MOMENT EXPLOSION IN THE LIBOR MARKET MODEL

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Abstract. In the LIBOR market model, forward interest rates are log-normal under their respective forward measures. This note shows that their distributions under the other forward measures of the tenor structure have approximately log-normal tails.

1. Introduction

The LIBOR market model [2] is one of the most popular models for pricing and hedging interest rate derivatives. Its state variables are forward interest rates $F_n(t) := F(t; T_{n-1}, T_n)$, spanning time periods $[T_{n-1}, T_n]$, where

$$0 < T_0 < T_1 < \cdots < T_M$$

is a fixed tenor structure. Under the T_M-forward measure $\mathbb{Q}^M$, which has as numeraire the zero coupon bond maturing at T_M, the dynamics of the forward rates are

$$\begin{aligned} \mathrm{d}F_n(t) &= -\sigma_n(t)F_n(t)\sum_{j=n+1}^{M}\frac{\rho_{nj}\tau_j\sigma_j(t)F_j(t)}{1+\tau_jF_j(t)}\mathrm{d}t + \sigma_n(t)F_n(t)\mathrm{d}W_n(t), \\ &\qquad 1 \le n < M, \\ \mathrm{d}F_M(t) &= \sigma_M(t)F_M(t)\mathrm{d}W_M(t). \end{aligned}$$

Here, σ_n are some positive deterministic volatility functions, and W is a vector of standard Brownian motions with instantaneous correlations $\mathrm{d}W_i(t)\mathrm{d}W_j(t) = \rho_{ij}\mathrm{d}t$. Moreover, $\tau_n = \tau(T_{n-1}, T_n)$ denotes the year fraction between the tenor dates T_{n-1} and T_n.

Note that each rate F_n is a geometric Brownian Motion under its own forward measure, while it has a non-zero drift under the other forward measures. A popular approximation of the above dynamics is obtained by "freezing the drift":

$$\begin{aligned} \mathrm{d}F_n^{\mathrm{fd}}(t) &= -\sigma_n(t)F_n^{\mathrm{fd}}(t)\sum_{j=n+1}^{M}\frac{\rho_{nj}\tau_j\sigma_j(t)F_j(0)}{1+\tau_jF_j(0)}\mathrm{d}t + \sigma_n(t)F_n^{\mathrm{fd}}(t)\mathrm{d}W_n(t), \\ &\qquad 1 \le n < M, \\ \mathrm{d}F_M^{\mathrm{fd}}(t) &= \sigma_M(t)F_M^{\mathrm{fd}}(t)\mathrm{d}W_M(t). \end{aligned}$$

Since the drifts are now deterministic, the new rates F_n^{fd} are just geometric Brownian motions, which allows for explicit pricing formulas for many interest-linked products. As a piece of evidence for the quality of this approximation, we show in the present note that, for fixed $t > 0$, the distribution of $F_n^{\mathrm{fd}}(t)$ has roughly the same tail heaviness as the distribution of $F_n(t)$.

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2. Main Result

If X is any log-normal random variable, so that $\log X \sim \mathcal{N}(\mu, \sigma^2)$ for some real μ and positive σ, then

$$\sup\{v : \mathbf{E}[\mathrm{e}^{v \log^2 X}] < \infty\} = \frac{1}{2\sigma^2}. \tag{1}$$

This follows from

$$\mathbf{E}[\mathrm{e}^{v \log^2 X}] = \frac{1}{\sqrt{1-2\sigma^2 v}} \exp\left(\frac{\mu^2 v}{1-2\sigma^2 v}\right), \qquad v < \frac{1}{2\sigma^2}.$$

Our main result shows that $F_n(t)$ has approximately log-normal tails, in the sense that the left-hand side of (1) is finite and positive if X is replaced by $F_n(t)$. Furthermore, this "critical moment" is the same for $F_n(t)$ and the frozen drift approximation $F_n^{\mathrm{fd}}(t)$.

Theorem 1. *In the log-normal LIBOR market model, we have for all $t > 0$ and all $1 \le n \le M$*

$$\sup\{v : \mathbf{E}^M[\mathrm{e}^{v \log^2(F_n(t))}] < \infty\} = \sup\{v : \mathbf{E}^M[\mathrm{e}^{v \log^2(F_n^{\mathrm{fd}}(t))}] < \infty\}$$
$$= \frac{1}{2\int_0^t \sigma_n(s)^2 \mathrm{d}s}.$$

Proof. Note that the latter equality is obvious from (1), since $F_n^{\mathrm{fd}}(t)$ is log-normal with log-variance parameter $\sigma^2 = \int_0^t \sigma_n(s)^2 \mathrm{d}s$. We now show the first equality. Recall that the measure change from the T_n-forward measure to the T_{n-1}-forward measure is effected by the likelihood process [1]

$$\left.\frac{\mathrm{d}\mathbb{Q}^n}{\mathrm{d}\mathbb{Q}^{n-1}}\right|_{\mathcal{F}_t} = \frac{1+\tau_n F_n(0)}{1+\tau_n F_n(t)}.$$

Therefore, putting $\phi(x) = \exp(\log^2 x)$, we obtain

$$\begin{aligned}
\mathbf{E}^M[\phi(F_n(t))^v] &= \mathbf{E}^{M-1}\left[\phi(F_n(t))^v \times \frac{1+\tau_M F_M(0)}{1+\tau_M F_M(t)}\right] \\
&= \cdots = \\
&= \mathbf{E}^n\left[\phi(F_n(t))^v \prod_{i=n+1}^{M} \frac{1+\tau_i F_i(0)}{1+\tau_i F_i(t)}\right] \\
&\le \mathbf{E}^n[\phi(F_n(t))^v] \prod_{i=n+1}^{M} (1+\tau_i F_i(0)),
\end{aligned}$$

hence

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \le \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}.$$

On the other hand, for $1 < k \le M$ we have

$$\begin{aligned}
\mathbf{E}^{k-1}[\phi(F_n(t))^v] &= \mathbf{E}^k\left[\phi(F_n(t))^v \times \frac{1+\tau_k F_k(t)}{1+\tau_k F_k(0)}\right] \\
&= \frac{1}{1+\tau_k F_k(0)}\left(\mathbf{E}^k[\phi(F_n(t))^v] + \tau_k \mathbf{E}^k[F_k(t)\phi(F_n(t))^v]\right).
\end{aligned}$$

Now let $\varepsilon > 0$ be arbitrary, and define q by $\frac{1}{q} + \frac{1}{1+\varepsilon} = 1$. Then Hölder's inequality yields

$$\mathbf{E}^k[F_k(t)\phi(F_n(t))^v] \le \mathbf{E}^k[F_k(t)^q]^{1/q} \times \mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}]^{1/(1+\varepsilon)}.$$

By the finite moment assumption, we obtain the implication

$$\mathbf{E}^k[\phi(F_n(t))^{v(1+\varepsilon)}] < \infty \quad \Longrightarrow \quad \mathbf{E}^{k-1}[\phi(F_n(t))^v] < \infty, \qquad v \in \mathbb{R}.$$

(Note that the left-hand side implies $\mathbf{E}^k[\phi(F_n(t))^v] < \infty$.)

Inductively, this leads to the implication

$$\mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty \quad \Longrightarrow \quad \mathbf{E}^n[\phi(F_n(t))^v] < \infty, \qquad v \in \mathbb{R}.$$

Therefore, we find

$$\begin{aligned} \sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} &\geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^{v(1+\varepsilon)^{M-n}}] < \infty\} \\ &= \frac{1}{(1+\varepsilon)^{M-n}} \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}. \end{aligned}$$

Since ε was arbitrary,

$$\sup\{v : \mathbf{E}^n[\phi(F_n(t))^v] < \infty\} \geq \sup\{v : \mathbf{E}^M[\phi(F_n(t))^v] < \infty\}$$

follows, which finishes the proof. □

References

[1] T. Björk, *Arbitrage theory in continuous time*, Oxford university press, second ed., 2004.

[2] A. Brace, D. Gatarek, and M. Musiela, *The market model of interest rate dynamics*, Math. Finance, 7 (1997), pp. 127–155.

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