Strategic Insights From Playing the Quantum Tic-Tac-Toe

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Abstract. In this paper, we perform a minimalistic quantization of the classical game of tic-tac-toe, by allowing superpositions of classical moves. In order for the quantum game to reduce properly to the classical game, we require legal quantum moves to be orthogonal to all previous moves. We also admit interference effects, by squaring the sum of amplitudes over all moves by a player to compute his or her occupation level of a given site. A player wins when the sums of occupations along any of the eight straight lines we can draw in the 3×3 grid is greater than three. We play the quantum tic-tac-toe first randomly, and then deterministically, to explore the impact different opening moves, end games, and different combinations of offensive and defensive strategies have on the outcome of the game. In contrast to the classical tic-tac-toe, the deterministic quantum game does not always end in a draw. In contrast also to most classical two-player games of no chance, it is possible for Player 2 to win. More interestingly, we find that Player 1 enjoys an overwhelming quantum advantage when he opens with a quantum move, but loses this advantage when he opens with a classical move. We also find the quantum blocking move, which consists of a weighted superposition of moves that the opponent could use to win the game, to be very effective in denying the opponent his or her victory. We then speculate what implications these results might have on quantum information transfer and portfolio optimization.

PACS numbers: 03.65.-w, 03.67.-a

1. Introduction

Since Bouwmeester *et al.*'s 1997 empirical demonstration of quantum teleportation [1], first proposed theoretically by Bennett *et al.* [2], there has been a surge of interest in quantum information transfer between two parties, Alice and Bob (see for example, [3, 4, 5, 6, 7, 8], and the reviews [9, 10, 11]). At the same time, quantum cryptography research has been focussed on devising ways to prevent a third party, Eve, from intercepting and reading the message transmitted over a quantum channel, or for Alice or Bob to detect any attempt at eavesdropping [12, 13, 14, 15] (see review by Gisin *et al.* [16]). But what if Eve, frustrated at failing in every attempt to decipher Alice's message to Bob, turn her attention to foiling all transmissions? Should this quantum jamming scenario develop, Alice will be forced to explore various strategies to get her message through to Bob, knowing that Eve will attempt to interrupt the transmission, but not knowing beforehand how she plan to do so.

In essence, cutting the measurements Bob has to make out of the picture, the ding-dong decisions made by Alice and Eve have the flavour of a two-player game. Naturally, because information is transferred across quantum channels, this is a quantum game, not a classical game. Adding quantum-mechanical elements to a classical game always lead to surprises. In 1999, Meyer constructed a quantum game of penny flip, and concluded that quantum strategies increase a player's payoff beyond what is possible with classical strategies [17]. Eisert *et al* later analyzed non-zero-sum games and found for the famous Prisoner's Dilemma that the the classical dilemma no longer arise if quantum strategies are allowed [18]. Since these pioneering works, there have been further studies on the exact nature of quantum advantages [19, 20, 21], whether these advantages persist when the games are noisy [22, 23, 24, 25], and how entanglement influences the choice of quantum strategies [26, 27, 28]. These works also spawned a series of in-depth studies into the game-theoretic structure of quantum games [29, 30, 31, 32, 33, 34].

The quantum information transfer scenario described above is an asymmetric two-player quantum game, because the moves available to Alice are not the same as those available to Eve. In the financial arena, portfolio optimization can also be viewed as a symmetric N-player quantum game, in the sense that the same set of moves are available to all N players. Here, stocks are the classical states, and portfolios made up of linear combinations of long and short positions on these stocks are the quantum states. When one fund manager optimizes his portfolio, the optimalities of all other portfolios are affected, forcing the other fund managers to also adjust their portfolios. In this sense, the stock market is a gigantic real-time multiplayer game where a large number of fund managers reacts to price changes induced by other fund managers, making adjustments to keep their portfolios optimal. This is an area where the relatively young field of quantum game theory can potentially make important contributions.

To understand at a deeper level how quantum mechanics influence the choice of strategies for such games, and eventually their outcomes, we analyze the simplest two-player game of tic-tac-toe. In Section 2, we will define the quantum moves and winning

condition that we have adopted, and explain how these are similar to or different from existing quantizations of the game. In Section 3, both players make random moves allowed by our rules, to simulate a benchmark situation where there is total absence of strategy, for comparison against the random classical game. We find that Player 1 wins about 60% of the time in both random games, but Player 2 is at a greater disadvantage in the quantum game. We then study the impacts of different opening moves on the random games, to find classical opening moves being most favourable towards Player 2. We also study end-game situations, where Player 1 is on the verge of winning, i.e. Player 1 will surely win on the next move, if Player 2 forfeits his or her move. Here we find that Player 2 can effectively deny Player 1 of his victory, by playing a blocking move comprising a weighted superposition of the best moves that Player 1 can make to win. Based on our understanding derived from the random games, we then analyze in Section 4 the effectiveness of different strategies that the two players can adopt in deterministic games. For all strategy pairs, the outcomes are very similar: Player 2 wins more deterministic games than Player 1, when Player 1 opens with a classical opening move. On the other hand, when quantum opening moves are used, the natural advantage to Player 1 is restored, with Player 2 winning only a small, but non-zero, proportion of deterministic games. Finally, we summarize our most important findings in the Section 5.

2. Quantum moves and winning condition

The classical tic-tac-toe is a childhood game played on a 3×3 grid. It is a two-player game of no chance, as no randomizing devices (for example, a dice) are used. In addition, it is also a game with no hidden information (unlike, for example, the hands of opponents in most card games). Both players know what moves have been played, and what moves are available to themselves, as well as to their opponents. In this game, the two players take turn occupying empty sites on the 3×3 grid. A player wins whenever he succeeds in occupying a straight line consisting of three sites, be it horizontally, vertically, or diagonally. Alternatively, if all nine sites are occupied and no player succeeded in making a line of three sites, then the game ends in a draw (also called a tie). In fact, if both players make no mistakes, it can be proven mathematically that the classical tic-tac-toe always ends in a draw [35].

To quantize games for two or more players, generalized quantization schemes have been proposed [31, 33]. These game-theoretic quantization schemes allow us to very quickly construct payoff matrices, but they are not convenient for implementing iterated play where the space of moves diminishes with every move made. The quantization scheme we chose is very similar to that defined by Goff *et al.* [36, 37], but differs in important aspects of iterated play. Goff *et al.* developed their version of the quantum tic-tac-toe as a teaching metaphor for entanglement and measurement in quantum mechanics, and thus their main interest is in introducing measurement, and the ensuing wave function collapse, into the game. However, when we play by Goff *et al.*'s rules, the

quantum tic-tac-toe does not properly reduce to the classical game upon the restriction to classical moves. In the subsections to follow, we will introduce a set of rules that embodies part of the essence of being 'quantum', but at the same time properly reduces to the classical rules when only classical moves are used.

2.1. The quantum move

As with Goff *et al.*, we map the nine possible classical moves to basis vectors in a nine-dimensional vector space, as shown in Figure 1. However, in contrast to Goff *et al.*, whose quantum moves partially occupy only two sites, we define our *quantum move*

$$|m\rangle = \sum_{i=1}^{9} v_i |b_i\rangle, \quad \sum_{i=1}^{9} |v_i|^2 = 1 \tag{1}$$

to be any normalized linear combination of the classical moves $\{|b_i\rangle\}$, i.e. we allow simultaneous partial occupation of any number of sites. In general, the amplitudes v_i can be complex. In this paper, we restrict ourselves to real v_i, to make the numerical studies presented in Sections 3 and 4 simpler.
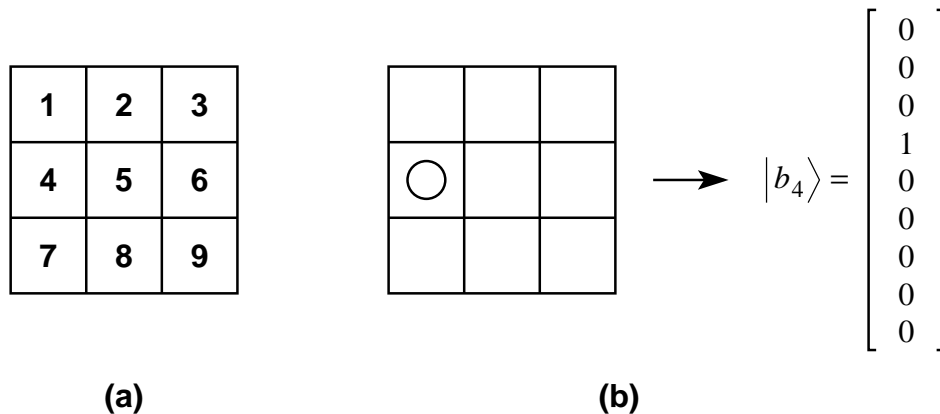

(a) **(b)**

Figure 1: The (a) sites on the 3×3 grid for tic-tac-toe, numbered from 1 through 9, and (b) an example of how a classical move is mapped to a basis vector in the nine-dimensional vector space.

For our quantum tic-tac-toe to properly reduce to the classical tic-tac-toe, we must impose the following restriction onto our quantum moves. In the classical game, a player may not play the classical move $|b_i\rangle$, if it has already been played earlier. This would correspond to him or her trying to occupy an already occupied site. Instead, he or she must play a classical move $|b_j\rangle$, with $j \neq i$, if it has not been played. Noting that $|b_j\rangle$ is by construction orthogonal to $|b_i\rangle$, we require a legal quantum move to be orthogonal to all previous quantum moves. If we use $|m_{k\sigma}\rangle$ to denote the kth quantum move made by player σ, then the orthogonality requirement can be written as

$$\langle m_{l\sigma}|m_{k\sigma}\rangle = 0, \quad \langle m_{l'\sigma'}|m_{k\sigma}\rangle = 0, \tag{2}$$

for $l, l' < k$ and $\sigma' \neq \sigma$. Here $\sigma = 1, 2$, and $1 \leq k \leq 5$ for Player 1 and $1 \leq k \leq 4$ for Player 2.

2.2. The winning condition

In Goff *et al.*'s version of the quantum tic-tac-toe, the two players take turns playing quantum moves of the form $|m\rangle = \frac{1}{\sqrt{2}}|b_i\rangle + \frac{1}{\sqrt{2}}|b_j\rangle$, where $i \neq j$, until a closed loop of moves have been made by one of the players. The other player must then perform a measurement on one site within the closed loop of moves, to collapse the state of the game onto a classical state. The classical state is then checked against the classical winning condition, to see if one or the other player wins. Else the game continues, with the restriction that future quantum moves cannot occupy any site on the collapsed loop. The outcome of the game depends on which site on the closed loop the wave function collapse started, and is thus not deterministic. For the quantum information transfer and portfolio optimization scenarios outlined in Section 1, we prefer to have no wave function collapse. More importantly, we would like to define a deterministic winning condition that is compatible with the quantum moves defined in the previous subsection, and will also properly reduce to the classical winning condition. At the same time, we want to admit the possibility of quantum-mechanical interference in our quantum game.

To define the winning condition, let us first define the *weight* $W_{pqr}^{k\sigma}$ Player σ has along the straight line through sites p, q, and r after k quantum moves. In spite of the orthogonality constraint described earlier, he or she is likely to have played nonzero amplitudes at all sites for all k moves. To compute the different occupation levels of the nine sites, we sum all k moves of Player σ,

$$|m_{1\sigma}\rangle + |m_{2\sigma}\rangle + \cdots + |m_{k\sigma}\rangle = \sum_{i=1}^{9} v_{i1\sigma}|b_i\rangle + \sum_{i=1}^{9} v_{i2\sigma}|b_i\rangle + \cdots + \sum_{i=1}^{9} v_{ik\sigma}|b_i\rangle \qquad (3)$$

$$= \sum_{i=1}^{9} \left(v_{i1\sigma} + v_{i2\sigma} + \cdots + v_{ik\sigma}\right)|b_i\rangle \qquad (4)$$

$$= \sum_{i=1}^{9} \left(\sum_{l=1}^{k} v_{il\sigma}\right)|b_i\rangle, \qquad (5)$$

where $v_{il\sigma}$ denotes the amplitude contribution to site i by the lth quantum move. The term in the parentheses is the accumulated amplitude in site i. The weight $W_{pqr}^{k\sigma}$ Player σ has along the *direction pqr* can then be calculated as

$$W_{pqr}^{k\sigma} = \sum_{i=p,q,r} \left(\sum_{l=1}^{k} v_{il\sigma}\right)^2. \qquad (6)$$

Thus, Player σ wins after his or her kth move, if

$$W_{pqr}^{k\sigma} \geq 3 \qquad (7)$$

for some direction pqr. For the sake of clarity in the rest of the paper, we will refer to Player 1 in the masculine, and to Player 2 in the feminine.

3. Random games

Even though our quantum tic-tac-toe 'contains' the classical tic-tac-toe, it is a very different game from its classical counterpart. In fact, it is so different we did not know

how to play it at first. When two players play the game without any proper strategy, the game would look very much like a random game. Therefore, to start understanding our quantum tic-tac-toe, we played random classical and quantum games, to see how different they really are from each other. This will also serve as a benchmark study of the quantum game played in the absence of any strategy, for later comparison against the deterministic strategic plays studied in Section 4.

In a random classical game, the nine classical moves $\{|b_i\rangle\}_{i=1}^9$ are played in random order. After each move, the maximum weight

$$W_{\max} = \max_{pqr} W_{pqr} \tag{8}$$

of the active player is calculated. If this weight is equal to three, the active player wins. Otherwise, the game continues, until one player wins, or the game ends in a draw. In a random quantum game, we first construct nine random vectors which are neither normalized nor orthogonal. We then apply the Gram-Schmidt orthonormalization procedure on the nine vectors to obtain a set of nine orthonormal *random (quantum) moves*. These random moves are then played sequentially, until one player wins according to the quantum winning condition in Eqn. (7), or the game ends in a draw.

3.1. Winning proportions

After playing 10,000 random classical games and 10,000 random quantum games, we tabulate the outcomes in Table 1. In both the random classical and random quantum games, Player 1 wins about 60% of the time. However, Player 2 is at a greater disadvantage in the random quantum game, in the sense that she wins only 14.2% of the time, as opposed to 28.5% of the time in the random classical game. Furthermore, we see that in the random classical game, both Player 1 and Player 2 win about 9% of the time after their third move. In the random quantum game, no player wins after the third move.

Table 1: Outcomes of 10,000 random classical games and 10,000 random quantum games. Here we show the proportions of wins by Player 1 and Player 2 after move k for both games. Player 2 has only four moves, so the number shown for $k = 5$ is the proportion of games ending in a draw.

Move k	Claissical Game (%)		Quantum Game (%)	
	Player 1	Player 2	Player 1	Player 2
1	0	0	0	0
2	0	0	0	0
3	9.4	9.0	0	0
4	26.5	19.5	21.8	14.2
5/draw	22.4	13.2	38.5	25.5

To understand why this is so, let us sum up the k moves that Player σ has made,

$$|m_\sigma\rangle = |m_{1\sigma}\rangle + |m_{2\sigma}\rangle + \cdots + |m_{k\sigma}\rangle \tag{9}$$

and check the weights

$$W_{pqr} = |\langle m_\sigma | b_{pqr}\rangle|^2 = |\langle m_\sigma | b_p\rangle|^2 + |\langle m_\sigma | b_q\rangle|^2 + |\langle m_\sigma | b_r\rangle|^2 \tag{10}$$

along the eight straight lines on the 3×3 grid, where $|b_{pqr}\rangle = |b_p\rangle \times |b_q\rangle \times |b_r\rangle$ is the hypersurface spanned by $|b_p\rangle$, $|b_q\rangle$, and $|b_r\rangle$. These can be viewed as the squares of the scalar projections of the resultant vector $|m_\sigma\rangle$ onto the eight three-dimensional subspaces spanned by $|b_p\rangle$, $|b_q\rangle$, and $|b_r\rangle$. Since all quantum moves have to be normalized and orthogonal to each other, the resultant vector is the diagonal of a k-dimensional cube, as shown in Figure 2.
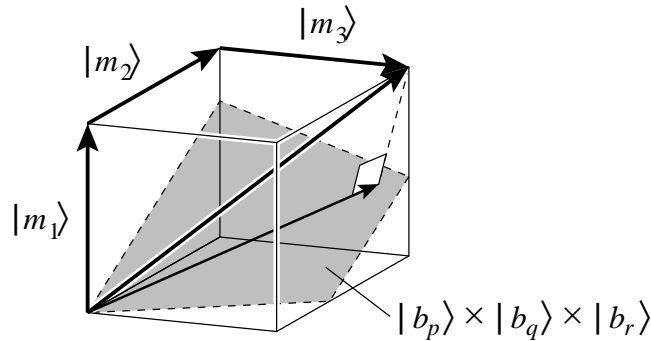

Figure 2: Schematic diagram showing the resultant vector for three orthonormal quantum moves $|m_1\rangle$, $|m_2\rangle$, and $|m_3\rangle$, and its vector projection onto the $|b_{pqr}\rangle = |b_p\rangle \times |b_q\rangle \times |b_r\rangle$ subspace.

For $k = 3$ moves, the resultant vector $|m_\sigma\rangle$ has a length of $\sqrt{3}$. Thus, the only way for the square of its scalar projection to be equal to three is for $|m_\sigma\rangle$ to lie entirely within one such three-dimensional subspace. It is also impossible for the maximum weight of three quantum moves to be greater than three. Since a quantum game offers infinitely many more moves than the classical game, the set of three successive moves with resultant vector lying exactly on one of the eight three-dimensional subspaces is of measure zero. This explains why no player was found to win after the third move in our simulations.

3.2. Opening moves

To someone learning to play chess formally, the first order of business is always to learn the various opening moves, and understand the relative advantages they confer. An *opening move* is the first move played in the game. It is an important move, as it influences the middle game, and thus also the end game. In this subsection, we investigate different opening moves, to better understand the advantages they confer to Player 1.

For concreteness, let us compare three opening moves: (i) the *classical opening move*; (ii) the *uniform opening move*; and (iii) the *random opening move*. In (i), Player 1 always plays the classical move $|b_5\rangle$ as his first move, whereas in (ii), Player 1 always start by playing the quantum move $\frac{1}{\sqrt{9}}|b_1\rangle + \frac{1}{\sqrt{9}}|b_2\rangle + \cdots + \frac{1}{\sqrt{9}}|b_9\rangle$, which has uniform contribution from all classical moves. In (iii), Player 1 plays a random opening move. For each opening move, we played 10,000 games for which all subsequent moves are random quantum moves. The outcomes are shown in Table 2.

Table 2: Outcomes of 10,000 random quantum games each for three different opening moves: (i) classical; (ii) uniform; and (iii) random. Here we show the proportions of wins by Player 1 and Player 2 after move k for both games. Player 2 has only four moves, so the number shown for $k = 5$ is the proportion of games ending in a draw.

Move k	Opening Move					
	Classical (%)		Uniform (%)		Random (%)	
	Player 1	Player 2	Player 1	Player 2	Player 1	Player 2
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	0
4	7.6	28.9	23.4	16.4	21.8	14.2
5	27.0	36.5	35.0	25.2	38.5	25.5

As we can see from Table 2, the proportions of games won by Player 1, Player 2, and ending in a tie are very similar for the uniform and random opening moves, down to the breakdown of proportions of games won after the fourth and fifth moves. The situation for the classical opening move, however, is very different. While Player 1 still wins more games, Player 2 wins nearly twice as many games opened with a classical move compared to games opened with a uniform move or a random move. This tells us that in the absence of strategies adopted by Players 1 and 2, a quantum opening move significantly improves the advantage enjoyed by Player 1.

The geometrical picture behind this quantum advantage is very simple. The three-dimensional winning subspace $|b_{pqr}\rangle$ is spanned by the classical moves $|b_p\rangle$, $|b_q\rangle$, and $|b_r\rangle$. The moment Player 1 plays the classical move $|b_p\rangle$, the scalar projection of $|m_1\rangle$ onto $|b_p\rangle$ saturates at $\langle b_p|m_1\rangle = 1$. However, if Player 1 avoids playing $|b_p\rangle$, the scalar projection $\langle b_p|m_1\rangle$ can grow with the number of moves made. In fact, with an appropriate choice of quantum moves, we can make $\langle b_p|m_1\rangle > 1$ after Player 1's second move. By opening with $|b_5\rangle$, Player 1 has thus eroded the natural advantage he enjoys in the game, by limiting the rates at which he is accumulating weights along four of the eight straight lines.

3.3. End games

Besides the opening moves, we also learn a game by studying the end games, whereby the combinatorial complexity of the game is reduced because there are only a few moves left. In particular, we studied end games in which Player 1 is on the verge of winning. To arrive at an end-game situation, we played random quantum games, and kept those games where Player 1 wins after his fourth move. We then discard the moves after Player 1's third move, to obtain an *end game* where Player 1 has made three moves and Player 2 has made two moves.

Because Player 1 can win on his next move, it is evident that Player 2 must play a *blocking move*. To stop Player 1 from winning, Player 2 can play the move Player 1 would use to win, i.e. Player 1's *winning move*. Thereafter, Player 1 can no longer play it, because he is forced to play moves orthogonal to all previous moves. However, just like in the classical game, Player 1 may have more than one winning move. In fact, Player 1 has infinitely many winning moves within the four-dimensional space of all legal quantum moves remaining.

Clearly, this manifold of winning moves should be densely distributed about moves that maximize Player 1's weight along one or more of the eight straight lines. To find the *maximizing move* $|x\rangle$ that maximizes Player 1's weight

$$W_{pqr} = |\left(\langle m_1| + \langle x|\right)|b_{pqr}\rangle|^2 \tag{11}$$

along the direction pqr, subject to the condition that it orthonormal to all previous moves, we use the method of Lagrange multipliers. Here, $|m_1\rangle = |m_{11}\rangle + |m_{21}\rangle + |m_{31}\rangle$ is the sum of the three moves Player 1 has made. Writing out the constraints

$$\langle x|x\rangle = 1, \tag{12}$$

$$\langle m_{l\sigma}|x\rangle = 0, \tag{13}$$

explicitly, for $l\sigma = 11, 12, 21, 22, 31$, the simultaneous equations we need to solve are (see Appendix A for detail derivations)

$$-2\alpha |x\rangle + M\beta + 2\sum_{s=p,q,r} |b_s\rangle \left(\langle b_s|m_1\rangle + \langle b_s|x\rangle\right) = 0, \tag{14}$$

$$1 - \langle x|x\rangle = 0, \tag{15}$$

$$M^T |x\rangle = 0, \tag{16}$$

where α is the Lagrange multiplier for enforcing normalization, β is a 5×1 vector of Lagrange multipliers for enforcing orthogonalization, and

$$M = \left[\ |m_{11}\rangle\ \big|\ |m_{12}\rangle\ \big|\ |m_{21}\rangle\ \big|\ |m_{22}\rangle\ \big|\ |m_{31}\rangle\ \right] \tag{17}$$

is a 9×5 matrix compiling the five previous moves. Here, 0 denotes either the scalar, the 5×1 or the 9×1 null vectors depending on the context.

After finding Player 1's eight maximizing moves, and the maximum weights they are associated with, Player 2 can play the maximizing move with the largest maximum weight overall as her blocking move. However, if Player 1 can win along multiple

directions, then Player 2 is sure to lose in the classical tic-tac-toe. In the quantum tic-tac-toe, Player 2 might be able to take advantage of the 'quantumness' of the game, to simultaneously block all of Player 1's winning directions. We evaluated the effectiveness of one such quantum blocking move, by first sorting the end games according to their *pre-winning weight*. For end games of a given pre-winning weight ω, we then let Player 2 play a *weighted blocking move*,

$$|y\rangle = \mathcal{N}\left(W_1 |x_1\rangle + W_2 |x_2\rangle + W_3 |x_3\rangle\right), \tag{18}$$

consisting of the three best moves $|x_1\rangle$, $|x_2\rangle$, and $|x_3\rangle$ by Player 1, i.e. the three maximizing moves that gives the largest *winning weights* W_1, W_2, and W_3. Here, $\mathcal{N}$ is a normalization constant we need to compute each time $|y\rangle$ is constructed, because $|x_1\rangle$, $|x_2\rangle$, and $|x_3\rangle$ are not necessarily orthogonal to each other. Finally, after Player 2 has played $|y\rangle$, we let Player 1 play the maximizing move $|z_1\rangle$ along the direction ω is obtained.

In our simulations, we generated 100,000 end games, and group them into bins with width $\Delta\omega = 0.05$. For each bin, we had Player 2 play the weighted blocking move, as well as a random move not specifically intended for blocking. Thereafter, we let Player 1 play $|z_1\rangle$, before checking whether he has won the game. As shown in Figure 3, we see that the weighted blocking move is statistically more effective than the random move, not only in terms of the proportion of end games successfully blocked, but also in terms of how this proportion falls off as we approach $\omega = 3$.

4. Deterministic games

After analyzing the end games, we realized that the basic element for playing the quantum tic-tac-toe is the maximizing move. We also understood strategic differences between how Players 1 and 2 were using such a move in the end games. In essence, Player 2 played a *defensive* third move, seeking only to deny Player 1 from successfully maximizing his weight. Following this, Player 1 played an *offensive* fourth move, seeking only to maximize his own weight. With this insight, we are now able to play the game deterministically, after the opening move by Player 1. Our goal is to examine how the outcomes, subject to different opening moves, depend on following strategies adopted by Players 1 and 2:

(i) *Win/block (WB).* Player 1 aims to win by playing only offensive moves, whereas Player 2 plays only blocking moves;

(ii) *Win-block/block (WBB).* Player 1 plays offensive moves, but will respond with a blocking move if (i) Player 2 will win after the next move, *and* (ii) he will not win after the present move. We implement this blocking condition approximately, by making Player 1 block whenever Player 2's current pre-winning weight ω_2 exceeds two (and is thus is likely to exceed three in the next move), and simultaneously his's current pre-winning weight ω_1 is smaller than ω_2. Player 2 plays only blocking moves;
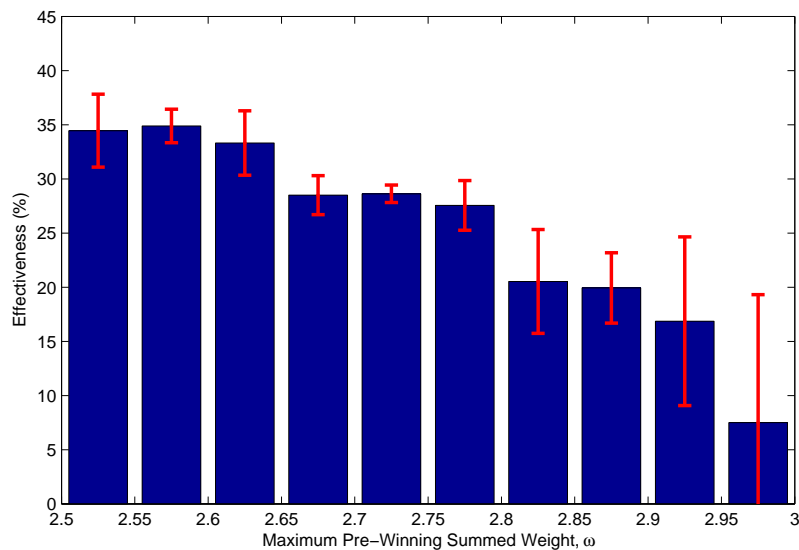
(a)
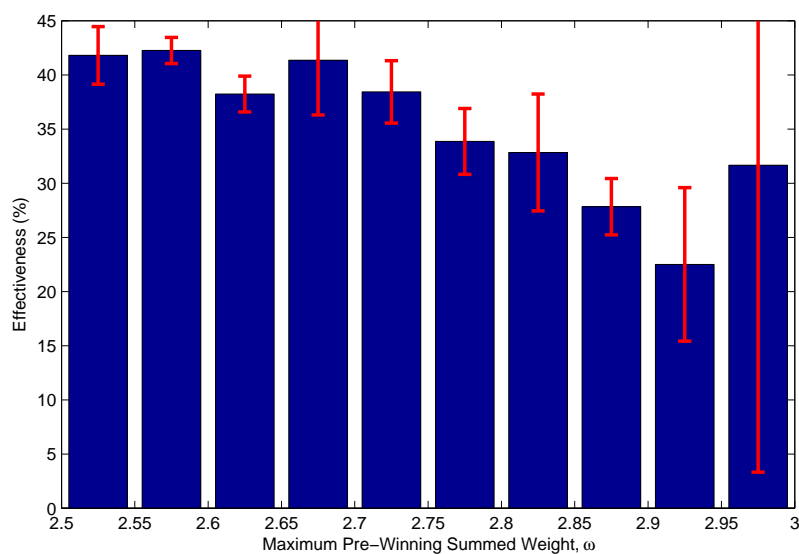
(b)

Figure 3: Effectiveness of (a) the random blocking move, and (b) the weighted blocking move, measured in terms of the proportions of end games successfully blocked for each pre-winning weight ω. The weighted blocking move is about 10% more effective than the random blocking move. More importantly, the weighted blocking move remains highly effective as we approach $\omega = 3$.

(iii) *Win/win-block (WWB)*. Player 1 plays only offensive moves. Player 2 plays offensive moves, but will respond with a blocking move if (i) Player 1 will win after the next move, *and* (ii) she will not win after the present move. Again, we approximate this blocking condition as $\omega_1 > 2$ and $\omega_1 > \omega_2$ simultaneously;

(iv) *Win-block/win-block (WBWB)*. Players 1 and 2 start by playing offensive moves, but switch over to defensive moves whenever the opponent is on the verge of winning, and they themselves are not.

To properly define the *offensive move*, let us note that for a given move, the active player can play eight maximizing moves, one each for directions $pqr = 123, 456, 789, 147, 258, 369, 159, 357$. After each of these maximizing moves are played, the maximum weights that the active player can attain are $W_{123}, W_{456}, W_{789}, W_{147}, W_{258}, W_{369}, W_{159}, W_{357}$ respectively. The offensive move is the maximizing move associated with the largest maximum weight overall,

$$W_{\max} = \max\{W_{123}, W_{456}, W_{789}, W_{147}, W_{258}, W_{369}, W_{159}, W_{357}\}. \tag{19}$$

As defined in the previous section, the *defensive move* is the weighted superposition of the opponent's three best maximizing moves.

Because of the normalization constraint, we have to solve a nonlinear system of simultaneous equations to find each maximizing move. This is done numerically using a nonlinear optimization routine in MATLAB, using random initial guesses. Depending on our initial guess, we can converge to a global maximizing move, or to stationary solutions that do not maximize the active player's weight along the given direction. Therefore, for each direction, we solve for stationary moves starting with 20 initial guesses. We then select the stationary move with the maximum weight, and perform a second-derivative test on it. If it is locally maximum, we accept the stationary move as our maximizing move. Although this procedure is not guaranteed to always find the globally maximizing move, we find it giving reliable results in practice. Details on the second-derivative test can be found in Appendix B.

Before we move on to discuss our results, we would like to remark that though the strategies are deterministic, the games do not progress deterministically, because of the random initial guesses used to solve for maximizing moves. This probabilitistic progress of the games is most prominent for highly degenerate games, like those opened with a classical move or a uniform move. Play-by-play analysis of the deterministic quantum games for different strategies can be found at Ref. [38]. In this paper, we will focus on generic outcomes shown in Table 3 for the different strategies, subject to different opening moves.

4.1. Comparison against the deterministic classical game

From Table 3, we see that the deterministic quantum tic-tac-toe do not always end up in a draw, even for the classical opening move, when the proportions of games ending in a draw is highest (around 70%), whatever the strategy pair. This is a clear departure

Table 3: Outcomes of deterministic quantum games each for the Win/Block (WB), Win-Block/Block (WBB), Win/Win-Block (WWB), and Win-Block/Win-Block (WBWB) strategies, subject to the classical, uniform, and random opening moves. The move number is not listed, but increases from $k = 1$ to $k = 5$ downwards. Player 2 has only four moves, so the proportional shown in the fifth row under Player 2 is the proportion of games that ended in a draw. Also, not all 10,000 games were played to completion for each strategy pair and opening move, because the active player fails to find maximizing moves at some point in the game. The number at the last row of each strategy pair indicates how many games ended prematurely because of this problem. The proportions shown in the table are computed from the successfully completed games.

Strategy	Opening Move					
	Classical (%)		Uniform (%)		Random (%)	
	Player 1	Player 2	Player 1	Player 2	Player 1	Player 2
WB	0	0	0	0	0	0
	0	0	0	0	0	0
	0	0	0	0	0	0
	0	22.4	68.4	2.9	40.5	6.2
	5.1	72.5	6.2	22.5	5.0	48.3
	183 games		2186 games		5279 games	
WBB	0	0	0	0	0	0
	0	0	0	0	0	0
	0	0	0	0	0	0
	0	21.2	68.1	3.1	21.5	15.0
	6.3	72.5	6.1	22.6	10.6	52.9
	176 games		2139 games		5543 games	
WWB	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0	0.0
	13.3	0.5	0.0	0.0	0.0	0.0
	0.9	15.5	44.6	1.5	51.4	6.0
	4.8	65.0	2.8	51.2	3.6	39.0
	420 games		1605 games		2083 games	
WBWB	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0	0.0
	0.0	0.0	0.0	0.0	0.0	0.0
	2.0	24.8	15.7	6.7	43.7	6.3
	10.9	62.3	41.5	36.1	12.8	37.1
	442 games		1575 games		3719 games	

from the classical tic-tac-toe, where all deterministic games must end in a draw [35]. Between the two quantum opening moves, the proportion of tied games is generally lower for games opened with the uniform move than for games opened with the random move. We expect this, because the uniform opening move confers the maximum quantum advantage on Player 1, who would go on to win most of these deterministic games.

What is perhaps more surprising, is Player 2 winning more deterministic games than Player 1, whatever the strategy pair, when these games are opened with the classical move! We know of no classical two-player games whereby Player 2 owns the advantage. It turns out that the reason Player 1 does poorly, after opening with the classical move, is the same for deterministic games as it is for random games. After saturating the scalar projection $\langle m_1|b_5\rangle$ with the opening move, Player 1 effectively traded away his ability to more rapidly increase his weights along four out of eight directions with further moves. This loss of advantage by Player 1 is extremely pronounced in the WB and WBB games, from winning over 30% of random quantum games opened with the classical move $|b_5\rangle$, to about 5% in deterministic games opened with $|b_5\rangle$. Since Player 2 is playing defensively in these two class of games, her winning proportions did not increase over that in the random games. The sharp drop in Player 1's winning proportions is thus a testimony on how effective the quantum blocking move is.

4.2. Comparison between opening moves

In contrast to the classical opening move, the uniform and random opening moves confer immense advantage onto Player 1, when we compare their outcomes against those of random quantum games opened with the same moves. Player 2 went from winning about 15% of the random games to winning about 3–6% in the deterministic games. The only exception is WBB games opened with a random move, where Player 2 apparently suffers no further quantum disadvantage. Comparing Tables 2 and 3, we find Player 1 wins more of his random games after $k = 5$ moves, but most of his WB, WBB, WWB games after $k = 4$ moves. This shows that the quantum opening move is an effective move for Player 1, when playing strategically.

We were also surprised to find Player 1 winning 13.3% of the WWB games opened with the classical move after the third move. Upon checking the games play by play for this strategy pair, we found that the pre-winning weight of Player 1 should always be $\omega_1 = 2$. Depending on numerical truncation errors, the numerical value of ω_1 either just fails or just succeeds to trigger the criteria for Player 2 to start blocking. In the former, Player 2 plays an offensive second move, leaving Player 1 unhampered to play a winning third move. In the latter, Player 2 plays a blocking second move, effectively denying Player 1 of his third-move win. Because of the integer nature of the classical opening move, the numerical truncation errors associated with ω_1 is smaller than those associated with ω_2, after the same number of moves. Thus, Player 1's third move in WBWB games opened with the classical move is almost always a blocking move. This explains why Player 1 is not observed to win after three moves in such games.

4.3. Comparison between different strategies

With the classical opening move, Player 1 seriously disadvantaged himself. His winning proportion is lowest when he plays to win, while Player 2 plays to block. We might be tempted to think that this is because he fails to block Player 2 when she is on the verge of winning. But when Player 1 plays to win, but also block Player 2 whenever necessary, his winning proportion increases only slightly, from 5.1% to 6.3%. In contrast, when Player 2 decides to start with an offensive move, and block only when necessary, in the WWB and WBWB games, Player 1 is no longer quite as disadvantaged. This tells us that the major factor affecting Player 1's fortune is whether Player 2 choose to start defensively or offensively.

This same pattern is repeated for the quantum opening moves. Player 1 does no worse, or slightly better when he also blocks, than when he single-mindedly plays to win, for the same Player 2 strategy. On the other hand, Player 2 is worse off if she also plays to win, than when she single-mindedly blocks, if she is playing against a purely offensive Player 1. She fares better with mixed offensive-defensive moves, than with purely defensive moves, however, if Player 1 also plays mixed offensive-defensive moves.

5. Conclusions

To conclude, we have in this paper introduced a minimalistic quantization of the classical tic-tac-toe, by admitting quantum moves which are arbitrary superpositions of the classical moves. We require our quantum moves to be orthonormal to all previous moves, and also for the sum of squares of resultant amplitudes to exceed three along any straight line of three cells for a player to win, so that our quantum tic-tac-toe reduces properly to the classical tic-tac-toe. Playing the quantum game first randomly and then deterministically, we find that unlike the classical game, the deterministic quantum tic-tac-toe does not always end in a draw. Furthermore, unlike most classical two-player games of no chance, both players can win in the deterministic quantum game. More interestingly, in both random and deterministic quantum games, we see that Player 1 enjoys an overwhelming quantum advantage when he opens with a quantum move. This advantage, which is lost when Player 1 opens with a classical move, has a very simple geometrical interpretation in terms of the projection of the resultant move onto the classical winning subspaces. Finally, the biggest contrast between the classical and quantum tic-tac-toes must surely be the effective quantum blocking move that the defending player can play. In fact, a defensive strategy based solely on such a quantum blocking move is the strategy of choice for Player 2, for most strategies that Player 1 adopts.

While the quantum tic-tac-toe does not properly describe the quantum information transfer scenario developed in the Introduction, we believe some generic results obtained for the former should also apply in the latter. For instance, we believe Alice will also enjoy a huge quantum advantage with a uniform opening move, if we imagine she has

multiple quantum channels through which she can transmit to Bob. This move is the least informative, and Eve would have to guess which quantum channels will ultimately be used to transmit the message to Bob, in order to come up with a blocking move. Certainly, Alice should not first attempt to transmit a classical bit utilizing just one channel, because she will almost certainly lose the advantage she naturally enjoys as Player 1. Eve can learn something from this paper as well. If the transmissions by Alice as to be understood as purely offensive moves, Eve should adopt a pure quantum jamming strategy by playing quantum blocking moves. She should not succumb to the temptation to also intercept the message, which we can interpret as an offensive move, because she is not likely to do any better with such a mixed strategy.

Like the quantum information transfer scenario, the multiplayer portfolio optimization game idealized in the Introduction differs from the quantum tic-tac-toe in many important aspects. In particular, both the multiplayer portfolio optimization game and the quantum information transfer game are not subjected to stringent orthonormality constraints. Nevertheless, we believe the generic lessons learnt from the quantum tic-tac-toe will apply even in this significantly more complex quantum game. To prevent competitors from concerted or inadvertent sabotage, a fund manager should play a uniform move by maximally diversifying his portfolio. This is because adjustments to such a portfolio yields the least information for other fund managers to act upon, and therefore its optimality is least susceptible to malicious attacks. Should a fund manager suspect intentional attacks to his portfolio by multiple players, we also expect the quantum blocking move to be highly effective. We believe such a 'defensive' strategy will help a fund fare better during a financial crisis, where the cascading loss-cutting measures adopted by other funds can be seen as a coordinated assault on its position.

Finally, we note that in the duel between grandmasters, there is the additional element of timing in the strategic game play. For example, an effective move can be planted ahead of time, and its effectiveness enhanced by subsequent moves. Another example would be, at times where a defensive move seems inevitable, a grandmaster can force his opponent's hand by playing an offensive move elsewhere, and then return leisurely to play the defensive move. In our quantum tic-tac-toe, the game complexity is not high enough for such situations to arise. A future topic of research would be to quantize a more complex two-player game, where these timing situations do arise, and then explore game-theoretically how different the outcome might be if quantum moves are made available.

Acknowledgments

Appendix A. Method of Lagrange multipliers

In Section 3.3, the method of Lagrange multipliers was used to find the maximizing move $|x\rangle$ along a given direction pqr. In this appendix, we will describe how we obtain the simultaneous equations (14), (15), and (16). In the method of Lagrange multipliers, if $f(x, y)$ is the function we wish to maximize, subject to the constraints, $g(x, y) = c$ and $h(x, y) = d$, we introduce the *Lagrange function*,

$$\Lambda(x, y, \alpha, \beta) = f(x, y) + \alpha(g(x, y) - c) + \beta(h(x, y) - d) \tag{A.1}$$

where α and β are the *Lagrange multipliers*. To maximize $\Lambda(x, y, \alpha, \beta)$, we partial differentiate $\Lambda(x, y, \alpha, \beta)$ with respect to x and y, as well as α and β, and set the partial derivatives $\partial\Lambda/\partial x$, $\partial\Lambda/\partial y$, $\partial\Lambda/\partial\alpha$, $\partial\Lambda/\partial\beta$ to zero.

In the end-game situation discussed in Section 3.3, Player 1 has made his third move, and we would like to maximize his weight along the direction pqr, using a normalized move orthogonal to all previous moves. In this situation, the function we would like to maximize is the weight W_{pqr}, given in Eqn. (11), and the normalization and orthogonality constraints are given by Eqn. (12) and Eqn. (13) respectively. Using Eqn. (6), we can write the weight of Player 1 along pqr after the maximizing move explicitly as

$$W_{pqr}^{41} = \sum_{i=p,q,r} \left(\sum_{l=1}^{3} v_{il1} + x_i \right)^2. \tag{A.2}$$

We can also write the normalization and orthogonality constraints out explicitly as

$$\sum_{i=1}^{9} x_i^2 = 1, \quad \sum_i v_{il\sigma} x_i = 0, \tag{A.3}$$

where x_i is the ith amplitude of $|x\rangle$, and $l\sigma = 11, 12, 21, 22, 31$. With these, our Lagrange function becomes

$$\Lambda = \sum_{i=p,q,r} \left(\sum_{l=1}^{3} v_{il\sigma} + x_i \right)^2 + \alpha \left(1 - \sum_{i=1}^{9} x_i^2 \right) + \sum_{\{l\sigma\}} \beta_{l\sigma} \left(\sum_i v_{il\sigma} x_i \right), \tag{A.4}$$

using a total of six Lagrange multipliers, α to enforce normalization, and five $\beta_{l\sigma}$ to enforce orthogonality with respect to each of the five previous moves.

Differentiating the Lagrange function with respect to x_i, we find

$$\frac{\partial\Lambda}{\partial x_i} = -2\alpha x_i + \sum_{\{l\sigma\}} \beta_{l\sigma} v_{il\sigma} = 0 \tag{A.5}$$

if $i \neq p, q, r$. If i is p, q, or r, then $\partial\Lambda/\partial x_i$ has an extra term $2\left(\sum_{l=1}^{3} v_{il\sigma} + x_i\right)$ arising from the first term in Eqn. (A.1). We combine these two types of partial derivatives by writing

$$\frac{\partial\Lambda}{\partial x_i} = -2\alpha x_i + \sum_{\{l\sigma\}} \beta_{l\sigma} v_{il\sigma} + 2 \left(\sum_{l=1}^{3} v_{il\sigma} + x_i \right)_{pqr} = 0 \tag{A.6}$$

where the subscript pqr in the last term indicates that we only add the last term if $i = p, q$ or r. This becomes Eqn. (14) when written in matrix-vector form. Eqn. (15) and Eqn. (16) are simply $\partial\Lambda/\partial\alpha = 0$, the normalization constraint, and $\partial\Lambda/\partial\beta_{l\sigma} = 0$, the orthogonality constraints, written in matrix-vector form.

Appendix B. Hessian matrix and second-derivative test

To do the second-derivative test for the maximizing move, we first evaluate the Hessian matrix

$$
H(\Lambda) =
\begin{bmatrix}
\frac{\partial^2 \Lambda}{\partial x_1^2} & \frac{\partial^2 \Lambda}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial x_1\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial x_1\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial x_1\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial x_1\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial x_1\,\partial \beta_{l\sigma}} \\[2mm]
\frac{\partial^2 \Lambda}{\partial x_2\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial x_2^2} & \cdots & \frac{\partial^2 \Lambda}{\partial x_2\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial x_2\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial x_2\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial x_2\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial x_2\,\partial \beta_{l\sigma}} \\[2mm]
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\[2mm]
\frac{\partial^2 \Lambda}{\partial x_9\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial x_9\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial x_9^2} & \frac{\partial^2 \Lambda}{\partial x_9\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial x_9\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial x_9\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial x_9\,\partial \beta_{l\sigma}} \\[2mm]
\frac{\partial^2 \Lambda}{\partial \alpha\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial \alpha\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial \alpha\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial \alpha^2} & \frac{\partial^2 \Lambda}{\partial \alpha\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial \alpha\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial \alpha\,\partial \beta_{l\sigma}} \\[2mm]
\frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial \beta_{11}^2} & \frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{11}\,\partial \beta_{l\sigma}} \\[2mm]
\frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial \beta_{12}^2} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{12}\,\partial \beta_{l\sigma}} \\[2mm]
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\[2mm]
\frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial x_1} & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial x_2} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial x_9} & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial \alpha} & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial \beta_{11}} & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}\,\partial \beta_{12}} & \cdots & \frac{\partial^2 \Lambda}{\partial \beta_{l\sigma}^2}
\end{bmatrix}
\tag{B.1}
$$

of the Lagrange function given in Eqn. (A.4).

Since the Lagrange function $\Lambda(x_1, x_2, \ldots, x_9, \alpha, \beta_{11}, \beta_{12}, \ldots, \beta_{l\sigma})$ does not contain cross terms of the form $x_i x_j$, the 9×9 submatrix in $H(\Lambda)$ is diagonal, with diagonal matrix elements

$$
H_{ii}(\Lambda) = \frac{\partial^2 \Lambda}{\partial x_i^2} = -2\alpha + (2)_{pqr}.
\tag{B.2}
$$

Differentiating Eqn. (A.6) with respect to α and β, we will also get

$$
\frac{\partial^2 \Lambda}{\partial \alpha \partial x_i} = -2x_i,
\tag{B.3}
$$

$$
\frac{\partial^2 \Lambda}{\partial \beta_{l\sigma} \partial x_i} = v_{il\sigma}
\tag{B.4}
$$

respectively. Finally, we see that there are neither quadratic or cross terms involving α and β in the Lagrange function, Eqn. (A.4), and thus the second partial derivatives of $\Lambda(x_1, x_2, \ldots, x_9, \alpha, \beta_{11}, \beta_{12}, \ldots, \beta_{l\sigma})$ with respect to the Lagrange multipliers are always zero. The Hessian matrix is thus

$$
H(\Lambda) =
\begin{bmatrix}
A & -2|x\rangle & M \\
-2|x\rangle^T & & \\
M^T & & \mathcal{O}
\end{bmatrix},
\tag{B.5}
$$

where A is a 9×9 diagonal matrix, with all the diagonal entries being -2α, except the pth, qth and rth diagonal entries, which are $-2\alpha + 2$. The matrix M is the matrix compiling all previous moves defined in Eqn. (17), while $\mathcal{O}$ is a $(k+1) \times (k+1)$ null matrix, k being the total number of moves made by both players.

We then evaluate the Hessian matrix $H(\Lambda)$ at the optimal values $(x_1^*, x_2^*, \ldots, x_9^*;$ $\alpha^*, \beta_{11}^*, \beta_{12}^*, \ldots, \beta_{l\sigma}^*)$ of the maximizing move, before diagonalizing it to check if the

maximizing move does indeed maximize the weight of the active player. In unconstrained optimization within a d-dimensional space of parameters, we must have d negative eigenvalues, for the given optimal point to be locally maximum. In constrained optimization, each constraint defines a hypersurface. The constrained optimal point need not be locally maximum along directions normal to these constraint hypersurfaces, since we are not allowed to venture off these hypersurfaces anyway. If k moves have already been played, there will be k normal directions. The eigenvalues of $H(\Lambda)$ associated with eigenvectors lying within the space spanned by these k normal vectors need not be negative. Hence, a maximizing move is locally maximum if $H(\Lambda)$ has at least $n = 9 - k$ negative eigenvalues, where n is the number of moves remaining. Only deterministic quantum games for which all moves are locally maximizing are reported in this paper (see Table 3).

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