to $\tilde{w}$ to obtain a partial differential equation the value function U satisfies.

Assumption 4.1. *We assume that θ_i and σ_{ij} are, for all $i, j \in \{1, \cdots, d\}$, locally Lipschitz.*

Assumption 4.2. *We also make the following ellipticity assumption: For every compact subset $K \subset (0, \infty)^d$, there exists a positive constant C_K such that*

$$\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq C_K \|\xi\|^2,$$

for all $\xi \in \mathbb{R}^d$ and $x \in K$.

We have the following result from [30, Theorem 2], which generalizes the results of [7] to several dimensions. A similar result when a is assumed to have linear growth appears in [20]. However, as in [30], we are interested in relaxing the linear growth assumption so that we can consider cases in which multiple solutions to (4.5) may exist.

Lemma 4.1. *Under Assumptions 2.1, 4.1, and 4.2, we have that $\widetilde{w}$ is a classical solution to*

$$-v_t - \frac{1}{2}\mathit{Trace}(\sigma\sigma' D_x^2 v) - \frac{1}{2}|\theta|^2 q^2 D_q^2 v - q\,\mathit{Trace}(\sigma\theta D_{xq} v) = 0, \tag{4.5}$$

$(t, x, q) \in [0, T) \times (0, \infty)^d \times (0, \infty)$, with the boundary condition

$$v(T, x, q) = (q - g(x))^+. \tag{4.6}$$

¿From the smoothness of $\tilde{w}$, we can deduce the differentiability of F as follows.

Proposition 4.1. *Assume that Assumptions 2.1, 4.1 and 4.2 hold. Then the cumulative distribution function F is differentiable.*

Proof. By Lemma 4.1, $\tilde{w}$ is twice differentiable in q. Thus, to prove that F is differentiable, it's enough to show that

$$D_q\tilde{w}(t, x, q) = \mathbb{P}[Z^{t,x,1}(T)g(X^{t,x}(T)) \leq q] = F(q). \tag{4.7}$$

For any $\varepsilon > 0$ and $q \in (0,\infty)$, let $E := \{\omega : q < Z^{t,x,1}(T)g(X^{t,x}(T)) \le q+\varepsilon\}$. Then $\bar{A}_q$ and E are disjoint, and $\bar{A}_{q+\varepsilon} = \bar{A}_q \cup E$. It follows that

$$\begin{aligned}
& \frac{1}{\varepsilon}[\tilde{w}(t,x,q+\varepsilon) - \tilde{w}(t,x,q)] \\
= \; & \frac{1}{\varepsilon}\left\{\mathbb{E}[\big(q+\varepsilon - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_{\bar{A}_{q+\varepsilon}}] - \mathbb{E}[\big(q - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_{\bar{A}_q}]\right\} \\
= \; & \frac{1}{\varepsilon}\left\{\mathbb{E}[\big(q+\varepsilon - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_{\bar{A}_q}] + \mathbb{E}[\big(q+\varepsilon - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_E]\right\} \\
- \; & \frac{1}{\varepsilon}\mathbb{E}[\big(q - Z^{t,x,1}(T)g(X^{t,x}(T))1_{\bar{A}_q}\big)] \\
= \; & \mathbb{P}[\bar{A}_q] + \frac{1}{\varepsilon}\mathbb{E}[\big(q+\varepsilon - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_E].
\end{aligned}$$

By the definition of E,

$$\begin{aligned}
0 \;\le\; & \frac{1}{\varepsilon}\mathbb{E}[\big(q+\varepsilon - Z^{t,x,1}(T)g(X^{t,x}(T))\big)\, 1_E] \le \frac{1}{\varepsilon}\mathbb{E}[\varepsilon 1_E] \\
=\; & \mathbb{P}[q < Z^{t,x,1}(T)g(X^{t,x}(T)) \le q+\varepsilon)] \to 0, \text{ as } \varepsilon \downarrow 0.
\end{aligned}$$

Thus, we get the right derivative of $\tilde{w}$ with respect to q as

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}[\tilde{w}(t,x,q+\varepsilon) - \tilde{w}(t,x,q)] = \mathbb{P}[Z^{t,x,1}(T)g(X^{t,x}(T)) \le q]. \tag{4.8}$$

We can derive the left derivative of $\tilde{w}$ with respect to q in the same manner and thus have (4.7) proved. □

¿From the definition of $\tilde{w}$, it's obvious that $\tilde{w}$ is convex in the q variable. To further show its strict convexity, we make the following assumption.

Assumption 4.3. *For any $(t,x) \in [0,T]\times(0,\infty)^d$, $F(\cdot)$ is strictly increasing.*

Proposition 4.2. *Under Assumptions 2.1, 4.1, 4.2 and 4.3, $\tilde{w}$ is strictly convex in the q variable.*

Proof. By Lemma 4.1, $\tilde{w}$ is smooth. Then by direct computation

$$D_q\tilde{w}(t,x,q) = \mathbb{P}[Z^{t,x,1}(T)g(X^{t,x}(T)) \le q] = F(q).$$

Since $F(\cdot)$ is assumed to be strictly increasing, we see that $\tilde{w}$ is strictly convex in q. □

Now we want to relate w to $\tilde{w}$. We first present another representation for $\tilde{w}$ in the following result.

Lemma 4.2. *For any $(t,x,q) \in [0,T]\times(0,\infty)^d\times(0,\infty)$, we have*

$$\max_{a\ge 0}\mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \tilde{w}(t,x,q).$$

Proof. First consider the case when $a < q$. On the set $\{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$, we have

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

On the other hand, on $\{\omega : a < Z^{t,x,1}(T)g(X^{t,x}(T)) \leq q\}$, we have

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = 0 \leq (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

Finally, on $\{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) > q\}$, it holds that

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = 0 = (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

Thus, we conclude that when $a < q$

$$\mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] \leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] = \tilde{w}(t,x,q). \tag{4.9}$$

Next, consider the case when $a > q$. On the set $\{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) \leq q\}$, we have

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

On the other hand, on $\{\omega : q < Z^{t,x,1}(T)g(X^{t,x}(T)) \leq a\}$, it holds that

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = q - Z^{t,x,1}(T)g(X^{t,x}(T)) < 0 = (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

Finally, on $\{\omega : Z^{t,x,1}(T)g(X^{t,x}(T)) > a\}$,

$$(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a} = 0 = (q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}.$$

Thus, we conclude that as $a > q$

$$\mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] \leq \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_q}] = \tilde{w}(t,x,q). \tag{4.10}$$

Now, the claim follows combining (4.9) and (4.10). $\square$

Next, we will argue that w and $\tilde{w}$ are equal.

Proposition 4.3. *Under Assumptions 2.1, 4.1, 4.2,* $w(t,x,q) = \tilde{w}(t,x,q)$.

Proof. The continuity of F implies

$$w(t,x,q) = \sup_{p \in [0,1]} \{pq - U(t,x,p)\} = \sup_{a \geq 0} \{F(a)q - U(t,x,F(a))\}. \tag{4.11}$$

It follows from (3.6) that

$$\begin{aligned} F(a)q - U(t,x,F(a)) &= F(a)q - \mathbb{E}[Z^{t,x,1}(T)g(X^{t,x}(T))1_{\bar{A}_a}] \\ &= \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}]. \end{aligned} \tag{4.12}$$

Now, using (4.11), (4.12), and Lemma 4.2, we obtain

$$w(t,x,q) = \max_{a \geq 0} \mathbb{E}[(q - Z^{t,x,1}(T)g(X^{t,x}(T)))1_{\bar{A}_a}] = \tilde{w}(t,x,q).$$

□

Thanks to Proposition 4.3, we can conclude from Lemma 4.1 and Proposition 4.2 that w is a classical solution to the PDE (4.5) with the boundary condition (4.6), and that it is strictly convex in q.

Now we are ready to present one of the main results in this section.

Proposition 4.4. *Under Assumptions 2.1, 4.1, 4.2, and 4.3, the function U satisfies*

$$0 = -\partial_t U - \frac{1}{2} Trace[\sigma\sigma' D_x^2 U] - \inf_{a\in\mathbb{R}^d}\left((D_{xp}U)'\sigma a + \frac{1}{2}|a|^2 D_p^2 U - \theta' a D_p U\right) \tag{4.13}$$

in classical sense, with the boundary condition

$$U(T,x,p) = pg(x). \tag{4.14}$$

Moreover, U is strictly convex in p.

Proof. Since w is smooth and strictly convex in q, its Legendre-Fenchel dual U is also smooth and strictly convex in p. We can therefore express U as

$$U(t,x,p) = \sup_{q\in(0,\infty)}\{pq - w(t,x,q)\} = pH(t,x,p) - w(t,x,H(t,x,p)),$$

where $p \mapsto H(\cdot,p)$ is the inverse function of $q \mapsto D_q w(\cdot,q)$. Then by direct calculations,

$$\begin{aligned}
D_p U(t,x,p) &= H(t,x,p),\\
D_p^2 U(t,x,p) &= D_p H(t,x,p) = \frac{1}{D_q^2 w(t,x,H(t,x,p))},\\
D_x U(t,x,p) &= -D_x w(t,x,H(t,x,p)),\\
D_x^2 U(t,x,p) &= -D_x^2 w(t,x,H(t,x,p)) + \frac{1}{D_p^2 U(t,x,p)}(D_{px}U)(D_{px}U)',\\
D_{px} U(t,x,p) &= -D_{px} w(t,x,H(t,x,p)) D_p U(t,x,p),\\
\partial_t U(t,x,p) &= -\partial_t w(t,x,H(t,x,p)).
\end{aligned}$$

It follows that for $(t,x,p) \in [0,T) \times (0,\infty)^d \times [0,1]$, by setting $q := H(t,x,p)$, we have

$$\begin{aligned}
0 &= -\partial_t w - \frac{1}{2}\mathrm{Trace}[\sigma\sigma' D_x^2 w] - \frac{1}{2}|\theta|^2 q^2 D_q^2 w - q\mathrm{Trace}[\sigma\theta D_{xq} w]\\
&= \partial_t U + \frac{1}{2}\mathrm{Trace}[\sigma\sigma' D_x^2 U] - \frac{1}{2}\mathrm{Trace}[\sigma\sigma'(D_{px}U)(D_{px}U)'] - \frac{1}{2}|\theta|^2\frac{(D_p U)^2}{D_p^2 U} + \frac{D_p U}{D_p^2 U}\mathrm{Trace}[\sigma\theta D_{px} U]\\
&= \partial_t U + \frac{1}{2}\mathrm{Trace}[\sigma\sigma' D_x^2 U] + \left((D_{xp}U)'\sigma a^* + \frac{1}{2}|a^*|^2 D_p^2 U - \theta' a^* D_p U\right)\\
&= \partial_t U + \frac{1}{2}\mathrm{Trace}[\sigma\sigma' D_x^2 U] + \inf_{a\in\mathbb{R}^d}\left((D_{xp}U)'\sigma a + \frac{1}{2}|a|^2 D_p^2 U - \theta' a D_p U\right),
\end{aligned}$$

where

$$a^*(t,x,p) := \frac{D_pU(t,x,p)}{D_p^2U(t,x,p)}\theta(x) - \frac{1}{D_p^2U(t,x,p)}\sigma'(x)D_{px}U(t,x,p).$$

Finally, observe that for any $p \in [0,1]$, the maximum of $pq - (q-g(x))^+$ is attained at $q = g(x)$. Therefore, by (4.6)

$$U(T,x,p) = \sup_{q\in\mathbb{R}_+}\{pq - w(t,x,p)\} = \sup_{q\in\mathbb{R}_+}\{pq - \tilde{w}(t,x,p)\} = \sup_{q\in\mathbb{R}_+}\{pq - (q-g(x))^+\} = pg(x).$$

□

A few remarks are in order:

Remark 4.1. *Results similar to Proposition 4.4, but in viscosity sense, were proved by* [5], *with stronger assumptions (such as the existence of an equivalent martingale measure and the existence of a unique strong solution to* (2.1)*), using the stochastic target formulation. Here, we use the observation that the dual function of U is equal to $\tilde{w}$ and that $\tilde{w}$ is a classical solution to a linear PDE that is strictly convex in q, to obtain a nonlinear PDE that U satisfies in classical sense. Under our assumptions this solution may not be unique as pointed out in the next remark.*

Remark 4.2. (i) *Let us consider the PDE satisfied by the superhedging price $U(t,x,1)$:*

$$0 = v_t + \frac{1}{2}\mathit{Trace}(\sigma\sigma' D_x^2 v), \quad \text{on } (0,T)\times(0,\infty)^d, \tag{4.15}$$

$$v(T-,x) = g(x), \quad \text{on } (0,\infty)^d. \tag{4.16}$$

Unless additional boundary conditions are specified, this PDE may have multiple solutions, see e.g. the volatility stabilized model of [9]. *Even when additional boundary conditions are specified, the growth of σ might lead to the loss of uniqueness. In the one-dimensional case one can determine an explicit condition which is sufficient and necessary for uniqueness (non-uniqueness); see* [4].

(ii) *Let $\Delta U(t,x,1)$ be the difference of two solutions of* (4.15)-(4.16). *Then both $U(t,x,p)$ and $U(t,x,p)+\Delta U(t,x,1)$ are solutions of* (4.13) *(along with its boundary conditions). As a result when* (4.15) *and* (4.16) *has multiple solutions so does the PDE for the function U.*

Remark 4.3. *Instead of relying on the Legendre-Fenchel duality we could directly apply the dynamic programming principle of* [17] *for weak solutions to the formulation in Section 3.1. The problem with this approach is the growth assumptions on the coefficients of* (2.1), *which would force us to rule out the cases that are of interest to us.*

4.3. **Characterizing the value function.**

Proposition 4.5. *Assume that Assumptions 2.1, 4.1, 4.2 hold. Suppose u is a nonnegative classical subsolution to the PDE* (4.13) *which satisfies* (4.14)*, and is strictly convex in p. Then $u \leq U$.*

Proof. Define the Legendre-Fenchel dual of u with respect to p

$$w^u(t,x,q) := \sup_{p\in[0,1]} \{pq - u(t,x,p)\}.$$

By the definition of w^u and (4.14), it's easy to see that w^u satisfies (4.6). Also, since u is strictly convex in p, we can redo the calculation in the proof of Proposition 4.4 (but going backward) and obtain that w^u is a classical supersolution to (4.5). Now define the process $N(\cdot)$ by

$$N(s) = Z^{t,x,1}(s)w^u(s, X^{t,x}(s), Q^{t,x,q}(s)). \tag{4.17}$$

Using Itô's rule and the supersolution property of w^u, it can be seen that $N(\cdot)$ is a supermartingale. Then by the supermartingale property and (4.6), we have

$$\begin{aligned} w^u(t,x,q) &= N(t) \geq \mathbb{E}[N(T)] = \mathbb{E}[Z^{t,x,1}(T)w^u(T, X^{t,x}(T), Q^{t,x,q}(T))] \\ &= \mathbb{E}[Z^{t,x,1}(T)(Q^{t,x,q}(T) - g(X^{t,x}(T)))^+] = \tilde{w}(t,x,q) \end{aligned}$$

Now by Proposition 4.3, we conclude that $w^u(t,x,p) \geq w(t,x,p)$ and thus $u(t,x,q) \leq U(t,x,q)$. ☐

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