Equations of motion in General Relativity and Quantum Mechanics

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Abstract

In a previous article a relationship was established between the linearized metrics of General Relativity associated with geodesics and the Dirac Equation of quantum mechanics. In this paper the extension of that result to arbitrary curves is investigated. The Dirac equation is derived and shown to be related to the Lie derivative of the momentum along the curve. In addition,the equations of motion are derived from the Hamilton-Jacobi equation associated with the metric and the wave equation associated with the Hamiltonian is then shown not to commute with the Dirac operator. Finally, the Maxwell-Boltzmann distribution is shown to be a consequence of geodesic motion.

KEY WORDS: non-geodesic motion, Dirac equation, equations of motion, Maxwell-Boltzmann distribution.

1 Introduction

In a recent paper Frank Tipler [10] gives a derivation of the Schrodinger equation using the Hamilton-Jacobi principle of action. In doing so, he is able to transpose Erwin Schrodinger's "*purely* formal procedure"[1] of replacing $\frac{\partial W}{\partial t}$ in the Hamilton-Jacobi equation with $\pm\frac{h}{2\pi i}\frac{\partial}{\partial t}$ with a more direct derivation of the wave equation. Likewise, in a paper by Marie-Noelle Celerier and Laurent Nottale [4] in 2003 they derive the Dirac equation as the square root of the Klein-Gordan equation by using the "bi-quaternionic action" associated with geodesic motion and the Hamilton-Jacobi principle of action. Other approaches not directly based on the use of the action principle have also been used. For example, Ng and Dam used a geometrical derivation of the Dirac equation by exploiting "rotational invariance" and "the explicit use of the spin-$\frac{1}{2}$ property of ψ"[5], while Martin Rivas quantized a Poincare invariant Hamiltonian in which the spin angular momentum of the particle is constant with respect to the center of mass observer to derive the Dirac equation.[9]

This paper contains parallels with the approach of Tipler, and Celerier and Nottale in that it makes use of the Hamilton Jacobi equation but it also differs in that it emphasizes the fundamental role of the metric in enabling us to derive two equations associated with non-geodesic motion: one a generalized Dirac equation, which will capture the kinematics, and the other an equation of motion describing the dynamics.

Throughout the paper, $(\mathcal{M}, g)$ will denote a connected four dimensional Hausdorff manifold, with metric g of signature $+2$. At every point p on the space-time manifold $\mathcal{M}$ we erect a local tetrad $e_0(p), e_1(p), e_2(p), e_3(p)$ such that a point x has coordinates $x = (x^0,\ x^1,\ x^2,\ x^3) = x^a e_a$ in this tetrad coordinate system, while the spinor Ψ can be written as $\Psi = \psi^i e_i(p)$, where ψ^i represent the coordinates of the spinor with respect to the tetrad at p. This is permissable since a spinor is also an element of a vector space. In particular, when each ψ^i is equal for each i then we can write $\Psi = \psi\xi$ where the spinor $\xi = (e_i)$. When this occurs, we will say that ψ is a scalar field. Such a scalar field will occur when Ψ is Fermi transported along a geodesic. For example, in the case of the Dirac equation, the plane wave solution can be written as

$$\Psi(x) = e^{-ipx/\hbar} u_+(p), \tag{1}$$

where $\psi = e^{-ipx/\hbar}$ and $u_+(p) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and allows for two independent solutions for each momentum p. In particular, for motion along a geodesic, the spinor $u(p)$ when expressed in Fermi normal coordinates remains constant along a geodesic. This also reflects Rivas result in that the derivation of the Dirac equation presupposes the motion of a particle "in a plane orthogonal to S, which is constant in this frame," [9] where S refers to spin-angular momentum. In terms of quantum mechanics each constant can be identified with a quantum number. Also at each point p we can establish a tangent vector space $T_p(\mathcal{M})$, with basis $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ and a dual 1-form space, denoted by T_p^* with basis $\{dx_0, dx_1, dx_2, dx_3\}$ at p, defined by

$$dx^\mu \partial_\nu \equiv \partial_\nu x^\mu = \delta^\mu_\nu. \tag{2}$$

We refer to the basis $\{dx^0, dx^1, dx^2, dx^3\}$ as the basis of one forms dual to the basis $\{\partial_0, \partial_1, \partial_2, \partial_3\}$ of vectors at p.

With notation clarified, we note that in a previous article [8], the Dirac equation was derived as a dual of a linearized metric for geodesics. The linkage was accomplished in a natural way by associating a generalized Dirac equation with those operators which are duals of differential one-forms, obtained by linearizing the metrics of General Relativity (expressed locally as a Minkowski metric). Specifically if

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b \tag{3}$$

where a and b refer to local tetrad coordinates and η to a rigid Minkowski metric of signature +2, then associated with this metric and the vector **ds** is the scalar ds and a matrix $\tilde{ds} \equiv \gamma_a dx^a$ respectively, where $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$.

In addition, the $\tilde{ds}$ matrix can be considered as the dual of the expression $\tilde{\partial}_s \equiv \gamma^a \frac{\partial}{\partial x^a}$ which in turn enabled us to define a generalized Dirac equation

$$\tilde{\partial}_s \psi \xi \equiv \gamma^a \frac{\partial \psi}{\partial x^a} \xi = \frac{\partial \psi}{\partial s} \xi, \tag{4}$$

associated with the motion of a particle along a geodesic, with a fixed spinor ξ.

This paper is an extension of that previous work. In the first part of the article, we will re-investigate this result by identifying a natural motion along a curve with a Hamilton-Jacobi function, and then derive the Dirac equation as a consequence. In the second part, we will introduce accelerations and derive an appropriate equation of motion, as well as a corresponding wave equation, which can be extended to statistical mechanics. Indeed, the Maxwell-Boltzmann distribution will be discussed in this context. Also, I would hope that the derivation will give a deeper understanding of the meaning of the phrase "wave-particle" duality.

2 Generalized Dirac Equation

The generalized Dirac equation defined above relies on the definition of the four-momentum in special relativity and upon the fact that $\tilde{\partial}_s\psi$ and $\tilde{ds}$ are parallel along geodesics, and consequently by the chain rule their product is $\frac{d\psi}{ds}ds$. In contrast, when accelerations are introduced we will find that in general

$$\frac{\tilde{ds}}{ds}.\tilde{\partial}_s\psi = \frac{1}{2}\left\{\frac{\tilde{ds}}{ds},\tilde{\partial}_s\psi\right\} + \frac{1}{2}\left[\frac{\tilde{ds}}{ds},\tilde{\partial}_s\psi\right] \quad (5)$$

$$= \frac{d\psi}{ds} + \frac{\vec{ds}}{ds}\wedge\frac{\partial\vec{\Psi}}{\partial s}, \quad (6)$$

and that it is the dot product of $\tilde{ds}$ and its dual $\tilde{\partial}_s$ that coincides with $\frac{d\psi}{ds}ds$. This term can also be directly related to the Hamilton-Jacobi characteristic function associated with a coherent set of natural motion [3]. Indeed, to remove any ambiguity, we begin with the following definitions:

Definition 1 *A* coherent set *of natural motions associated with a particle in* $(\mathcal{M}, g)$ *is a set of curves* $S = \{\sigma(\lambda;\nu)\}$ *with momentum* $\mathbf{p}(\lambda) = \mathbf{p}(\lambda;\nu)$ *and energy* $H(\lambda) = H(\lambda;\nu)$*,*$\lambda$ *a parameter along the curve, such that*

$$\oint(\mathbf{pdx} - \mathbf{Hdt}) = \mathbf{0}. \quad (7)$$

Equivalently, $dW = \mathbf{pdx} - Hdt$ is an exact differential associated with the motion of a particle with momentum $\mathbf{p}$ and energy H, where $\mathbf{pdx} = (p_1dx_1 + p_2dx_2 + p_3dx_3)$. W is usually referred to as the Hamilton-Jacobi function associated with the coherent set of motions. Note, also, that a natural motion presupposes the existence of such functions. Later, in the next section we will show that such natural motions always obey Hamilton's equations of motion. However, for the moment we will focus on the kinematics associated with the Dirac equation. This brings us to the following lemmas and corollaries.

Lemma 1 *Let $F(\mathbf{x}, t)$ and $\psi(F)$ be differentiable functions then $p^a = \frac{\partial F}{\partial x^a}$ iff $p^a \psi' = \frac{\partial \psi}{\partial x^a}$ where $\psi' = \frac{d\psi}{dF}$.*

Proof: Trivial. It is sufficient to substitute and use the chain rule.

The following corollary immediately follows:

Corollary 1 *If $F = W(\lambda) = \int \eta^{ab} p_a dx_b = \int \boldsymbol{p} d\boldsymbol{x} - H dt$ is the Hamilton-Jacobi function for a natural motion along a curve $\sigma(\lambda)$, then*

$$\gamma^a \frac{\partial W}{\partial x_a} = \gamma^a p_a \qquad \Longleftrightarrow \qquad \gamma^a \frac{\partial \psi(W)}{\partial x_a} = \gamma^a p_a \psi'(W). \tag{8}$$

Proof: By definition of the Hamilton-Jacobi function $\frac{\partial W}{\partial x_a} = p_a$. Writing this in spinor notation means $\gamma^a \frac{\partial W}{\partial x_a} = \gamma^a p_a$. The result follows from the lemma.

In addition, based on the definition of a natural motion, it is possible to prove the existence of a Hamilton-Jacobi function which is directly related to the metric structure $ds^2 = dx^a dx_a$ of the space.

Lemma 2 *Let $ds^2 = dx^a dx_a$ define a metric structure along a curve $x^a = x^a(\lambda)$, and let*

$$W = \int_\sigma m \frac{ds}{d\lambda} ds = \int \boldsymbol{p} d\boldsymbol{x} - mc^2 dt, \qquad \textit{where} \quad p^a = m \frac{dx^a}{d\lambda}, \; m = m_o \frac{d\lambda}{ds} \tag{9}$$

then there exists a natural motion associated with W such that $\frac{\partial W}{\partial x^a} = p^a$ along the curve $(x^a(\lambda))$.

Proof: It is sufficient to show that such a function can be constructed locally at each point on the Manifold. First, parameterize the curve in terms of its arc length, so that $\frac{\partial W}{\partial x^a} = \frac{dx^a}{ds}$. Taking a Taylor expansion about x_o gives $x^a = x_0^a(\nu) + \left(\frac{dx^a}{ds}\right)_o ds + o(ds^2)$ and by definition of differential $dx^a = \left(\frac{dx^a}{ds}\right)_o ds$ in the tangent plane at $x_0(\nu)$, for each ν. Therefore, on integrating we obtain $W = \int_o (p_o^a dx^a) + w_o = p_o^a x^a + w_o(\nu)$ at x_o, where $w_o(\nu)$ is a constant along a given curve. The result follows.

Remark: The lemma has shown the existence of W at every point on the manifold by showing that W can also be constructed in terms of a local tetrad as a linear function. In effect, this means that we are defining the Hamilton-Jacobi along a geodesic passing through the point x_o. In practice as we will see in the example below, it is sufficient to integrate $\frac{\partial W}{\partial x^a} = \frac{dx^a}{ds}$ to obtain W. Moreover, although $p_o = p(x_o)$ is a constant along the geodesic passing through x_o, this does not mean that $p^a = \frac{\partial W}{\partial x^a}$ is constant along the curve. Indeed, this latter condition only follows, if motion is along a geodesic.

Example: Consider a motion of a particle of rest mass m_o fired into the air without resistance. The Equation of Motion for such a projectile in Minkowski space will be given by

$$\mathbf{F} = -m_o g \mathbf{j}, \tag{10}$$

or equivalently, denoting $\frac{dx}{d\tau} = \dot{x}$,

$$m_o \ddot{x} = 0 \qquad m_o \ddot{y} = -m_o g. \tag{11}$$

Solving the equations gives

$$x = x_0 + u_x \tau \quad \text{and} \qquad y = y_0 + u_y \tau - \frac{1}{2} g \tau^2 \tag{12}$$

.

Conversely beginning with the metric of the particle associated with this motion, and using equation (12), we obtain

$$ds^2 \;\; = \;\; dx^2 + dy^2 - c^2 dt^2$$

$$\text{iff} \qquad m_o \frac{ds}{d\tau} ds = m_o \dot{x} dx + m_o \dot{y} dy - m_o c^2 \dot{t} dt$$
$$\text{iff} \qquad m_o \frac{ds}{d\tau} ds = m_o u_x dx + m_o (u_y - gt) dy - m_o c^2 \dot{t} dt.$$

If we now let $dW = m(ds/d\tau)ds$ and require that W be an exact differential, we obtain $W = m_o u_x x + m_o(u_y - gt)y - m_o c^2 t + w_o$. Indeed, on taking partial derivatives, we find $\frac{\partial W}{\partial x} = m_o u_x = p_x$, $\frac{\partial W}{\partial y} = m_o(u_y - gt) = p_y$ and $\frac{\partial W}{\partial t} = -m_o g y - m_o c^2 = -m_o c^2 \dot{t}$, which is the correct result.

Equation (6) represents the most general form of a "wave-equation" with respect to a tetrad coordinate system associated with a particle moving along a curve with tangent vector $(\frac{dt}{ds}, -\frac{dx^1}{ds}, -\frac{dx^2}{ds}, -\frac{dx^3}{ds})$ and arbitrary wave function ψ. In the case where $[\tilde{\partial}_s \psi, \tilde{ds}] = 0$, then $\tilde{\partial}_s \psi$ and $\tilde{ds}$ are parallel or equivalently $(\partial_a \psi) = g(s)\frac{dx^a}{ds}$, for some function $g(s)$. In particular, if there exists an action W such that $g(s) = \frac{d\psi(W)}{dW} = \psi'(W)$, this leads to the following lemma:

Lemma 3 *If $\psi(W)$ is a differentiable function with respect to an action W such that $[\tilde{\partial}_s \psi, \tilde{ds}] = 0$ then*

$$(\tilde{\partial}_s \psi)\xi(p) = (\partial_s \psi)\xi(p), \tag{13}$$

which in the case of geodesic motion reduces to

$$\tilde{\partial}_s \Psi = \frac{d\Psi}{ds}, \qquad \textit{where} \qquad \Psi = \psi\xi. \tag{14}$$

Remark: *$(\tilde{\partial}_s \psi)\xi = \tilde{\partial}_s \Psi$ in general, since x is independent of p in phase space. However, $\frac{d\Psi}{ds} \neq \psi'(p)\xi$ unless motion is along a geodesic.*

Proof: First note that $\tilde{ds}\xi = ds\xi$ if and only if $\gamma_a p^a \xi = mc\xi$, and since $[\tilde{\partial}_s \psi, \tilde{ds}] = 0$ there exists simultaneous eigenvectors $\xi = \xi(p)$ such that $\tilde{ds}\xi = ds\xi$ and $(\tilde{\partial}_s \psi)\xi = \gamma^a p_a \psi' \xi(p) = mc\psi'(p) = (\partial_s \psi)\xi(p)$.
Also, $\xi(p)$ is constant along a geodesic and therefore

$$\tilde{\partial}_s \Psi = \frac{d\Psi}{ds}, \qquad \text{where} \qquad \Psi = \psi\xi.$$

The result follows.

Corollary 2 *Let $ds^2 = dx^a dx_a$ and W be as defined in Lemma (2). Let $\psi = \psi(W)$ then for a natural motion along the curve*

$$(\tilde{\partial}_s \psi)\xi(p) = (\partial_s \psi)\xi(p).$$

Proof: By Lemma (2) $p^a = \frac{\partial W}{\partial x^a}$. Therefore,

$$[\tilde{\partial}_s \psi, \tilde{ds}] = [\gamma^a \frac{\partial W}{\partial x^a}\psi'(W), \gamma^a dx_a] = \psi'(W)[\gamma^a p_a, \gamma^a dx_a] = 0. \tag{15}$$

The result follows by Lemma (3)

Remark: We refer to Equation (13) as a generalized Dirac equation. It reduces to the usual form of the Dirac equation if we let $\psi = e^{kW}$, where $k = \frac{i}{\hbar}$:

Corollary 3 *Let $\psi = e^{kW}$ where W is the action function defined in Cor. 1 and $k = \frac{i}{\hbar}$ then*

$$\gamma^a \frac{\partial \Psi}{\partial x^a} = \frac{i}{\hbar} m \Psi. \tag{16}$$

Proof: $\tilde{\partial}_s \psi \xi = \gamma^a p_a \psi \xi = \frac{\partial \psi}{\partial s}\xi = \frac{i}{\hbar} m \psi \xi$ and Equation (14) reduces to the conventional Dirac equation

$$\gamma^a \frac{\partial \Psi}{\partial x^a} = \frac{i}{\hbar} m \Psi. \tag{17}$$

The above equation can be rewritten in the conventional form, if we multiply across by $i\hbar\gamma^0$ and define $\alpha^0 = -\gamma^0$, $\alpha^a = -\gamma^0\gamma^a$ to get

$$\left[-i\hbar\left(\alpha_1 \frac{\partial}{\partial x_1} + \alpha_2 \frac{\partial}{\partial x_2} + \alpha_3 \frac{\partial}{\partial x_3}\right) + \alpha_0 m\right]\Psi = E\Psi \qquad \text{where} \qquad \Psi = \psi\xi. \tag{18}$$

Next, we prove a theorem that relates the Lie derivative and the Dirac equation for a particle in a closed system.

Theorem 1 *Consider a coherent set of motions $\{\sigma(\lambda;\nu)\}$ with tangent vectors $\frac{dx^a}{ds}$ associated with the metrics $ds^2 = dx^a dx_a$ then the Lie derivative $\mathcal{L}_u(p) = 0$ iff there exists a natural motion with action W such that $[\gamma^a \frac{\partial W}{\partial x^a}, \tilde{ds}] = 0$ and*

$$(\tilde{\partial}_s \psi)\xi(p_o) = (\partial_s \psi)\xi(p_o).$$

Proof:($\Rightarrow$) Since $\mathcal{L}_u(p) = 0$ there exists a coordinate system [2] such that $u^a = \delta^a_0$ and $p^a_{,0} = 0$. Therefore $p^a = mu^a + mv^a$ where m is constant and $\frac{\partial v^a}{\partial x^0} = 0$. Denote mu^a by p^a_o. Then $p^a_o = \frac{\partial W}{\partial x^a}$ defines a naural motion by Lemma 2 and consequently for $\psi = \psi(W)$, $[\tilde{\partial}_s\psi, \tilde{ds}] = 0$ for a natural motion with action W. It follows from Lemma 3 that

$$(\tilde{\partial}_s\psi)\xi(p_o) = (\partial_s\psi)\xi(p_o).$$

($\Leftarrow$) Recall $\psi = \psi(W)$. Therefore $\frac{\partial\psi}{\partial x^a} = \psi' \frac{\partial W}{\partial x^a}$. Now $[\gamma^a \frac{\partial W}{\partial x^a}, \tilde{ds}] = 0$ implies $\gamma^a \frac{\partial W}{\partial x^a} = g(s)\tilde{ds}$ for some function $g(s)$. But $(\tilde{\partial}_s\psi)\xi(p_o) = (\partial_s\psi)\xi(p_o)$. It follows that $\gamma^a \frac{\partial W}{\partial x^a} = \tilde{ds}$. Let $p^a_o = \frac{\partial W}{\partial x^a} = m\frac{dx^a}{d\lambda}$. Define $p^a = p^a_o + mv^a$, where u^a is a killing vector for v^a. It now follows that

$$\begin{aligned} \mathcal{L}_u(p^a) &= \mathcal{L}_u(p_o) + \mathcal{L}_u(mv^a) \\ &= (p^a_o)_{;b}u^b - u^a_{;b}p^b_o \\ &= mu^a_{;b}u^b - mu^a_{;b}u^b \\ &= 0. \end{aligned}$$

The theorem has been proven.

Remark: The theorem states that the Dirac equation associated with a particle exists and is defined locally if and only if the wave function is Lie transported along the curve whose action is W.

By way of concluding this section, we make some final observations:

- In the case of a function ψ such that $[\tilde{\partial}_s\psi, \tilde{ds}] \neq 0$, it is possible to replace $\tilde{\partial}_s\psi$ with the projected cosine of $\tilde{\partial}_s\psi$ along $\tilde{ds}$, which by definition is parallel to $\tilde{ds}$ and varies along the curve such that $\tilde{\partial}_s\psi = g(s)\tilde{\partial}_s\phi(s) + \gamma_a k^a$, $\frac{dx^a}{ds}$ is parallel to $\tilde{\partial}_s\phi(s)$ and a killing vector for k^a and $(\tilde{\partial}_s\phi)\xi(p_o) = (\partial_s\phi)\xi(p_o)$, where $p^a_0 = \frac{\partial W}{\partial x^a}$ and $\phi = \phi(W)$.
- The Hamilton-Jacobi function can be re-written in covariant form for a general coordinate system as follows:

$$dW = g^{\mu\nu}p_\mu dx_\nu, \tag{19}$$

with the corresponding wave equation

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu}\xi \quad = \quad \gamma^\mu p_\mu \psi' \xi \tag{20}$$

associated with the action along a curve, provided $2g^{\mu\nu} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$.

- The generalized Dirac equation

$$\gamma^\mu \frac{\partial \psi}{\partial x^\mu}\xi \quad = \quad \frac{\partial \psi}{\partial s}\xi. \tag{21}$$

can be defined along an arbitrary curve and is always covariant.

- The Schroedinger equation can be derived as an approximation to the Dirac equation (14) defined along a geodesic. To see this multiply Eqn. (14) by $\hbar^2/2m\tilde{\partial}_s$ to get

$$\frac{-\hbar^2}{2m}\left[\sum_{i=1}^{3}\frac{\partial^2 \Psi}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2}\right] \quad = \quad \frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial s^2} \tag{22}$$

$$\text{iff} \qquad \frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{\partial^2 \Psi}{\partial x_i^2} - E\Psi \quad = \quad -V\Psi \tag{23}$$

$$\text{iff} \qquad \frac{\hbar^2}{2m}\sum_{i=1}^{3}\frac{\partial^2 \Psi}{\partial x_i^2} + V\Psi \quad = \quad E\Psi, \tag{24}$$

where Ψ is a simultaneous eigenvector of $\tilde{\partial}_s$, $\frac{\partial^2 \Psi}{\partial t^2}$ and $\frac{\partial^2 \Psi}{\partial s^2}$. In general such simultaneous eigenvectors do not exist. However, in the limit as the velocity of the particle in the tetrad frame approaches 0 then $\frac{\partial \Psi}{\partial s} \to \frac{\partial \Psi}{\partial t}$ and the equation will then be approximately correct.

- The above approach deepens our understanding of the Principle of Complementarity. The particle properties should be directly associated with the metric. The wave properties emerge from Equation (13).

- It should be clear that the strict form of the Dirac equation (13) pertains to the kinematics and not the dynamics of the motion. It describes the kinematics with respect to a local tetrad. The dynamics requires further work, which we will do in the next section. Indeed, the restriction of the motion to geodesics also explains why we obtain distinct energy and momentum levels. Geodesic motion presupposes constant momentum and Energy. Quantum mechanics associates these constants with quantum numbers.

3 Non-geodesic Motion associated with the Hamiltonian

In the previous section we related the Hamilton-Jacobi characteristic function directly to the general form of the wave equation of quantum mechanics. At the same time because of the equivalence principle it was noted that the general form of the wave equation is determined only locally and not globally, especially when we consider motion along a non-geodesic. Indeed, the existence of non-geodesics suggests that other factors other than gravity are at work. The dynamics in such cases is usually analyzed in terms of test particles. We will continue then for the purpose of this article to use a somewhat "classical" approach to quantum mechanics, in that we will continue to associate a wave ψ and a generalized Dirac equation with a particle moving along a curve. Moreover, from a mathematical perspective ψ can be an L^p function. However, for the purpose of quantum mechanics, we will take $\psi \in L^2$ or $\psi \in L^2 \otimes H$ where H is a Hilbert Space.

3.1 The Physics Interpretation

Although the wave function can be given a precise mathematical meaning both as an L^2 function and in terms of probability of the state of the system, from a physics perspective things are more nuanced. The word "state" can be assigned multiple interpretations depending on the physics of the system and on the question been asked. Indeed, in Lemma 1 no restrictions were put on the wave function, other than the fact that it was differentiable and could be written as $\psi = \psi(F)$. The state, therefore, may refer to position, momentum, force, temperature, potential, electric and magnetic fields etc., where F represents one of these characteristic of the system. For example, in the previous section we worked primarily with the action function W, while in this section we will work mainly with the Hamiltonian H.

In will also be helpful at this stage to say something about eigenvectors and their physical meaning. To work with with eigenvector equations essentially means to look for the invariant states of the system. For example, if A is an operator such that $A\psi = \pm\psi$ then ψ can be interpreted as either an axis of rotation or as the axis of reflection which remain invariant under the operation. These axes correspond to the stable states or the states of equilibrium of the system associated with the operator. One of the challenges then for physics is to find the equilibrium conditions associated with the relevant operators (such as the Hamiltonian or Spin operators), solve their eigenvector equations and then interpret their results. From a methodological perspective, it should be noted that when physical states are not stationary or invariant then they are more difficult to access. This can be seen in the uncertainty principle, where both the position operator x and the momentum operators p_x do not have the same eigenfunctions. Consequently, the physical system cannot be in both invariant states simultaneously and therefore both cannot be measured at the same time. Similarly, we will see below that the vectors $\gamma^a p_a$ and $\gamma^a \dot{p}_a$ do not commute, which is a way of saying that we cannot simultaneously observe the quantum states of the system associated with the Dirac equation and the dynamics associated with the Hamiltonian.

3.2 Hamilton's Equations of Motion

With these observations in mind, we derive Hamilton's equations from the action, and then reconsider Lemma 1 from the perspective of the Hamiltonian function. First note that if we are restricted to fixed tetrads in Minkowski space then we need to distinguish between differentiating with respect to the parameter t (which will be denoted by .) and with respect to τ. In this regard, for any differentiable function W, $\frac{dW}{d\tau} = \dot{W}\frac{dt}{d\tau}$. To derive the Hamiltonian equations proper, from the action, two (equivalent) approaches are possible. One approach can be found in standard texts on theoretical mechanics, like Synge and Griffith [3]. In the approach, presented here we exploit the fact that for a natural motion the action W should be an exact differential. Specifically,

$$W = \int \mathbf{p}d\mathbf{x} - Hdt = \int d(\mathbf{px} - Ht) - \mathbf{x}d\mathbf{p} + tdH. \tag{25}$$

The requirement that W be an exact differential gives

$$\frac{\partial W}{\partial x^a} = p^a \quad \text{and} \quad \frac{\partial W}{\partial t} = -H, \tag{26}$$

$$\frac{\partial W}{\partial p^a} = - x^a \quad \text{and} \quad \frac{\partial W}{\partial H} = t. \tag{27}$$

Differentiating this with respect to t gives

$$\frac{\partial \dot{W}}{\partial x^a} = \dot{p}^a \quad \text{and} \quad \frac{\partial \dot{W}}{\partial t} = -\dot{H}, \tag{28}$$

$$\frac{\partial \dot{W}}{\partial p^a} = -\dot{x}^a \quad \text{and} \quad \frac{\partial \dot{W}}{\partial H} = 1. \tag{29}$$

But $\frac{\partial \dot{W}}{\partial H} = 1$ implies $\dot{W} = H$ from which it follows that for a natural motion

$$\dot{x}^a = \frac{\partial H}{\partial p^a}, \qquad \dot{p}^a = -\frac{\partial H}{\partial x^a} \qquad \text{and} \qquad \dot{H} = -\frac{\partial H}{\partial t}. \tag{30}$$

Before proceeding any further, we note that we have until now worked with a manifold of signature $+2$, in other words within a space-like environment. On the other hand, if we switch to a time-like event, the signature of the manifold becomes -2, and we obtain the usual form of Hamilton's equations:

$$\dot{x}^a = -\frac{\partial H}{\partial p^a}, \qquad \dot{p}^a = \frac{\partial H}{\partial x^a} \qquad \text{and} \qquad \dot{H} = \frac{\partial H}{\partial t}. \tag{31}$$

We also note that both forms of the equations are valid and give a deeper understanding of the relationship between positive and negative energy levels in quantum mechanics, in terms of time-like and space-like motion. Furthermore, these canonical equations of motion can be re-expressed in terms of the proper time τ with respect to a local tetrad coordinate system as follows:

$$\frac{dx^a}{d\tau} = \eta^{ab}\frac{\partial K}{\partial p^b}, \qquad \frac{dp^a}{d\tau} = -\eta^{ab}\frac{\partial K}{\partial x^b}, \tag{32}$$

where $K \equiv H\frac{\partial t}{\partial \tau}$. These are the same equations assumed by Horwitz and Schieve in their work on the Gibb's ensemble[6]. In terms of a generalized coordinate system both the space-like and time-like equations of motion can be subsumed into the covariant form:

$$\frac{dx^\mu}{d\tau} = g^{\mu\nu}\frac{\partial K}{\partial p^\nu}, \qquad \frac{Dp^\mu}{d\tau} = -g^{\mu\nu}\frac{\partial K}{\partial x^\nu}, \tag{33}$$

where $\frac{Dp^\mu}{d\tau} = p^\mu_{;}u^\nu$ is the covariant derivative.

3.3 Statistical Mechanics

When W is associated with a metric (see Lemma 2) then $H = mc^2$ can be identified with the Hamiltonian as it appears in the Hamiltonian-Jacobi equation (26). It now follows from Lemma 1 and Lemma (2) that there exists a differentiable function $\psi = \psi(H)$ such that

$$\frac{\partial \psi}{\partial x^a} = -\psi'\dot{p}^a \tag{34}$$

which can be re-written in spinor notation as

$$\gamma^a\frac{\partial \psi}{\partial x^a} = -\gamma^a\psi'\dot{p}^a. \tag{35}$$

Taking the inner product $\frac{1}{2}\{\gamma^a\frac{\partial\psi(H)}{\partial x^a}, \gamma^a p_a\}$ gives

$$\frac{d\psi(H)}{dt} = -\psi'(H)\dot{p}^a p_a. \tag{36}$$

In particular when $\psi' = k\psi$, an eigenvector, and k a constant, (as will happen when motion is along a geodesic) solving for ψ gives

$$\psi = e^{-\frac{k}{2}p^a p_a}. \tag{37}$$

We shall call $\psi(H)$ the stationary state of the system. In the case of n independent particles the the joint wave function becomes

$$\psi = e^{-\frac{k}{2}\sum_1^n p^a p_a}. \tag{38}$$

Moreover, if the motion is along a geodesic then $H = constant$ and $p^a p_a = mc^2$ for each particle Consequently,

$$\psi = c \exp\left[-\frac{k}{2} m \sum_n (-\dot{t}^2 + \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\right] = e^{\frac{k}{2} nmc^2}. \tag{39}$$

Defining $T = \frac{1}{k_B k}$, to be the temperature, where k_B is Boltzmann's constant, gives for any t the Maxwell-Boltzmann statistics for free particles:

$$\psi_t = c \exp\left(\frac{-\frac{1}{2}m}{kT} \Sigma_n (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\right). \tag{40}$$

Note, that c represents the speed of light in a vacuum. This equation can be interpreted to mean that the system is in equilibrium, with total conserved energy $\sum_n mc^2$. In other words the same Maxwell distribution occurs for each t. Moreover, if T is not constant then $k = k(T)$ also varies and in this case, on solving for ψ one obtains

$$\psi = e^{\int k(T) p_a \dot{p}^a dt}. \tag{41}$$

Also from equation (36) we should note that $\frac{\psi}{\psi'}$ is always an exact differential, and incorporates $k(T)$. If $k = k(T)$ has no explicit time dependence then ψ will define a stationary non-equilibrium state. On the other hand if there is an explicit time dependence then the system would be also non-stationary. Note in that in the case of very large n, T is proportional to the sample variance of the velocities $\sum_n \frac{(u_i - \bar{u})^2}{n-1}$.

3.4 Non-commutative Operators

Although Lemma 1 can be applied both to the action of the system W and the Hamiltonian H it should be noted that in general the operators $[\tilde{\partial}_s \psi(W), \tilde{\partial}_s \psi(H)] \neq 0$. To see this, note that

$$\tilde{\partial}\psi(W) = \gamma^a \frac{\partial \psi(W)}{\partial x^a} = \gamma^a p_a \psi'(W) \tag{42}$$

and

$$\tilde{\partial}\psi(H) = \gamma^a \frac{\partial \psi(H)}{\partial x^a} = \gamma^a \dot{p}_a \psi'(H) \tag{43}$$

and unless $\dot{p}^a = g(t)p^a$ for some function $g(t)$ then they cannot commute. Indeed, the condition $\dot{p}^a = g(t)p^a$ is equivalent to $x^a = \exp(\int^t f(\omega)d\omega)$ independently of a, which defines a non affine parameter along a geodesic. Consequently, there do not exist simultaneous eigenvectors, and both operators cannot be simultaneously measured, except along a geodesic. This in itself explains some of the difficulty with dynamics in quantum mechanics.

4 Conclusion

The article has attempted to establish a relationship between the metrics of General Relativity and Quantum Mechanics. This has been achieved by first relating the metric structure of spacetime to the Hamilton-Jacobi function and then using this relationship to derive a Generalized Dirac equation. In addition we have derived Hamilton's equations of motion directly from the Hamilton-Jacobi equation both for space-like and time-like regions, and then used these equations to determine a dynamical wave equation for the evolution of the system. These equations also give a deeper understanding of the relationship between positive and negative energy levels in quantum mechanics, in terms of time-like and space-like motion respectively. In addition, the dynamical equations derived do not commute with the generalized Dirac equation and consequently cannot be measured simultaneously. The dynamical equations also permit the derivation of the statistical mechanics of the system. Indeed, in this paper the Maxwell-Boltzmann distribution was derived directly from the equations of motion.

Finally, it should be noted that we have restricted ourselves to scalar fields as defined in the introduction. However, in its most general form, we can write

$$\gamma^a \frac{\partial \psi}{\partial x^a} = \phi \tag{44}$$

where ϕ would be defined by the physics of the problem. For example, Maxwell's equations in Minkowski space can be written in spinor form as

$$i\alpha^a \frac{\partial \psi}{\partial x^a} = -4\pi\phi, \tag{45}$$

where $\phi_0 = \rho$ is charge density, and $\phi_a = j_a,\ a \in \{1, 2, 3\}$ is a current density. Also in this case, $\phi_0 = 0$ and $\phi_a = H_a - iE_a$, where H_a and E_a

are the magnetic and electric fields respectively [7]. Such cases would need further study.

Acknowledgement: I would like to thank Prof. Lamberto Rondoni from the *Politecnico di Torino* for his valuable suggestions and input while I was writing this paper.

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