Entanglement of Indistinguishable Particles

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We present a general criterion for entanglement of N indistinguishable particles decomposed into arbitrary s subsystems based on the unambiguous measurability of correlation. Our argument provides a unified viewpoint on the entanglement of indistinguishable particles, which is still unsettled despite various proposals made so far primarily for the $s = 2$ case. Even though entanglement is defined only with reference to the measurement setup, we find that the so-called i.i.d. states form a special class of bosonic states which are universally separable.

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Since its first recognition in the seminal EPR and Schrödinger's papers [1, 2], quantum entanglement has long been seen as the most distinctive trait of quantum theory. Notably, it underlies nonlocal correlation in composite physical systems, invoking various conceptual questions on the foundation of physics and, at the same time, offers a key resource for quantum information sciences. In view of this, we find it rather puzzling that the very notion of entanglement still eludes a formal, let alone intuitive, understanding, especially when the system admits no apparent decomposition into subsystems. This occurs typically in systems of indistinguishable particles (*i.e.*, fermions or bosons) with which actual realizations of entanglement – via photons, electrons or composite particles such as hydrogen atoms – have been implemented mostly today.

To see the nontrivial nature of entanglement, take, for example, the familiar $N = 2$ particle Bell states,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_2 \pm |1\rangle_1 |0\rangle_2), \tag{1}$$

with $|0\rangle_k$ and $|1\rangle_k$ being the orthonormal qubit states of particle $k = 1, 2$. These are prototypical entangled states for distinguishable particles, but if the particles are indistinguishable, the labels k are no longer usable for classifying the measurement outcomes to define the correlation. Consideration of remotely separated particles by introducing the spatial degrees of freedoms directly for Eq.(1) does not yield any nontrivial correlation – a property known as the cluster separability [3]. Clearly, a physically motivated and mathematically solid definition of entanglement is needed for general composite systems including those of indistinguishable particles.

Recently, Ghirardi *et al.* [4, 5] gave a possible definition of separability (non-entanglement) for N particle systems based on the criterion that, if one can deduce a complete set of physical properties (CSP) pertaining to a subsystem, then the state is separable with respect to the subsystem and the rest. This criterion derives from the demand that, in a separable state, all physical quantities in the subsystem have elements of reality in the EPR sense [1]. Independently, Zanardi *et al.* [6] presented a criterion for uncovering a tensor product structure (TPS) in the Hilbert space upon which entanglement can be defined. The criterion demands the existence of subalgebras representing the observables of the subsystems which are measurable, independent and complete to form the entire set of observables in the system (see also [7]). Yet another criterion has been proposed by Schliemann *et al.* [8] and others [9] particularly for indistinguishable particles using the (Schmidt or Slater) rank of state decomposition, which is related to the standard measures of entanglement such as the von Neumann entropy.

These criteria for (non)entanglement are rather different from each other and, not surprisingly, do not completely agree on deciding which states are separable, with an example being the $N = 2$ bosonic 'independently and identically distributed' (i.i.d.) state $|\phi\rangle_1 |\phi\rangle_2$ (for an attempt of reconciliation, see [10]). More recently, the present authors furnished a criterion for the decomposition of an N fermionic system into s arbitrary subsystems [11], where we find that the orthogonal structure introduced to distinguish the subsystems in [4] corresponds precisely to the choice of observables with which correlation is defined. In other words, entanglement can be defined only *relative* to the measurement setup and it is highly non-unique [6]. Under these circumstances, one is naturally led to ask if there is any coherent picture of entanglement prevalent among these criteria.

The purpose of this Letter is to provide a positive answer to this. Namely, we show that all these criteria can be put into a larger perspective consisting of two descriptions of the system, one for the measurement outcomes and the other for the provisional states of the system. The gap between the two descriptions, which lies at the root of the apparent disagreement, can be filled by an isomorphism between the two, providing a unified viewpoint of entanglement for indistinguishable particles. Unlike the previous analyses, entanglement can be treated equally for the fermionic and bosonic cases here. We also find that the i.i.d. states for general N form a special class of bosonic states which are universally separable irrespective of the choice of measurement setup.

To define entanglement as an attribute to generate nontrivial correlation among subsystems, we first need an appropriate set of physical observables associated with the subsystems for which the correlation in their measurement outcomes can be evaluated unambiguously. To discuss the situation explicitly, we consider the case where the total system breaks into s subsystems $\Gamma_1, \ldots, \Gamma_s$ and assume that to each Γ_i we have a complete set of commuting observables $\mathcal{C}_i$ which are all implementable in the measurement to determine the state of the subsystem. Let $\mathcal{L}_i$ be the set of observables (self-adjoint operators) containing the set $\mathcal{C}_i$. The collection of states of the subsystem Γ_i describing the measurement outcomes form a Hilbert space $\mathcal{H}^{\rm mes}(\Gamma_i)$ in which $\mathcal{L}_i$ is represented irreducibly. Assuming further that the measurements of the observables $\mathcal{L}_i$ can be performed independently for all $i = 1, \ldots, s$, we find that the set $\mathcal{L}$ of observables in the total system is given by $\mathcal{L} = \otimes_i \mathcal{L}_i$. Accordingly, the state space of the system describing the measurement outcomes is given by the tensor product,

$$\mathcal{H}^{\rm mes} = \bigotimes_{i=1}^{s} \mathcal{H}^{\rm mes}(\Gamma_i). \tag{2}$$

The TPS of the space $\mathcal{H}^{\rm mes}$ in Eq.(2) allows us to define the entanglement by the conventional way, that is, if the measured state $|\Psi\rangle \in \mathcal{H}^{\rm mes}$ admits the product form

$$|\Psi\rangle = \bigotimes_{i=1}^{s} |\psi_i\rangle_{\Gamma_i}, \quad |\psi_i\rangle_{\Gamma_i} \in \mathcal{H}^{\rm mes}(\Gamma_i), \tag{3}$$

then it is separable; if not, it is entangled. Evidently, since the separable state Eq.(3) yields definite outcomes for the measurement of observables in a properly chosen set $\mathcal{C}_i$ in $|\psi_i\rangle_{\Gamma_i}$ for all i, it possesses a CSP [4].

Meanwhile, in the total space $\mathcal{H}^{\rm mes}$ the observable $O_i \in \mathcal{L}_i$ is expressed by

$$\widehat{O}_i = \bigotimes_{j=1}^{i-1} \mathbb{1}_j \otimes O_i \otimes \bigotimes_{j=i+1}^{s} \mathbb{1}_j \tag{4}$$

where $\mathbb{1}_j$ is the identity operator in $\mathcal{H}^{\rm mes}(\Gamma_j)$. The aforementioned independence is then assured trivially by

$$[\widehat{O}_i, \widehat{O}_j] = 0 \qquad \text{for} \qquad i \neq j. \tag{5}$$

The observable $\widehat{O} \in \mathcal{L}$ corresponding to the simultaneous measurement for the subsystems is then given by

$$\widehat{O} = \prod_{i=1}^{s} \widehat{O}_i = \bigotimes_{i=1}^{s} O_i. \tag{6}$$

Denoting the set of such operators by $\mathcal{T}^{\rm mes} \subset \mathcal{L}$, we see that any $\widehat{O} \in \mathcal{T}^{\rm mes}$ has a factorized expectation value for the separable state $|\Psi\rangle$ in Eq.(3):

$$\langle\Psi|\, \widehat{O}\, |\Psi\rangle = \prod_{i=1}^{s} \langle\Psi|\, \widehat{O}_i\, |\Psi\rangle . \tag{7}$$

The properties Eqs.(5) and (6), together with the implementability assumption, constitute the formal conditions to realize a TPS in [6]. Note that, in our measurement-based description, the TPS appears as a direct consequence of the construction.

The entanglement in the measurement-based description is related with the measurement outcomes directly, but the conventional treatment of indistinguishable particles employ the framework of the provisional Hilbert space of distinguishable particles for the description of states with appropriate restriction required by the statistics of the particles. Here, the description is not directly related to the measurement outcomes, and the restricted space of states does not admit a TPS in any obvious manner. In physical terms, the measurement outcomes of observables, such as spin, cannot be attributed to those of a particular particle due to the indistinguishability, and the formal structure of the state fails to signify the correlation as exemplified by Eq.(1). To fill the gap, we need a prescription to bridge the two descriptions.

For definiteness, let us label the N particles by the integer set $\mathcal{N} = \{1, 2, \cdots, N\}$. Each of the particles is characterized by an n-level state, *i.e.*, the state space of the k-th particle is $\mathcal{H}_k \cong \mathbb{C}^n$ for all k. Let $\{|e_i\rangle\}$ be a complete orthonormal basis in $\mathbb{C}^n$. By the isomorphism among the constituent spaces $\mathcal{H}_k$, any pure state $|\Psi\rangle$ in the provisional space $\mathcal{H} = \bigotimes_{k\in\mathcal{N}} \mathcal{H}_k$ of the total system can be written as

$$|\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} \bigotimes_{k=1}^{N} |e_{i_k}\rangle_k , \tag{8}$$

where $\Psi_{i_1 i_2 \cdots i_N} \in \mathbb{C}$ and $\{|e_{i_k}\rangle_k\}$ is the complete orthonormal basis in $\mathcal{H}_k$ isomorphic to $\{|e_i\rangle\}$.

To incorporate the indistinguishability of the particles, consider an element $\sigma \in \mathfrak{S}_N$ of the symmetric group $\mathfrak{S}_N$ associated with the permutation $k \to \sigma(k)$. In $\mathcal{H}$, the element is represented by a self-adjoint operator π_σ with

$$\pi_\sigma |\Psi\rangle = \sum_{i_1, i_2, \cdots, i_N} \Psi_{i_1 i_2 \cdots i_N} \bigotimes_{k=1}^{N} |e_{i_k}\rangle_{\sigma(k)} . \tag{9}$$

From π_σ, both the symmetrizer and the antisymmetrizer are defined as

$$\mathcal{S} = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma\in\mathfrak{S}_N} \pi_\sigma, \quad \mathcal{A} = \frac{1}{|\mathfrak{S}_N|} \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma)\pi_\sigma, \tag{10}$$

where $|\mathfrak{S}_N| = N!$ is the cardinality of $\mathfrak{S}_N$, and $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ. The Hilbert space of the total system of N bosons (fermions) is the subspace of $\mathcal{H}$ consisting of symmetric (antisymmetric) states. With $\mathcal{X} = \mathcal{S}$ for bosons and $\mathcal{X} = \mathcal{A}$ for fermions, they are

$$\mathcal{H}_\mathcal{X} = [\mathcal{H}]_\mathcal{X} := \left\{ \mathcal{X} |\Psi\rangle \,\middle|\, |\Psi\rangle \in \mathcal{H} \right\}. \tag{11}$$

To introduce the decomposition into subsystems in the total system, we consider a partition Γ of the integer set $\mathcal{N}$ into non-empty and mutually exclusive sets $\Gamma_i \subseteq \mathcal{N}$,

$$\Gamma = \{\Gamma_i\}_{i=1}^{s}, \quad \bigcup_{i=1}^{s} \Gamma_i = \mathcal{N}, \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{for } i \neq j. \tag{12}$$

For specifying the subsystems of indistinguishable particles, only the cardinality $|\Gamma_i|$ of Γ_i matters. Note that there is no apparent TPS in $\mathcal{H}_{\mathcal{X}}$ with respect to Γ, and we need to somehow find an embedding of the measurement-based description in the provision-based description.

This embedding is handled usually by considering the positions of individual particles to gain a fictitious distinction among the particles. For the distinction to be unambiguous, the measurements of the subsystems should be performed remotely from each other, and this amounts to introducing an orthogonal decomposition in the 1-particle Hilbert space (after accommodating the spatial degrees of freedoms). More generally, the embedding requires an orthogonal structure V which is a set of subspaces $V_i \subset \mathbb{C}^n$ mutually orthogonal to each other with respect to the innerproduct of $\mathbb{C}^n$,

$$V = \{V_i\}_{i=1}^{s}, \qquad V_i \perp V_j \quad \text{for } i \neq j. \tag{13}$$

Together with the orthogonal complement,

$$V_0 = \left(V_1 \oplus V_2 \oplus \cdots \oplus V_s\right)^{\perp}, \tag{14}$$

the set V furnishes an orthogonal decomposition of $\mathbb{C}^n$. The physical idea behind this is that these orthogonal spaces correspond to mutually independent measurement of subsystems such that, given a measurement setup, the subsystem Γ_i is susceptible only for the measurement of particles $k \in \Gamma_i$ residing in V_i. If we denote the subspace V_i in $\mathcal{H}_k$ by $V_i(\mathcal{H}_k) \subset \mathcal{H}_k$, then the actual Hilbert space describing the measurement outcomes for Γ_i is given by

$$\mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i) = \left[\bigotimes_{k\in\Gamma_i} V_i(\mathcal{H}_k)\right]_{\mathcal{X}}. \tag{15}$$

Here, as in Eq.(11), the symbol $[*]_{\mathcal{X}}$ implies the subspace of $*$ invariant under $\mathcal{X}$ which is defined now for the symmetry group $\mathfrak{S}_K$ with K being the cardinality of $*$ (we use the same $\mathcal{X}$ by abuse of notation); namely, in Eq.(15) we have $K = |\Gamma_i|$. Clearly, $\mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)$ is the actual space of states determined from the measurement and, therefore, corresponds to $\mathcal{H}^{\rm mes}(\Gamma_i)$ in the measurement-based description Eq.(2) where the state $|\psi_i\rangle_{\Gamma_i} \in \mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)$ is identified with $|\psi_i\rangle_{\Gamma_i} \in \mathcal{H}^{\rm mes}(\Gamma_i)$.

From the description for the subsystems, we obtain the Hilbert space of the total system by

$$\mathcal{H}_{\mathcal{X}}(\Gamma, V) = \left[\bigotimes_{i=1}^{s} \mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)\right]_{\mathcal{X}}. \tag{16}$$

Note that, due to the (anti)symmetrization $\mathcal{X}$, the resultant space $\mathcal{H}_{\mathcal{X}}(\Gamma, V)$ has no TPS with respect to the decomposition Γ and, hence, no obvious correspondence with $\mathcal{H}^{\rm mes}$ in Eq.(2). In spite of this, the two spaces can be made isomorphic based on the identification $\mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i) \cong \mathcal{H}^{\rm mes}(\Gamma_i)$ mentioned above. Indeed, it is attained, with this identification, by the linear map,

$$f_{\mathcal{X}} : \mathcal{H}^{\rm mes} \cong \bigotimes_{i=1}^{s} \mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i) \mapsto \mathcal{H}_{\mathcal{X}}(\Gamma, V), \tag{17}$$

defined by

$$f_{\mathcal{X}}\left(\bigotimes_{i=1}^{s} |\psi_i\rangle_{\Gamma_i}\right) = \sqrt{M}\mathcal{X}\bigotimes_{i=1}^{s} |\psi_i\rangle_{\Gamma_i}, \tag{18}$$

with the normalization factor $M := N!/\prod_{i=1}^{s}|\Gamma_i|!$. Obviously, the map $f_{\mathcal{X}}$ is surjective by construction, and to see the injectivity, we note that, thanks to the orthogonal structure V in Eq.(13), the innerproduct is invariant under the map [12]. It follows that $\| \otimes_i |\psi_i\rangle_{\Gamma_i} \| = \| \sqrt{M}\mathcal{X} \otimes_i |\psi_i\rangle_{\Gamma_i} \|$, which ensures the injectivity of the map and hence the isomorphism (see Fig.1).

The isomorphism Eq.(17) induces the correspondence between the observables such that, if O_i are the observables in $\mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)$ for $i = 1, \ldots, s$, then the observable for their simultaneous measurement in $\mathcal{H}_{\mathcal{X}}(\Gamma, V)$ reads

$$\widetilde{O} = f_{\mathcal{X}}\left(\bigotimes_{i=1}^{s} O_i\right) f_{\mathcal{X}}^{-1} = M\mathcal{X}\left(\bigotimes_{i=1}^{s} O_i\right)\mathcal{X}. \tag{19}$$

The set of all such operators $\widetilde{O}$ defines a subset $\mathcal{T}(\Gamma, V)$ of observables in $\mathcal{H}_{\mathcal{X}}(\Gamma, V)$. With the identification of the observables O_i between $\mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)$ and $\mathcal{H}^{\rm mes}(\Gamma_i)$, the isomorphism Eq.(19) implies the isomorphism between $\mathcal{T}^{\rm mes}$ and $\mathcal{T}(\Gamma, V)$ through $\widehat{O} \leftrightarrow \widetilde{O}$ with $\widehat{O}$ in Eq.(6).

In the provision-based description, the criterion on the entanglement of indistinguishable particles then emerges as follows. Given an arbitrary (normalized) state $|\Psi\rangle \in \mathcal{H}_{\mathcal{X}}$, we first decompose it as

$$|\Psi\rangle = |\Psi(\Gamma, V)\rangle + \left|\Psi(\Gamma, V)^{\perp}\right\rangle, \tag{20}$$

according to the orthogonal decomposition,

$$\mathcal{H}_{\mathcal{X}} = \mathcal{H}_{\mathcal{X}}(\Gamma, V) \oplus \ \mathcal{H}_{\mathcal{X}}(\Gamma, V)^{\perp}. \tag{21}$$

Since the piece $\left|\Psi(\Gamma, V)^{\perp}\right\rangle$ has a vanishing support for the observables in $\mathcal{T}(\Gamma, V)$ and is filtered out by the measurement, the only part significant for correlation is the piece $|\Psi(\Gamma, V)\rangle$. Thus, for describing the measurement outcomes, one may renormalize it as $\| \, |\Psi(\Gamma, V)\rangle \, \| = 1$. It is now evident that, if the piece takes the form,

$$|\Psi(\Gamma, V)\rangle = \sqrt{M}\mathcal{X}\bigotimes_{i=1}^{s} |\psi_i\rangle_{\Gamma_i}, \tag{22}$$

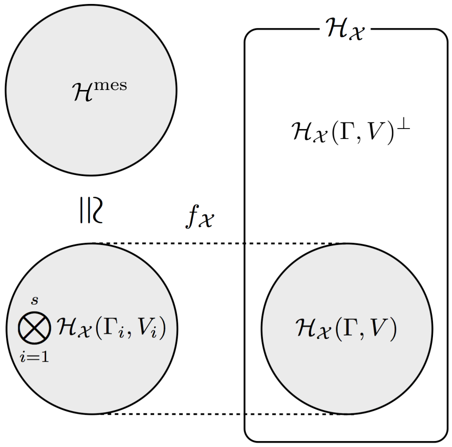

FIG. 1: A diagrammatical representation of the spaces mentioned in Eqs.(17) and (21). In the total space $\mathcal{H}_{\mathcal{X}}$, we find the subspace $\mathcal{H}_{\mathcal{X}}(\Gamma, V)$ isomorphic to $\otimes_i \mathcal{H}_{\mathcal{X}}(\Gamma_i, V_i)$ which has a TPS. The latter is then identified with the space $\mathcal{H}^{\rm mes}$ describing the measurement outcomes.

then for any $\widetilde{O} \in \mathcal{T}(\Gamma, V)$ we have the factorization,

$$\langle\Psi|\, \widetilde{O}\, |\Psi\rangle = \prod_{i=1}^{s} \langle\Psi|\, \widetilde{O}_i\, |\Psi\rangle\,, \tag{23}$$

in analogy with Eq.(7). Since the converse is also true, we learn that the state $|\Psi\rangle$ is separable if and only if the piece $|\Psi(\Gamma, V)\rangle$ in Eq.(20) admits the (anti)symmetrized direct product form Eq.(22); if not, it is entangled. We stress that entanglement is determined only relatively with respect to the measurement setup, as is evident from the explicit dependence on V in $|\Psi(\Gamma, V)\rangle$.

Despite the relative nature of entanglement, there exists a special class of states in the bosonic case $\mathcal{X} = \mathcal{S}$ which are separable under all measurement choices. These are the i.i.d. pure states $|\Psi\rangle \in \mathcal{H}_{\mathcal{S}}$ defined by

$$|\Psi\rangle = \bigotimes_{k=1}^{N} |\phi\rangle_k\,, \qquad |\phi\rangle_k \in \mathcal{H}_k. \tag{24}$$

To see the universal separability of the state, we decompose $|\phi\rangle_k$ according to Eqs.(13) and (14) as

$$|\phi\rangle_k = \sum_{i=0}^{s} |\varphi_i\rangle_k\,, \qquad |\varphi_i\rangle_k \in V_i(\mathcal{H}_k). \tag{25}$$

Plugging this into Eq.(24), we obtain Eq.(20) with

$$|\Psi(\Gamma, V)\rangle = \sqrt{M}\mathcal{S} \bigotimes_{i=1}^{s} |\psi_i\rangle_{\Gamma_i}\,, \quad |\psi_i\rangle_{\Gamma_i} = \bigotimes_{k\in\Gamma_i} |\varphi_i\rangle_k\,. \tag{26}$$

Since the piece $|\Psi(\Gamma, V)\rangle$, if non-vanishing, belongs to the class Eq.(22), the i.i.d. states $|\Psi\rangle$ are separable. Further, since this is true for any choice of (Γ, V), the separability holds irrespective of the measurement setup. Interestingly, for $N = 2$ the converse is also true: states which are universally separable must be the i.i.d. states.

In summary, we have presented a general criterion for entanglement of an indistinguishable N particle system decomposed into arbitrary s subsystems based on the unambiguous measurability of correlation. The point is that, although the Hilbert space $\mathcal{H}_{\mathcal{X}}$ of the system does not admit a TPS, one can find a subspace $\mathcal{H}_{\mathcal{X}}(\Gamma, V) \subset \mathcal{H}_{\mathcal{X}}$ which has a TPS and is directly related to the space $\mathcal{H}^{\rm mes}$ describing the measurement outcomes. Since $\mathcal{H}^{\rm mes}$ has a common structure with the space of distinguishable particles, our approach allows us to treat indistinguishable particles on the equal basis with distinguishable ones. This implies that, under the unambiguous measurability, all the standard measures of entanglement devised so far can be used equally for the indistinguishable case. More importantly, the handling of states without considering the effect of (anti)symmetrization practiced regularly in quantum optics is found to be safe as long as it deals with the space $\mathcal{H}^{\rm mes}$.

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[12] The proof is given in Ref.[11] for the fermionic case, but the proof for the bosonic is analogous.