Entanglement and nonclassicality for multi-mode radiation field states

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Abstract

Nonclassicality in the sense of quantum optics is a prerequisite for entanglement in multi-mode radiation states. In this work we bring out the possibilities of passing from the former to the latter, via action of classicality preserving systems like beamsplitters, in a transparent manner. For single mode states, a complete description of nonclassicality is available via the classical theory of moments, as a set of necessary and sufficient conditions on the photon number distribution. We show that when the mode is coupled to an ancilla in any coherent state, and the system is then acted upon by a beamsplitter, these conditions turn exactly into signatures of NPT entanglement of the output state. Since the classical moment problem does not generalize to two or more modes, we turn in these cases to other familiar sufficient but not necessary conditions for nonclassicality, namely the Mandel parameter criterion and its extensions. We generalize the Mandel matrix from one-mode states to the two-mode situation, leading to a natural classification of states with varying levels of nonclassicality. For two–mode states we present a single test that can, if successful, simultaneously show nonclassicality as well as NPT entanglement. We also develop a test for NPT entanglement after beamsplitter action on a nonclassical state, tracing carefully the way in which it goes beyond the Mandel nonclassicality test. The result of three–mode beamsplitter action after coupling to an ancilla in the ground state is treated in the same spirit. The concept of genuine tripartite entanglement, and scalar measures of nonclassicality at the Mandel level for two-mode systems, are discussed. Numerous examples illustrating all these concepts are presented.

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I. INTRODUCTION

States of multi-mode quantized radiation fields are, by very definition, nonclassical in nature. However for many purposes it is useful to characterize certain states as displaying nonclassical features in a particularly prominent or manifest manner. The quantum optical concept of nonclassicality, based on the diagonal coherent state representation [1], and the notion of entanglement [2], are two such important nonclassical features displayed by some states but not by others. Whereas the former has played an important role from the early days of quantum optics [1, 3–12], the latter has received enormous attention more recently, with the development of the theory of quantum information [13–17]. Of course entanglement is meaningful only in the case of two or more modes of radiation, while nonclassicality is a useful concept even at the single-mode level.

The main aim of this work is to study the relationships between quantum optical nonclassicality and entanglement of various kinds, and the possibility of generating the latter given the former. In this context the asymmetry between these two features must be carefully appreciated. It is well known that quantum optical nonclassicality is a prerequisite for entanglement [18]. Whereas every entangled state is nonclassical, a nonclassical state may be separable (tensor product of nonclassical states, for instance) or entangled. Thus entangled states are a proper subset of nonclassical ones. Then the following questions become meaningful. For every signature of nonclassicality, how much further must one go and what additional conditions have to be met before one is assured of entanglement (of specified type)? Given a nonclassical separable state, can it be transformed into an entangled state through a physically realizable process using classicality preserving systems?

In speaking in this sense of converting nonclassicality into entanglement we have in mind the use of passive devices such as beamsplitters [19, 20, 34] which act on several modes of a given system to produce unitary linear combinations of the annihilation operators of the modes. Such systems preserve total photon number and, further, they cannot create or destroy nonclassicality. A general study must include developing and clearly specifying signatures of various levels of nonclassicality on the one hand, and of entanglement on the other, and then examining how one can reach the latter starting from the former, using these classicality preserving passive systems.

Two comments by way of clarification are in order before we describe the manner in which the contents of this paper are organized. First, it should be stressed that our concern in this paper is (almost exclusively) with non-Gaussian states. Nonclassicality of Gaussian states is necessarily of the squeezing type (quadrature squeezing), and a beam splitter can convert such a nonclassicality into entanglement. It is essentially this Gaussian scenario that largely formed the work-bench during the early years of the theory of entanglement for continuous variable (canonical) systems, elevating the role of the symplectic group of linear canonical transformations and its unitary (metaplectic) representation. The interplay between nonclassicality and entanglement in the Gaussian case has been long appreciated. For instance, the principal result of Ref.[15] can be phrased thus, as emphasised by Kimble recently [21]: A two-mode (mixed) Gaussian state is separable if and only if its nonclassicality can be removed by local linear canonical transformations — entanglement is the same as nonlocal nonclassicality.

More recently, however, there has emerged considerable interest in non-Gaussian states [22–33]. Ref. [31–33] may be consulted for a good introduction to the literature. Nonclassicality signature of non-Gaussian states is not restricted to quadrature squeezing — the state can be antibunched, for instance. A non-Gaussian state which is neither squeezed nor antibunched could exhibit nonclassicality signature in the higher moments of the photon count statistics or photon number distribution [9]. Non-Gaussian states thus exhibit a rich variety of nonclassicality signatures.

The second comment is in respect of beam splitters; we make extensive use of this gadget in the present work. In the domain of quantum optics, nonclassicality generating operations like squeezing are considered to be expensive resources whereas beam splitter, being a passive device, is rightly considered inexpensive. The latter is passive in the sense that it cannot alter the total number of photons in the pair of modes it couples, and also in the stronger sense that it cannot create or destroy nonclassicality. It thus makes sense to ask to what extent the expensive nonclassicality resource can be converted into the (expensive) entanglement resource using the inexpensive beam splitter resource. One should, however, appreciate that the beamsplitter effects a *joint* unitary transformation on the pair of modes it couples, and hence will be considered an expensive resource in a different context like a pair of nano mechanical oscillators. Our analysis thus applies to the specific context of quantum optics.

The contents of this paper are arranged as follows. Section II defines the concepts of quantum optical classicality — QO-cl and nonclassicality — QO-noncl, for general single-mode fields. In the phase invariant case they are entirely stated in terms of the photon number distribution [35]. This is possible thanks to the result of the classical Stieltjes moment problem [36]. It is then shown that if such a single-mode state is coupled to an ancilla in any coherent state and passed through any nontrivial $U(2)$ beamsplitter, the resulting two-mode output state shows NPT entanglement [37] precisely when the input single-mode state is quantum optically nonclassical [38, 39]. In this case the signatures of the two coincide exactly. Section III defines quantum optical classicality and nonclassicality for general two-mode states. We then describe a single test which if successful is able to establish simultaneously both the nonclassicality and NPT entanglement of a given two-mode state. Of course proof of entanglement automatically implies proof of nonclassicality, so the interest here is in the structure of the expressions used in the test. Since theorems of the moment type are not available for two or more modes, we describe in Section IV a sufficient criterion for one-mode nonclassicality originally due to Mandel [8], and extend it to two-mode systems as well. All the later considerations of this work are based on this Mandel level of nonclassicality which is of course weaker than the general notion of nonclassicality. Several levels or types of nonclassicality in this sense are described, and three examples to illustrate the ideas are presented.

In Section V we study states of two-mode systems for which both notions of nonclassicality and entanglement make sense. We find conditions for nonclassical states to become NPT entangled via BS action, and show in detail the way in which these conditions go beyond those that guarantee Mandel type nonclassicality. This analysis is applied to two examples to see the formalism in action. Distillability of the resulting state is demonstrated in one case. Section VI takes up the theme of Section II and extends it to higher number of modes. Thus we couple a two-mode state which is nonclassical to a third ancilla mode in vacuum, pass this three-mode state through a beamsplitter acting on the three modes, and develop tests for NPT entanglement in the output state. As in Section V, here too the precise manner in which testing for entanglement goes beyond testing for nonclassicality of the Mandel type is emphasized. Section VII discusses the subtle notion of genuine tripartite entanglement, while Section VIII brings out some features of two-mode Mandel level nonclassicality and beamsplitter action. All the ideas in Sections VI, VII, and VIII are illustrated through several examples with tractable analytical structures. Section IX contains some concluding remarks.

II. SINGLE-MODE FIELDS

Let the photon annihilation and creation operators for the concerned mode be written as $\hat{a}$ and $\hat{a}^\dagger$, obeying the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{2.1}$$

The familiar Fock states and coherent states are

$$\begin{aligned} |n\rangle &= (n!)^{-1/2}(\hat{a}^\dagger)^n|0\rangle, \quad n = 0, 1, 2, \cdots \;; \\ |z_a\rangle &= e^{-\frac{1}{2}|z_a|^2} \sum_{n_a=0}^{\infty} \frac{z_a^{n_a}}{\sqrt{n_a!}}|n_a\rangle\,, \quad \hat{a}|z_a\rangle = z_a|z_a\rangle, \quad z_a \in \mathcal{C}. \end{aligned} \tag{2.2}$$

The Fock states $\{|n\rangle\}$ form an orthonormal basis for the space of all single-mode states, while the coherent states $\{|z_a\rangle\}$ form a normalized nonorthogonal overcomplete system.

For a single mode there is no meaning to entanglement, only a separation of states into the quantum optical classical (QO-cl) and the quantum optical nonclassical (QO-noncl) types. This is based on the diagonal coherent state representation of a general (pure or mixed) state $\hat{\rho}^{(a)}$:

$$\hat{\rho}^{(a)} = \pi^{-1} \int_C d^2 z_a \, \phi(z_a)|z_a\rangle\langle z_a|. \tag{2.3}$$

The properties of the real diagonal representation weight $\phi(z_a)$ determine the nature of $\hat{\rho}^{(a)}$ [1] :

$$\begin{aligned} \phi(z_a) \geq 0 \;&\Leftrightarrow\; \hat{\rho}^{(a)} \text{ QO-cl;} \\ \phi(z_a) \ngeq 0 \;&\Leftrightarrow\; \hat{\rho}^{(a)} \text{ QO-noncl.} \end{aligned} \tag{2.4}$$

In the former case, the meaning is that $\phi(z_a)$ is pointwise nonnegative; it is then mathematically a valid probability density in phase space.

If one is interested only in the expectation values of number conserving observables, i.e., of operators commuting with $\hat{N}_a = \hat{a}^\dagger \hat{a}$, and hence diagonal in the Fock basis, it suffices to use the phase averaged form $\hat{\rho}_D^{(a)}$ of $\hat{\rho}^{(a)}$:

$$\begin{aligned} \hat{\rho}_D^{(a)} &= \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta \hat{N}_a} \hat{\rho}^{(a)} e^{-i\theta \hat{N}_a} \\ &= \frac{1}{\pi} \int d^2 z_a P(I_a)|z_a\rangle\langle z_a|, \quad I_a = |z_a|^2, \\ P(I_a) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \phi(z_a e^{i\theta})\,, \\ \hat{\rho}_D^{(a)} \hat{N}_a &= \hat{N}_a \hat{\rho}_D^{(a)}. \end{aligned} \tag{2.5}$$

Clearly $\hat{\rho}_D^{(a)}$ is a physical state, equivalent to $\hat{\rho}^{(a)}$ as far as expectation values of number conserving operators are concerned. Moreover, the quantity $P(I_a)$ is the diagonal representation weight for $\hat{\rho}_D^{(a)}$, so (2.4) leads to the coarse grained classification [11, 12]

$$\begin{aligned} P(I_a) \geq 0 \;&\Leftrightarrow\; \hat{\rho}_D^{(a)} \text{ QO-cl;} \\ P(I_a) \ngeq 0 \;&\Leftrightarrow\; \hat{\rho}_D^{(a)} \text{ QO-noncl.} \end{aligned} \tag{2.6}$$

Further, all information about $P(I_a)$ is contained in the photon number probabilities $p(n) \equiv \langle n|\hat{\rho}^{(a)}|n\rangle = \langle n|\hat{\rho}_D^{(a)}|n\rangle$ for $n = 0, 1, 2, \cdots$ [35]. It is convenient to call the set of probabilities $\{p(n)\}$ the photon number distribution (PND). Clearly, $\hat{\rho}_D^{(a)}$ can be explicitly written in terms of the PND and vice versa:

$$\begin{aligned} \hat{\rho}_D^{(a)} &= \sum_{n=0}^{\infty} p(n)|n\rangle\langle n|, \\ p(n) &= \int_0^{\infty} dI_a P(I_a) e^{-I_a} {I_a}^n/n! \geq 0. \end{aligned} \tag{2.7}$$

Therefore one can ask if the conditions (2.6) for the coarse grained QO-cl — QO-noncl divide can be explicitly given in terms of the PND $\{p(n)\}$. This is indeed possible [35], as a result of the classical analysis of the Stieltjes moment problem. It involves two infinite sequences of matrix positivity conditions, set up as follows. It is convenient to introduce the auxiliary quantities

$$q_n = n! p(n), \;\; n = 0, 1, 2, \cdots \tag{2.8}$$

Then define two real symmetric infinite dimensional matrices L and $\tilde{L}$ as follows:

$$\begin{aligned} L &= (L_{n'n}), \quad L_{n'n} = q_{n'+n}, \quad n', n = 0, 1, 2, \cdots; \\ \tilde{L} &= (\tilde{L}_{n'n}), \quad \tilde{L}_{n'n} = q_{n'+n+1}, \quad n', n = 0, 1, 2, \cdots. \end{aligned} \tag{2.9}$$

The diagonal elements L_{nn} of L are q_{2n}, while $\tilde{L}_{nn}$ of $\tilde{L}$ are q_{2n+1}. Using L and $\tilde{L}$, we can define a sequence of $(N+1)$-dimensional matrices $L^{(N)}$, $\tilde{L}^{(N)}$ as restrictions. Then the key result is [35]:

$$\begin{aligned} P(I_a) \geq 0 \;\;&\Leftrightarrow\;\; \{\hat{\rho}_D^{(a)} \text{ and PND } \{p(n)\} \text{ are QO-cl}\} \\ &\Leftrightarrow\;\; L^{(N)},\; \tilde{L}^{(N)} \geq 0, \;\; N = 0, 1, 2, \cdots; \\ L^{(N)} &= (L_{n'n} \;:\; n', n = 0, 1, 2, \cdots, N), \\ \tilde{L}^{(N)} &= (\tilde{L}_{n'n} \;:\; n', n = 0, 1, 2, \cdots, N). \end{aligned} \tag{2.10}$$

Both $L^{(N)}$ and $\tilde{L}^{(N)}$ are real symmetric $(N+1)$-dimensional matrices, made up of the intersections of the first $(N+1)$ rows and $(N+1)$ columns of L and $\tilde{L}$ respectively. Conversely we have:

$$\begin{aligned} P(I_a) \ngeq 0 \;&\Leftrightarrow\; \{\hat{\rho}_D^{(a)} \text{ and PND } \{p(n)\} \text{ are QO-noncl}\} \\ &\Leftrightarrow\; \text{some } L^{(N)} \ngeq 0 \text{ and or some } \tilde{L}^{(N)} \ngeq 0. \end{aligned} \tag{2.11}$$

As noted earlier, for one-mode states there is no concept of entanglement. However a QO-noncl state characterized by (2.11) can be converted to a state possessing NPT entanglement by adjoining it to a second ancilla mode, the b-mode, initially say in its vacuum state, and passing this two-mode state through a generic beamsplitter (BS) corresponding to an element of the group $U(2)$. This is seen as follows:

The b-mode brings in an added operator pair $\hat{b}$, $\hat{b}^\dagger$ obeying the same commutation relation (2.1); and each of $\hat{a}$, $\hat{a}^\dagger$ commutes with $\hat{b}$ and $\hat{b}^\dagger$. Let now

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \tag{2.12}$$

be a general element of the two-dimensional unitary group $U(2)$. The action of the corresponding BS on the operators of the two modes is given by conjugation with a unitary operator $\hat{U}$ acting on the two-mode Hilbert space, the dependence of $\hat{U}$ on u being left implicit [11, 34]:

$$\begin{aligned} &\hat{U}\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}\hat{U}^{-1} = u^\dagger \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \quad \hat{U}\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}\hat{U}^{-1} = u^T \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}; \\ &\hat{U}^{-1}\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}\hat{U} = u \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix}, \quad \hat{U}^{-1}\begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}\hat{U} = u^* \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix}. \end{aligned} \tag{2.13}$$

This unitary operator is 'passive' in the sense that

$$\hat{U}(\hat{N}_a + \hat{N}_b) = (\hat{N}_a + \hat{N}_b)\hat{U}. \tag{2.14}$$

It is also passive in another important sense in the present context: for any two-mode state the property of being QO-cl or QO-noncl is preserved upon passage through any BS, so BS action can neither create nor destroy nonclassicality. (The output state of a beam splitter is a coherent state if and only if the input is. This will be clarified and elaborated in the next Section).

We start with the a-mode state $\hat{\rho}_D^{(a)}$ which commutes with $\hat{N}_a$, and take as the two-mode input to the BS the separable (product) state

$$\hat{\rho}_{\rm in}^{(ab)} = \hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|. \tag{2.15}$$

This commutes with the total number operator $\hat{N}_a + \hat{N}_b$. Passage through the BS preserves this property and results in the output state

$$\begin{aligned}\hat{\rho}_{\rm out}^{(ab)} &= \hat{U}\hat{\rho}_{\rm in}^{(ab)}\hat{U}^{-1} \\ &= \hat{U}\sum_{n=0}^{\infty}\frac{p(n)}{n!}(\hat{a}^\dagger)^n|0,0\rangle\langle 0,0|(\hat{a})^n\hat{U}^{-1} \\ &= \sum_{n=0}^{\infty}\frac{p(n)}{n!}(u_{11}\hat{a}^\dagger + u_{21}\hat{b}^\dagger)^n|0,0\rangle\langle 0,0|(u_{11}^*\hat{a} + u_{21}^*\hat{b})^n.\end{aligned} \tag{2.16}$$

As a general notation let us now use $n, n', n'', \cdots$ to denote the eigenvalues of $\hat{N}_a$, and $m, m', m'', \cdots$ to denote eigenvalues of $\hat{N}_b$. Then the general Fock state matrix elements of $\hat{\rho}_{\rm out}^{(ab)}$ are:

$$\langle n', m'|\hat{\rho}_{\rm out}^{(ab)}|n, m\rangle = \delta_{n'+m', n+m}\, q_{n+m}\,\frac{u_{11}^{n'}u_{21}^{m'}}{\sqrt{n'!m'!}}\frac{u_{11}^{*n}u_{21}^{*m}}{\sqrt{n!m!}}. \tag{2.17}$$

To see whether $\hat{\rho}_{\rm out}^{(ab)}$ possesses NPT entanglement, we carry out the partial transpose (PT) operation by implementing it in the b-mode space in the Fock basis. This amounts to interchanging m and m' on the right hand side of Eq. (2.17), and gives:

$$\langle n', m'|\hat{\rho}_{\rm out}^{(ab)PT}|n, m\rangle = \delta_{n'+m, n+m'}\, q_{n+m'}\,\frac{u_{11}^{n'}u_{21}^{m}}{\sqrt{n'!m!}}\frac{u_{11}^{*n}u_{21}^{*m'}}{\sqrt{n!m'!}}. \tag{2.18}$$

The question now is whether this is a physical state i.e., if $\hat{\rho}_{\rm out}^{(ab)PT} \geq 0$

We now isolate two principal submatrices out of the matrix $\hat{\rho}_{\rm out}^{(ab)PT}$ in the Fock basis, which are closely related to the matrices L, $\tilde{L}$ of Eq. (2.9). The first one is the submatrix $H = (H_{n'n})$ obtained by taking $m' = n'$, $m = n$ in (2.18):

$$\begin{aligned}H_{n'n} &= \langle n', n'|\hat{\rho}_{\rm out}^{(ab)PT}|n, n\rangle \\ &= q_{n+n'}\frac{(u_{11}u_{21}^*)^{n'}}{n'!}\frac{(u_{11}^*u_{21})^n}{n!}, \;\; n', n = 0, 1, 2, \cdots; \\ H &= A^\dagger L A, \;\; A = {\rm diag}(\frac{(u_{11}^*u_{21})^n}{n!}, \;\; n = 0, 1, 2, \cdots).\end{aligned} \tag{2.19}$$

The second is the submatrix $\tilde{H} = (\tilde{H}_{n'n})$ obtained by taking $m' = n' + 1$, $m = n + 1$ in (2.18):

$$\tilde{H}_{n'n} = \langle n', n'+1|\hat{\rho}_{\rm out}^{(ab)PT}|n, n+1\rangle$$

$$
\begin{aligned}
&= q_{n+n'+1}\frac{(u_{11}u_{21}^*)^{n'}}{n'!}\frac{u_{21}^*}{\sqrt{n'+1}}\frac{(u_{11}^*u_{21})^n}{n!}\frac{u_{21}}{\sqrt{n+1}}\ ,\ \ n',n=0,1,2,\cdots;\\
\tilde{H} &= \tilde{A}^\dagger\tilde{L}\tilde{A},\ \tilde{A}=\mathrm{diag}(\frac{(u_{11}^*u_{21})^n}{n!}\frac{u_{21}}{\sqrt{n+1}},\ n=0,1,2,\cdots). \qquad (2.20)
\end{aligned}
$$

Invertibility of A and $\tilde{A}$ implies that H, $\tilde{H}\geq 0$ if and only if L, $\tilde{L}\geq 0$. We then see: every signature of $\hat{\rho}_D^{(a)}$ and the PND $\{p(n)\}$ being QO-noncl, such as some $L^{(N)}\ngeq 0$ or some $\tilde{L}^{(N)}\ngeq 0$, directly implies a corresponding signature of $\hat{\rho}_{\rm out}^{(ab)}$ being NPT entangled, since either $H\ngeq 0$ or $\tilde{H}\ngeq 0$, implying that $\hat{\rho}_{\rm out}^{(ab)PT}\ngeq 0$. It is in this precise sense that any nontrivial BS is able to convert input QO-nonclassicality of a single-mode to NPT entanglement of the output state $\hat{\rho}_{\rm out}^{(ab)}$, in a sense preserving the signature of nonclassicality.

A simple calculation shows that in the input state (2.15) of the ab-system, the b-mode need not be in the vacuum but could be in a general coherent state $|z_b\rangle_b$ for some complex nonzero z_b. Since

$$
\begin{aligned}
|z_b\rangle_b &= D^{(b)}(z_b)|0\rangle_b,\\
D^{(b)}(z_b) &= \exp(z_b\hat{b}^\dagger - z_b^*\hat{b}), \qquad (2.21)
\end{aligned}
$$

we have the following replacements for the previous Eqs. (2.15, 2.16) :

$$
\begin{aligned}
\hat{\rho}_{\rm in}^{(ab)} &= \hat{\rho}_D^{(a)}\otimes|z_b\rangle_{bb}\langle z_b|\\
&= D^{(b)}(z_b)\{\hat{\rho}_D^{(a)}\otimes|0\rangle_{bb}\langle 0|\}D^{(b)}(z_b)^{-1};\\
\hat{\rho}_{\rm out}^{(ab)} &= \hat{U}\hat{\rho}_{\rm in}^{(ab)}\hat{U}^{-1}\\
&= D^{(a)}(u_{12}z_b)D^{(b)}(u_{22}z_b)\{\hat{U}\hat{\rho}_D^{(a)}\otimes|0\rangle_{bb}\langle 0|\hat{U}^{-1}\}D^{(a)}(u_{12}z_b)^{-1}D^{(b)}(u_{22}z_b)^{-1}. \ (2.22)
\end{aligned}
$$

(In passing we note that this input state no longer commutes with $\hat{N}_a+\hat{N}_b$). Here we have used Eq. (2.13). Thus the matrix elements of this two-mode BS output state in the displaced orthonormal product basis $D^{(a)}(u_{12}z_b)D^{(b)}(u_{22}z_b)|n,m\rangle$ are identically the same as the matrix elements of $\hat{\rho}_{out}^{(ab)}$ of Eq. (2.16) in the Fock basis $|n,m\rangle$. Implementing the PT operation in this new product basis we recover the earlier results. Thus every signature of QO-noncl of $\hat{\rho}_D^{(a)}$ is transformed by a general BS action into a corresponding signature of NPT entanglement of the output two-mode state (2.22).

In both cases (2.16) and (2.22) we see that nontrivial BS action, while being passive and so maintaining any QO nonclassicality present in the input state, is able to convert an initially unentangled two-mode state into an NPT entangled state. This it can do provided there is some QO nonclassicality to begin with. Crudely speaking, what the beam splitter achieves in the present case is to take the input nonclassicality which resides 'locally' in the a-mode and convert it into nonclassicality residing 'nonlocally' as entanglement between the modes.

III. TWO-MODE FIELDS - GENERAL PROPERTIES AND AN ENTANGLEMENT TEST

We now consider specific new features encountered in the study of states of a two-mode system. A general two-mode state $\hat{\rho}^{(ab)}$ possesses the diagonal coherent state representation

$$\hat{\rho}^{(ab)} = \int\int \frac{d^2 z_a}{\pi} \frac{d^2 z_b}{\pi} \, \phi(z_a, z_b) |z_a, z_b\rangle\langle z_a, z_b| \tag{3.1}$$

in terms of the two-mode (product) coherent states $|z_a, z_b\rangle$. Analogous to Eq. (2.4), the properties of $\phi(z_a, z_b)$ determine the nature of $\hat{\rho}^{(ab)}$:

$$\begin{aligned} \phi(z_a, z_b) \geq 0 \;&\Leftrightarrow\; \hat{\rho}^{(ab)} \text{ QO-cl;} \\ \phi(z_a, z_b) \ngeq 0 \;&\Leftrightarrow\; \hat{\rho}^{(ab)} \text{ QO-noncl.} \end{aligned} \tag{3.2}$$

In the former case $\phi(z_a, z_b)$ has the properties of a probability distribution in the four dimensional phase space.

Now, apart from examining whether a given state $\hat{\rho}^{(ab)}$ is QO-cl or QO-noncl, we can also ask in the latter case whether it is entangled, and if so whether it is NPT type, distillable [40], etc. These added questions become meaningful. In fact we will develop later in this Section an interesting test or criterion which can witness simultaneously for QO-noncl assicality as well as NPT entanglement.

With respect to BS action (2.12, 2.13) representing general elements $u \in U(2)$, we note the following. Such action is nonlocal since the modes a and b get linearly mixed, in addition to being passive in the sense of conserving $\hat{N}_a + \hat{N}_b$. Since annihilation operators go into linear combinations of annihilation operators under this action, coherent states go into coherent states with the matrix u acting on the pair z_a, z_b as a column vector. Therefore convex sums of coherent states go into similar convex sums [41], $\phi(z_a, z_b)$ experiences a point transformation $\phi(z_a, z_b) \to \phi(u_{11} z_a + u_{12} z_b,\, u_{21} z_a + u_{22} z_b)$, and thus such BS action preserves the QO-cl or QO-noncl nature of the state $\hat{\rho}^{(ab)}$. This is another sense of BS action being passive.

On the other hand, while a QO-cl state has no entanglement, a QO-noncl state may be separable, i.e., unentangled, or may possess entanglement. Entangled states are a proper subset of QO-noncl states, and NPT entangled states are a further subset. Now BS action can cause a transition, within the QO-noncl subset, from a separable to an entangled state, in which case we can further enquire into the nature of the entanglement so obtained, whether it is NPT type etc. This is in fact the case in the transition from $\hat{\rho}_{\rm in}^{(ab)}$ of Eq. (2.15) to $\hat{\rho}_{\rm out}^{(ab)}$ of Eq. (2.16), or the transition in Eq. (2.22).

Continuing with the definitions of QO-cl and QO-noncl in Eq. (3.2), if we are interested only in the total number conserving matrix elements of various operators, i.e., of operators commuting with $\hat{N}_a+\hat{N}_b$, it suffices to work with the total phase averaged state

$$\begin{aligned} \hat{\rho}_D^{(ab)} &= \int_0^{2\pi}\frac{d\theta}{2\pi}e^{i\theta(\hat{N}_a+\hat{N}_b)}\hat{\rho}^{(ab)}e^{-i\theta(\hat{N}_a+\hat{N}_b)} \\ &= \int\frac{d^2z_a}{\pi}\frac{d^2z_b}{\pi}P(I_a,I_b,\theta)|z_a,z_b\rangle\langle z_a,z_b|, \\ I_a &= |z_a|^2,\ \ I_b=|z_b|^2,\ \ \theta=\arg z_a^*z_b, \\ P(I_a,I_b,\theta) &= \int_0^{2\pi}\frac{d\theta'}{2\pi}\phi(\sqrt{I_a}e^{-i\theta'},\sqrt{I_b}e^{i(\theta-\theta')}). \end{aligned} \tag{3.3}$$

This state is number conserving:

$$\begin{aligned} \hat{\rho}_D^{(ab)}(\hat{N}_a+\hat{N}_b) &= (\hat{N}_a+\hat{N}_b)\hat{\rho}_D^{(ab)}, \\ \langle n'm'|\hat{\rho}_D^{(ab)}|nm\rangle &= \delta_{n'+m',n+m}\langle n'm'|\hat{\rho}^{(ab)}|nm\rangle. \end{aligned} \tag{3.4}$$

Since the coarse grained $P(I_a,I_b,\theta)$ is the (real) diagonal representation weight of $\hat{\rho}_D^{ab}$, we have the following QO classification at this level [11, 12]:

$$\begin{aligned} P(I_a,I_b,\theta)\geq 0 \ &\Leftrightarrow\ \hat{\rho}_D^{(ab)} \text{ is QO-cl}, \\ P(I_a,I_b,\theta)\not\geq 0 \ &\Leftrightarrow\ \hat{\rho}_D^{(ab)} \text{ is QO-noncl}. \end{aligned} \tag{3.5}$$

We use this in the next Section.

We now revert to a general state $\hat{\rho}^{(ab)}$ and describe a test which, if it succeeds, *simultaneously establishes* both QO nonclassicality of $\hat{\rho}^{(ab)}$ and its NPT entanglement. It is clear of course, that any test which establishes the latter immediately implies the former; the interest here is in the structure of the test itself.

We set up an infinite matrix $\hat{N}$ with operator entries $\hat{N}_{jk,lm}$ where j,k,l,m run independently over $0,1,2,\cdots$. The pair jk denotes a 'row index' and takes in sequence the values

$00; 10, 01; 20, 11, 02; 30, 21, 12, 03; \cdots$. Similarly the 'column index' pair lm also takes these same values in the same sequence. We define the entries of $\hat{N}$ thus:

$$\hat{N}_{jk,lm} = \hat{N}^{\dagger}_{lm,jk} = \hat{a}^{\dagger j}\hat{b}^{\dagger k}\hat{a}^{l}\hat{b}^{m}. \tag{3.6}$$

Clearly $\hat{N} = \left(\hat{N}_{jk,lm}\right)$ is an infinite 'hermitian' matrix of operator entries. Note that these entries are in normal-ordered form. Starting with the diagonal representation (3.1), for any set of complex coefficients $\{c_{jk}\}$ and the associated positive semidefinite operator $\sum_{jk,lm} c^*_{jk}\hat{N}_{jk,lm}c_{lm}$, we always have:

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}^{(ab)}\sum_{jk,lm} c^*_{jk}\hat{N}_{jk,lm}c_{lm}) &= \mathrm{Tr}(\hat{\rho}^{(ab)}(\sum_{jk} c_{jk}\hat{a}^{j}\hat{b}^{k})^{\dagger}(\sum_{lm} c_{lm}\hat{a}^{l}\hat{b}^{m}))\\ &= \int\int \frac{d^2z_a}{\pi}\frac{d^2z_b}{\pi}\,\phi(z_a,z_b)|\sum_{lm} c_{lm}z_a^l z_b^m|^2 \geq 0.\end{aligned} \tag{3.7}$$

This is independent of $\phi(z_a, z_b)$ being classical or otherwise, because we have here the expectation value of a positive semidefinite hermitian operator. On the other hand, if we pass to the partial transpose $\hat{\rho}^{(ab)PT}$ of $\hat{\rho}^{(ab)}$, by performing transposition only in the space of states of the b-mode in the Fock basis, this will amount to everywhere replacing $\hat{b}^{\dagger j}\hat{b}^{m}$ by $\hat{b}^{\dagger m}\hat{b}^{j}$, since in the Fock basis $\hat{b}^{\dagger}$ and $\hat{b}$ are real [42]. Thus for the same positive semidefinite operator as in (3.7) we have:

$$\begin{aligned}\mathrm{Tr}(\hat{\rho}^{(ab)PT}\sum_{jk,lm} c^*_{jk}\hat{N}_{jk,lm}c_{lm}) &= \mathrm{Tr}(\hat{\rho}^{(ab)}\sum_{jk,lm} c^*_{jk}\hat{a}^{\dagger j}\hat{b}^{\dagger m}\hat{a}^{l}\hat{b}^{k}c_{lm})\\ &= \int\int \frac{d^2z_a}{\pi}\frac{d^2z_b}{\pi}\,\phi(z_a,z_b)\,|\sum_{lm} c_{lm}z_a^l z_b^{*m}|^2.\end{aligned} \tag{3.8}$$

Notice the difference in the integrands of the last integrals in (3.7) and (3.8); the latter integral is sure to be positive if $\hat{\rho}^{(ab)PT} \geq 0$, otherwise it could be negative.

Thus we arrive at a single step test for QO-nonclassicality and NPT entanglement of $\hat{\rho}^{(ab)}$. The above expression (3.8) being negative implies *two things simultaneously*:

$$\begin{aligned}&\text{(i) } \phi(z_a,z_b) \ngeq 0, \text{ hence } \hat{\rho}^{(ab)} \text{ is QO} - \text{noncl;}\\ &\text{(ii) } \hat{\rho}^{(ab)PT} \ngeq 0, \text{ and hence } \hat{\rho}^{(ab)} \text{ is NPT entangled.}\end{aligned} \tag{3.9}$$

As we said already, $\hat{\rho}^{(ab)}$ being NPT entangled already implies also its being QO-noncl. The point here is that the expression (3.8) being negative manifestly displays both properties of $\hat{\rho}^{(ab)}$ immediately.

This interesting result is an indication of the possibility, in suitable circumstances, of 'bridging the gap' between the characterization of QO nonclassicality and the further characterization of (NPT) entanglement for two-mode fields in an efficient manner. For the particular the test based on (3.8), there is no 'gap' at all, but as we will show this will not always be the case.

IV. MANDEL MATRICES AND STATE CLASSIFICATION FOR ONE AND TWO-MODE FIELDS

The discussion of single-mode states in Section II made use of the probabilities $p(n)$, $n = 0, 1, 2, \cdots$, individually. In terms of the moments of the PND, this means in principle that all its moments are involved. It is naturally possible to make much more limited statements about QO nonclassicality if one uses only the first and second moments, say, of the PND. This is the content of the Mandel criterion for QO nonclassicality stated in terms of the Mandel matrix (or Mandel Q parameter) associated with a given state. It is useful to begin by outlining this for single-mode states to set up notations, and then generalize to two modes.

Another motivation is that for two or more modes there are no moment theorems at all comparable in scope to the one stemming from the classical Stieltjes moment problem, and therefore no simple generalizations of Eqs. (2.10, 2.11) into necessary and sufficient conditions for QO classicality can be expected to be available.

A. Mandel matrices for single-mode states

Given a one-mode state $\hat{\rho}^{(a)}$, we construct by Eq. (2.5) the state $\hat{\rho}_D^{(a)}$ conserving $\hat{N}_a$. The fact that $\hat{\rho}_D^{(a)}$ is a physical state leads to a certain 2×2 real symmetric matrix, involving the first and second moments of the PND $\{p(n)\}$, being positive semidefinite:

$$\begin{aligned} \hat{\rho}_D^{(a)} \; &\geq \; 0 \Rightarrow \begin{pmatrix} 1 & \langle \hat{N}_a \rangle \\ \langle \hat{N}_a \rangle & \langle \hat{N}_a^2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & \langle \hat{a}^\dagger \hat{a} \rangle \\ \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle \end{pmatrix} \geq 0, \\ \langle \hat{N}_a \rangle \; &= \; \sum_{n=0}^{\infty} n p(n), \quad \langle \hat{N}_a^2 \rangle = \sum_{n=0}^{\infty} n^2 p(n). \end{aligned} \tag{4.1}$$

All expectation values here are in the state $\hat{\rho}_D^{(a)}$. The Mandel matrix associated with $\hat{\rho}_D^{(a)}$ is obtained by replacing the expectation value $\langle \hat{N}_a^2 \rangle$ by the expectation value of the normal ordered form $\hat{a}^{\dagger 2}\hat{a}^2$ of $\hat{N}_a^2$:

$$M^{(1)}(\hat{\rho}_D^{(a)}) = \begin{pmatrix} 1 & \langle \hat{a}^\dagger \hat{a} \rangle \\ \langle \hat{a}^\dagger \hat{a} \rangle & \langle \hat{a}^{\dagger 2}\hat{a}^2 \rangle \end{pmatrix}. \tag{4.2}$$

Here the superscript $^{(1)}$ indicates that we are considering states of a single-mode. While positivity of $\hat{\rho}_D^{(a)}$ implies the positivity of the 2×2 matrix in (4.1), it does not imply the positivity of $M^{(1)}(\hat{\rho}_D^{(a)})$. From Eq. (2.7) we easily obtain :

$$\begin{aligned} \langle \hat{a}^\dagger \hat{a} \rangle &= \sum_{n=0}^{\infty} n p(n) = \int_0^\infty dI_a P(I_a) I_a = \langle I_a \rangle, \\ \langle \hat{a}^\dagger \hat{a} \hat{a}^\dagger \hat{a} \rangle &= \sum_{n=0}^{\infty} n^2 p(n) = \int_0^\infty dI_a P(I_a) I_a (I_a + 1) = \langle I_a (I_a + 1) \rangle, \\ \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle &= \sum_{n=0}^{\infty} n(n-1) p(n) = \int_0^\infty dI_a P(I_a) I_a^2 = \langle I_a^2 \rangle, \\ \langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 &= \int_0^\infty dI_a P(I_a) (I_a - \langle I_a \rangle)^2. \end{aligned} \tag{4.3}$$

Clearly, the last expression cannot be negative if $P(I_a)$ is pointwise nonnegative. Thus we are led to the Mandel classification of states of a single-mode field [8, 12] :

$$\begin{aligned} P(I_a) \geq 0,\ \hat{\rho}_D^{(a)} \text{ is QO-cl} \ \Rightarrow\ & \{ M^{(1)}(\hat{\rho}_D^{(a)}) \geq 0 \Leftrightarrow \det M^{(1)}(\hat{\rho}_D^{(a)}) \geq 0 \\ & \Leftrightarrow (\Delta N_a)^2 - \langle N_a \rangle \geq 0 \} : \\ & \quad \hat{\rho}_D^{(a)} \text{ displays super-Poissonian statistics (super-PS);} \end{aligned}$$

$$\begin{aligned} \{ M^1(\hat{\rho}_D^{(a)}) \ngeq 0 \ \Leftrightarrow\ & \det M^{(1)}(\hat{\rho}_D^{(a)}) < 0 \Leftrightarrow (\Delta N_a)^2 - \langle N_a \rangle < 0 \} \Rightarrow \\ P(I_a) \ \ngeq\ & 0,\ \hat{\rho}_D^{(a)} \text{ is QO-noncl :} \\ & \quad \hat{\rho}_D^{(a)} \text{ displays sub-Poissonian statistics (sub-PS).} \end{aligned} \tag{4.4}$$

It is because the Mandel matrix is two dimensional with obviously positive trace that the positivity or nonpositivity of $M^{(1)}(\hat{\rho}_D^{(a)})$ reduces to that of its determinant, hence to that of $(\Delta N_a)^2 - \langle N_a \rangle$.

B. Single-mode squeezed vacuum example

As a useful and instructive example of this concept, we consider the case of a single-mode squeezed vacuum [3–5]. Such a state is obtained by applying a unitary (scaling) operator involving the exponential of a complex combination of $\hat{a}^{\dagger 2}$ and $\hat{a}^2$ to the Fock vacuum $|0\rangle_a$, and is parametrised by a complex variable $\xi = \xi_1 + i\xi_2$ or an equivalent complex variable ω:

$$\begin{aligned} |\psi^{(a)}(\omega)\rangle &= \exp\{\frac{1}{4}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2)\}|0\rangle_a \\ &= (1-|\omega|^2)^{\frac{1}{4}} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+1/2)}{n!\sqrt{\pi}}}\, \omega^n\, |2n\rangle_a, \\ \omega &= \frac{\xi}{|\xi|}\tanh(|\xi|/2). \end{aligned} \tag{4.5}$$

Since only even photon number states are present, the probabilities $p(1)$, $p(3)$, $p(5)$, $\cdots$ in the PND vanish. That is, $\tilde{L} \neq 0$ but all its diagonals vanish, implying $\tilde{L} \ngeq 0$, which is immediate evidence that these states are QO-noncl. Some important expectation values are:

$$\begin{aligned} &\langle\psi^{(a)}(\omega)|\{\hat{a}^\dagger, \hat{a},\ \hat{N}_a,\ \hat{N}_a^2,\ \hat{a}^{\dagger 2}\hat{a}^2,\ \hat{a}^2\}|\psi^{(a)}(\omega)\rangle = \\ &\qquad \{0,\ 0,\ S^2,\ S^2(S^2+2C^2),\ S^2(2S^2+C^2),\ \frac{\xi}{|\xi|}\}, \\ &S = \sinh(|\xi|/2),\ C = \cosh(|\xi|/2). \end{aligned} \tag{4.6}$$

The 2×2 Mandel matrix for this state is thus:

$$\begin{aligned} M^{(1)}(|\psi^{(a)}(\omega)\rangle) &= \begin{pmatrix} 1 & S^2 \\ S^2 & S^2(2S^2+C^2) \end{pmatrix}, \\ \det M^{(1)}(|\psi^{(a)}(\omega)\rangle) &= S^2(S^2+C^2) \geq 0, \end{aligned} \tag{4.7}$$

where S and C are given in Eq. (4.6). Thus these states have super-PS, and the QO-nonclassicality does not show up, or is missed, at the Mandel level.

C. Mandel matrices for two-mode states

The two-mode generalization of the Mandel matrix idea leads naturally to a more intricate classification of states. We consider only states $\hat{\rho}_D^{(ab)}$ conserving, i.e., commuting with, $\hat{N}_a + \hat{N}_b$. First we develop the analogue of the positivity property (4.1), the statement that the number fluctuation $(\Delta N_a)^2$ for a PND $\{p(n)\}$ can never be negative. Define a column and row vector with number conserving operator entries as follows

$$
\begin{aligned}
\hat{C} &= \begin{pmatrix} \hat{a}^\dagger \\ \hat{b}^\dagger \end{pmatrix} \otimes \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} \hat{N}_a \\ \hat{a}^\dagger \hat{b} \\ \hat{b}^\dagger \hat{a} \\ \hat{N}_b \end{pmatrix}, \\
\hat{C}^\dagger &= \begin{pmatrix} \hat{N}_a & \hat{b}^\dagger \hat{a} & \hat{a}^\dagger \hat{b} & \hat{N}_b \end{pmatrix}.
\end{aligned} \tag{4.8}
$$

With their help next define a 5×5 matrix with operator entries and which is 'hermitian' like $\hat{N}$ in Eq. (3.6), and also 'positive definite':

$$
\begin{aligned}
\hat{\Sigma} &= \begin{pmatrix} 1 \\ \hat{C} \end{pmatrix} \begin{pmatrix} 1 & \hat{C}^\dagger \end{pmatrix} = \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{C}\hat{C}^\dagger \end{pmatrix} \\
&= \begin{pmatrix} 1 & \hat{N}_a & \hat{b}^\dagger \hat{a} & \hat{a}^\dagger \hat{b} & \hat{N}_b \\ \hat{N}_a & \hat{N}_a^2 & \hat{N}_a \hat{b}^\dagger \hat{a} & \hat{N}_a \hat{a}^\dagger \hat{b} & \hat{N}_a \hat{N}_b \\ \hat{a}^\dagger \hat{b} & \hat{a}^\dagger \hat{b} \hat{N}_a & \hat{a}^\dagger \hat{b} \hat{b}^\dagger \hat{a} & (\hat{a}^\dagger \hat{b})^2 & \hat{a}^\dagger \hat{b} \hat{N}_b \\ \hat{b}^\dagger \hat{a} & \hat{b}^\dagger \hat{a} \hat{N}_a & (\hat{b}^\dagger \hat{a})^2 & \hat{b}^\dagger \hat{a} \hat{a}^\dagger \hat{b} & \hat{b}^\dagger \hat{a} \hat{N}_b \\ \hat{N}_b & \hat{N}_b \hat{N}_a & \hat{N}_b \hat{b}^\dagger \hat{a} & \hat{N}_b \hat{a}^\dagger \hat{b} & \hat{N}_b^2 \end{pmatrix}.
\end{aligned} \tag{4.9}
$$

Given a state $\hat{\rho}_D^{(ab)}$, by taking entrywise expectation values of the operators in $\hat{\Sigma}$ we get the 5×5 numerical hermitian matrix Σ which is clearly hermitian positive semidefinite and generalizes (4.1):

$$
\begin{aligned}
\Sigma = \langle \hat{\Sigma} \rangle &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{C}\hat{C}^\dagger \end{pmatrix}) \\
&= \begin{pmatrix} 1 & \langle \hat{C}^\dagger \rangle \\ \langle \hat{C} \rangle & \langle \hat{C}\hat{C}^\dagger \rangle \end{pmatrix} \geq 0\,.
\end{aligned} \tag{4.10}
$$

We now define the *two-mode Mandel matrix* for the state $\hat{\rho}_D^{(ab)}$ by replacing $\hat{C}\hat{C}^\dagger$ in Eq. (4.10) by its normal ordered expression (the entries in $\hat{C}$ and $\hat{C}^\dagger$ are already in the normal ordered form) [12]:

$$
\begin{aligned}
\hat{B} &= :\hat{C}\hat{C}^\dagger:, \\
M^{(2)}(\hat{\rho}_D^{(ab)}) &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix})
\end{aligned}
$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & \langle \hat{a}^\dagger\hat{a}\rangle & \langle \hat{b}^\dagger\hat{a}\rangle & \langle \hat{a}^\dagger\hat{b}\rangle & \langle \hat{b}^\dagger\hat{b}\rangle \\ \langle \hat{a}^\dagger\hat{a}\rangle & \langle \hat{a}^{\dagger 2}\hat{a}^2\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}^2\rangle & \langle \hat{a}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle \\ \langle \hat{a}^\dagger\hat{b}\rangle & \langle \hat{a}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle & \langle \hat{a}^{\dagger 2}\hat{b}^2\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{b}^2\rangle \\ \langle \hat{b}^\dagger\hat{a}\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}^2\rangle & \langle \hat{b}^{\dagger 2}\hat{a}^2\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle & \langle \hat{b}^{\dagger 2}\hat{a}\hat{b}\rangle \\ \langle \hat{b}^\dagger\hat{b}\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle & \langle \hat{b}^{\dagger 2}\hat{a}\hat{b}\rangle & \langle \hat{a}^\dagger\hat{b}^\dagger\hat{b}^2\rangle & \langle \hat{b}^{\dagger 2}\hat{b}^2\rangle \end{pmatrix} \\
&= \int_0^\infty dI_a \int_0^\infty dI_b \int_0^{2\pi} \frac{d\theta}{2\pi} P(I_a, I_b, \theta)\chi(I_a, I_b, \theta)\, \chi(I_a, I_b, \theta)^\dagger \\
\chi(I_a, I_b, \theta) &= \begin{pmatrix} 1 & I_a & \sqrt{I_a I_b} e^{+i\theta} & \sqrt{I_a I_b} e^{-i\theta} & I_b \end{pmatrix}^T \qquad (4.11)
\end{aligned}
$$

The superscript ${}^{(2)}$ indicates that we are dealing with a two-mode state, and this Mandel matrix is 5×5 hermitian but not necessarily positive semidefinite.

Before presenting a classification of two-mode states based on the 5×5 Mandel matrix, we introduce a useful derived object. This is the 2×2 Mandel matrix associated with a general single mode defined as a linear combination of the modes a and b, the reduced subsystem state of this chosen single mode being calculated with respect to the two-mode state $\hat{\rho}_D^{(ab)}$. The definition of the annihilation operator $\hat{A}$ of such a mode and then of its Mandel matrix are:

$$
\begin{aligned}
\hat{A} = \alpha\hat{a} + \beta\hat{b}, &\qquad |\alpha|^2 + |\beta|^2 = 1: \quad [\hat{A}\hat{A}^\dagger] = \mathbb{1} \\
M^{(2,1)}(\hat{\rho}_D^{(ab)}; \alpha, \beta) &= \begin{pmatrix} 1 & \langle \hat{A}^\dagger\hat{A}\rangle \\ \langle \hat{A}^\dagger\hat{A}\rangle & \langle \hat{A}^{\dagger 2}\hat{A}^2\rangle \end{pmatrix} \\
&= Y(\alpha, \beta)^\dagger M^{(2)}(\hat{\rho}_D^{(ab)}) Y(\alpha, \beta), \\
Y(\alpha, \beta) &= \begin{pmatrix} 1 & 0 \\ 0 & \\ 0 & \psi_0(\alpha, \beta) \\ 0 & \\ 0 & \end{pmatrix}, \\
\psi_0(\alpha, \beta) &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \begin{pmatrix} \alpha\alpha^* \\ \alpha\beta^* \\ \beta\alpha^* \\ \beta\beta^* \end{pmatrix}. \qquad (4.12)
\end{aligned}
$$

The dependence of $\hat{A}$ on α, β is left implicit. The superscript $(2, 1)$ at the start of the above equations indicates that we are dealing with a general single-mode Mandel matrix obtained from the two-mode Mandel matrix for the $a-b$ system in the state $\hat{\rho}_D^{(ab)}$, by focussing on a variable linear combination $\hat{A}$ of $\hat{a}$ and $\hat{b}$. As we will immediately see, for two-mode states both the 5×5 matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ and the 2×2 matrix $M^{(2,1)}(\hat{\rho}_D^{(ab)};\alpha,\beta)$ are important.

The two-mode definitions of Mandel-type nonclassicality, sub-Poissonian statistics (sub-PS), super-Poissonian statistics (super-PS), etc are now as follows:

$$\begin{aligned}
\{\hat{\rho}_D^{(ab)} \text{ is QO-cl} \ &\Leftrightarrow \ P(I_a, I_b, \theta) \geq 0\} \\
&\Rightarrow \{M^{(2)}(\hat{\rho}_D^{(ab)}) \geq 0 \ \Leftrightarrow \ \hat{\rho}_D^{(ab)} \text{has super-PS}\}; \\
M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0 \ &\Leftrightarrow \ \{\hat{\rho}_D^{(ab)} \text{is QO-noncl, has sub-PS}\}. \qquad (4.13)
\end{aligned}$$

In the definition of super-PS, we used Eq. (4.11). The sub-PS case can be usefully separated into two types, depending on whether or not the nonpositivity of the 5×5 matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ is visible already at the single-mode level for some choice of coefficients α, β. Thus we define:

$$\begin{aligned}
\hat{\rho}_D^{(ab)} \text{ has Type I sub-PS} \ &\Leftrightarrow \ M^{(2,1)}(\hat{\rho}_D^{(ab)};\alpha,\beta) \ngeq 0 \text{ for some } \alpha,\ \beta; \\
\hat{\rho}_D^{(ab)} \text{ has Type II sub-PS} \ &\Leftrightarrow \ M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0, \text{ but} \\
&\qquad M^{(2,1)}(\hat{\rho}_D^{(ab)};\alpha,\beta) \geq 0 \text{ for all } \alpha,\ \beta. \qquad (4.14)
\end{aligned}$$

The physical meaning is that in Type I sub-PS, the Mandel level of QO nonclassicality is easy to detect already in terms of a suitable single-mode combination; while in Type II sub-PS, such nonclassicality is hidden or intrinsically two-mode in character [12].

For calculational purposes one can pass from the 5×5 Mandel matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ to a slightly simpler 4×4 matrix as follows. From Eq. (4.11),

$$\begin{aligned}
M^{(2)}(\hat{\rho}_D^{(ab)}) &= \mathrm{Tr}(\hat{\rho}_D^{(ab)} \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix}) = \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix}, \\
C &= \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{C}),\ B = \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{B}). \qquad (4.15)
\end{aligned}$$

(When necessary the state will be indicated as argument of C, B). Then it is easy to see that

$$\begin{aligned}
M^{(2)}(\hat{\rho}_D^{(ab)}) \geq 0 \ &\Leftrightarrow \ \Gamma = B - CC^\dagger \geq 0, \\
M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0 \ &\Leftrightarrow \ \Gamma \ngeq 0. \qquad (4.16)
\end{aligned}$$

Thus the 4×4 hermitian matrix Γ determines whether we have super-PS or sub-PS. For the separation of the latter into Type I and Type II, we have for any complex 2-vector $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$:

$$\phi^\dagger M^{(2,1)}(\hat{\rho}_D^{(ab)};\alpha,\beta)\phi = \\ |\phi_1 + \phi_2 C^\dagger \psi_0(\alpha,\beta)|^2 + |\phi_2|^2 \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta). \tag{4.17}$$

So given $M^{(2)}(\hat{\rho}_D^{(ab)}) \not\geq 0$, we are able to say:

$$\begin{aligned} \text{Type I sub-PS} \quad &\Leftrightarrow \quad \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) < 0 \text{ for some } \alpha,\ \beta; \\ \text{Type II sub-PS} \quad &\Leftrightarrow \quad \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) \geq 0 \text{ for all } \alpha,\ \beta. \end{aligned} \tag{4.18}$$

Indeed we easily find from Eqs. (4.12, 4.16) that

$$\det M^{(2,1)}(\hat{\rho}^{(ab)};\alpha,\beta) = \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta). \tag{4.19}$$

D. Examples of two-mode states and Mandel matrices

We consider two examples. We have seen in Section II that a single-mode QO-noncl state, when combined with a second ancilla mode in vacuum (or in a coherent state) and then passed through a nontrivial $U(2)$ BS, always results at the output in a two-mode state exhibiting NPT entanglement. We study this as the first example.

The two-mode state in question is given in Eq. (2.16). It is understandable that its 5×5 Mandel matrix is obtainable from the 2×2 Mandel matrix associated with the single mode input state $\hat{\rho}_D^{(a)}$. Straightforward calculation shows that:

$$\begin{aligned} \hat{\rho}_D^{(ab)} &= \hat{U}(u)\{\hat{\rho}_D^{(a)} \otimes |0\rangle_{bb}\langle 0|\}\hat{U}(u)^{-1}, \quad u \in U(2): \\ M^{(2)}(\hat{\rho}_D^{(ab)}) &= W(u)^\dagger M^{(1)}(\hat{\rho}_D^{(a)}) W(u), \\ W(u) &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & u_{11}^* u_{11} & u_{21}^* u_{11} & u_{11}^* u_{21} & u_{21}^* u_{21} \end{pmatrix}, \\ W(u)W(u)^\dagger &= 1\!\!1_{2\times 2}. \end{aligned} \tag{4.20}$$

Next using (4.12) we can immediately obtain the variable single-mode projection of this two-mode Mandel matrix:

$$M^{(2,1)}(\hat{\rho}_D^{(ab)};\alpha,\beta) = Y(\alpha,\beta)^\dagger W(u)^\dagger M^{(1)}(\hat{\rho}_D^{(a)}) W(u) Y(\alpha,\beta)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & |\xi|^2 \end{pmatrix} M^{(1)}(\hat{\rho}_D^{(a)}) \begin{pmatrix} 1 & 0 \\ 0 & |\xi|^2 \end{pmatrix},$$

$$\xi = u_{11}\alpha + u_{22}\beta. \tag{4.21}$$

From these expressions and the results of Section II, we find that the two-mode states produced from single-mode states in the above manner have the following significant properties:

$$\begin{aligned} &(i)\ \hat{\rho}_D^{(a)} \text{ has QO-noncl PND } \Rightarrow \hat{\rho}_D^{(ab)} \text{ has NPT entanglement;} \\ &(ii)\ \hat{\rho}_D^{(a)} \text{ has super-PS } \Rightarrow \hat{\rho}_D^{(ab)} \text{ has super-PS;} \\ &(iii)\ \hat{\rho}_D^{(a)} \text{ has sub-PS } \Rightarrow \hat{\rho}_D^{(ab)} \text{ has Type I sub-PS,} \\ &\qquad M^{(2,1)}(\rho_D^{(ab)}; \alpha, \beta) \not\geq 0 \text{ for every } \alpha, \beta. \end{aligned} \tag{4.22}$$

Of course only properties (ii) and (iii) involve the Mandel matrix analysis; it is significant that in (iii), every single-mode combination of the modes a and b displays sub-PS. To this we can add the following: states $\hat{\rho}_D^{(ab)}$ obtained from states $\hat{\rho}_D^{(a)}$ via Eq. (4.20) can never display Type-II sub-PS; and any sub-PS in $\hat{\rho}_D^{(a)}$ leads to both Type-I sub-PS and NPT entanglement in $\hat{\rho}_D^{(ab)}$.

The second example is the two-mode generalization of the squeezed vacuum state defined for a single mode in Eq. (4.5). We take independent complex ξ, ξ' or ω, ω' and define:

$$|\psi^{(ab)}(\omega, \omega')\rangle = |\psi^{(a)}(\omega)\rangle \otimes |\psi^{(b)}(\omega')\rangle, \tag{4.23}$$

with the second factor involving an exponential in $\hat{b}^{\dagger 2}$ and $\hat{b}^2$ applied to $|0\rangle_b$. This pure state is clearly also QO-noncl, but it is manifestly a product state of Schmidt rank one. Unlike the single mode case in Eq. (4.7), however, now the QO nonclassicality shows up at the Mandel level. The 5×5 Mandel matrix for the state (4.23) is easily found using Eqs. (4.6) and their b-mode analogues:

$$M^{(2)}(|\psi^{(ab)}(\omega, \omega')\rangle) = \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix},$$

$$C^\dagger = \begin{pmatrix} S^2 & 0 & 0 & S'^2 \end{pmatrix},$$

$$B = \begin{pmatrix} S^2(2S^2 + C^2) & 0 & 0 & S^2 S'^{\,2} \\ 0 & S^2 S'^{\,2} & e^{i\eta} S C S' C' & 0 \\ 0 & e^{-i\eta} S C S' C' & S^2 S'^{\,2} & 0 \\ S^2 S'^{\,2} & 0 & 0 & S'^{\,2}(2S'^{\,2} + C'^{\,2}) \end{pmatrix},$$

$$\eta = \arg \xi' \xi^* \tag{4.24}$$

Here S' and C' are defined as in Eq. (4.6) but in terms of ξ'. The 4×4 matrix Γ of Eq. (4.16) is :

$$\Gamma = \begin{pmatrix} S^2(S^2+C^2) & 0 & 0 & 0 \\ 0 & S^2S'^{\,2} & e^{i\eta}SCS'C' & 0 \\ 0 & e^{-i\eta}SCS'C' & S^2S'^2 & 0 \\ 0 & 0 & 0 & S'^{\,2}(S'^{\,2}+C'^{\,2}) \end{pmatrix}. \tag{4.25}$$

The eigenvalues of Γ are $S^2(S^2+C^2)$, $S'^2(S'^2+C'^2)$, $SS'(SS'+CC')$ and $SS'(SS'-CC')$. Assuming that ξ, ξ' are both non zero, the last eigenvalue is negative, leading by Eq. (4.16) to the conclusion that $M^{(2)}(|\psi^{(ab)}(\omega,\omega')\rangle) \ngeq 0$ or that the state $|\psi^{(ab)}(\omega,\omega')\rangle$ has sub-PS. This is an interesting and somewhat nonintuitive result since we have seen in Eq. (4.7) that each factor in the product state $|\psi^{(ab)}(\omega,\omega')\rangle$ has super-PS. We must now see whether it is Type I or Type II. For this we must compute the 'expectation value' of Γ in Eq. (4.25) for the four-component column vector $\psi_0(\alpha,\beta)$ as required by Eq. (4.18) :

$$\begin{aligned} &\psi_0(\alpha,\beta)^\dagger\Gamma\psi_0(\alpha,\beta) = \\ &\quad (|\alpha|^2S^2+|\beta|^2S'^2)^2+|\alpha|^4S^2C^2+|\beta|^4S'^2C'^2+2SCS'C'\Re(e^{i\eta}(\alpha^*\beta)^2) \\ &\qquad \geq (|\alpha|^2S^2+|\beta|^2S'^2)^2+(|\alpha|^2SC-|\beta|^2S'C')^2 > 0, \end{aligned} \tag{4.26}$$

since $\Re(e^{i\eta}(\alpha^*\beta)^2) \geq -|\alpha|^2|\beta|^2$. It follows that the sub-PS of the product state $|\psi^{(ab)}(\omega,\omega')\rangle$ is of Type II, it is hidden or intrinsic. This is consistent with the fact that the individual factors $|\psi^{(a)}(\omega)\rangle$ and $|\psi^{(b)}(\omega')\rangle$ are both super-PS.

V. TWO-MODE MANDEL LEVEL NONCLASSICALITY TO ENTANGLEMENT BY BS ACTION

We now consider a two-mode state $\hat{\rho}_D^{(ab)}$ which is QO nonclassical and of such a nature that this property is seen at the level of the Mandel matrix, i.e., $M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0$. In such a case, even if $\hat{\rho}_D^{(ab)}$ is of product or separable form, its passage through a $U(2)$ BS could result in an entangled state, possibly of NPT type. Our aim is now to see quantitatively how much beyond the nonpositivity of the Mandel matrix one has to go to reach NPT entanglement.

We set up the general framework to examine this, then illustrate it by an example. For simplicity we use a 50:50 BS rather than one corresponding to a general $u \in U(2)$.

We choose the $U(2)$ element and corresponding unitary operator action as follows [11, 34] :

$$u_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \in U(2) : \ \hat{U}_0^{-1} \begin{pmatrix} \hat{a} & \hat{a}^\dagger \\ \hat{b} & \hat{b}^\dagger \end{pmatrix} \hat{U}_0 \ = u_0 \begin{pmatrix} \hat{a} & \hat{a}^\dagger \\ \hat{b} & \hat{b}^\dagger \end{pmatrix} . \tag{5.1}$$

At the operator level, action by conjugation on $\hat{C}$, $\hat{C}^\dagger$, $\hat{B}$ of Eqs. (4.8, 4.11) is :

$$\begin{aligned} \hat{U}_0^{-1} \hat{C} \hat{U}_0 &= V_0 \hat{C}, \ \hat{U}_0^{-1} \hat{C}^\dagger \hat{U}_0 = \hat{C}^\dagger V_0^T, \ \hat{U}_0^{-1} \hat{B} \hat{U}_0 = V_0 \hat{B} V_0^T, \\ V_0 &= u_0 \otimes u_0 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} . \end{aligned} \tag{5.2}$$

Then when the state $\hat{\rho}_D^{(ab)}$ is transformed by this BS action to

$$\hat{\rho}'^{(ab)}_D = \hat{U}_0 \hat{\rho}_D^{(ab)} \hat{U}_0^{-1}, \tag{5.3}$$

the new Mandel matrix is given by a transformation using V_0 :

$$\begin{aligned} M^{(2)}(\hat{\rho}_D^{(ab)}) &= \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \rightarrow \\ M^{(2)}(\hat{\rho}'^{(ab)}_D) &= \begin{pmatrix} 1 & C'^\dagger \\ C' & B' \end{pmatrix} = \mathrm{Tr}(\hat{\rho}'^{(ab)}_D \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix}) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_0^T \end{pmatrix}, \\ C' &= V_0 C, \ B' = V_0 B V_0^T. \end{aligned} \tag{5.4}$$

Thus Γ' is related to Γ by *congruence* :

$$\Gamma' = B' - C' C'^\dagger = V_0 \Gamma V_0^T. \tag{5.5}$$

To test next whether $\hat{\rho}'^{(ab)}_D$ is NPT entangled, we pass to its partial transpose $\hat{\rho}'^{(ab)PT}_D$ and evaluate the 'expectation value' of a suitably chosen nonnegative hermitian operator with respect to it. If this turns out to be negative, then the output state $\hat{\rho}'^{(ab)}_D$ is definitely NPT entangled. To construct such a test which involves as closely as possible the use of

$M^{(2)}(\hat{\rho}'^{(ab)}_D)$, hence of $M^{(2)}(\hat{\rho}^{(ab)}_D)$, plus something additional, we should use a 'matrix of operators' similar in structure to $\begin{pmatrix} 1 \\ \hat{C} \end{pmatrix} \begin{pmatrix} 1 & \hat{C}^\dagger \end{pmatrix}$, i.e, making up a 'hermitian nonnegative' matrix of operator entries, such that when the partial transpose operation is switched from $\hat{\rho}'^{(ab)PT}_D$ to this 'matrix', we obtain the expectation values of $\hat{C}$, $\hat{C}^\dagger$ and $\hat{B}$ in $\hat{\rho}'^{(ab)}_D$, plus something additional. Now we have seen in the passage from Eq. (3.7) to Eq. (3.8) that the PT operation converts $\hat{b}^{\dagger j}\hat{b}^k$ to $\hat{b}^{\dagger k}\hat{b}^j$, and $\hat{b}^j\hat{b}^{\dagger k}$ to $\hat{b}^k\hat{b}^{\dagger j}$. Keeping these motivations and facts in mind we construct a 5×5 matrix of operators as follows:

$$
\hat{E} = \begin{pmatrix} \hat{a}^\dagger\hat{a} \\ \hat{a}^\dagger\hat{b}^\dagger \\ \hat{a}\hat{b} \\ \hat{b}^\dagger\hat{b} \end{pmatrix}, \quad \hat{E}^\dagger = \begin{pmatrix} \hat{a}^\dagger\hat{a} & \hat{a}\hat{b} & \hat{a}^\dagger\hat{b}^\dagger & \hat{b}^\dagger\hat{b} \end{pmatrix} \rightarrow
$$

$$
\left\{ \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix} \right\}^{PT} = \begin{pmatrix} 1 & \hat{C}^\dagger \\ \hat{C} & \hat{B} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \hat{Y} \end{pmatrix},
$$

$$
\hat{Y} = \begin{pmatrix} \hat{a}^\dagger\hat{a} & 0 & \hat{a}^\dagger\hat{b} & 0 \\ 0 & 0 & 0 & 0 \\ \hat{b}^\dagger\hat{a} & 0 & \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 1 & \hat{b}^\dagger\hat{a} \\ 0 & 0 & \hat{a}^\dagger\hat{b} & \hat{b}^\dagger\hat{b} \end{pmatrix}. \tag{5.6}
$$

We see that in the process of expressing the various operators involved in normal ordered form, as anticipated an additional piece $\hat{Y}$ linear in the entries of $\hat{C}$ appears. Then a test for NPT entanglement of $\hat{\rho}'^{(ab)}_D$ is to evaluate

$$
\mathrm{Tr}(\hat{\rho}'^{(ab)PT}_D \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix}) = \mathrm{Tr}(\hat{\rho}'^{(ab)}_D \left\{ \begin{pmatrix} 1 \\ \hat{E} \end{pmatrix} \begin{pmatrix} 1 & \hat{E}^\dagger \end{pmatrix} \right\}^{PT})
$$

$$= M^{(2)}(\hat{\rho}'^{(ab)}_{D}) + \begin{pmatrix} 0 & \vdots & 0 & 0 & 0 & 0 \\ \cdots & & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & & & & \\ 0 & \vdots & & & & \\ 0 & \vdots & & Y' & & \\ 0 & \vdots & & & & \end{pmatrix},$$

$$Y' = \begin{pmatrix} C'_1 & 0 & C'_2 & 0 \\ 0 & 0 & 0 & 0 \\ C'_3 & 0 & C'_1 + C'_4 + 1 & C'_3 \\ 0 & 0 & C'_2 & C'_4 \end{pmatrix}, \tag{5.7}$$

and see if this matrix is indefinite. By Eq. (5.4), the complete 5×5 matrix here is a congruence transformation applied to the initial state Mandel matrix $M^{(2)}(\hat{\rho}^{(ab)}_D)$ plus a 4×4 piece coming from $\hat{Y}$, namely it is:

$$\begin{pmatrix} 1 & 0 \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} 1 & C^\dagger \\ C & B \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_0^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Y' \end{pmatrix} = \begin{pmatrix} 1 & C^\dagger V_0^T \\ V_0 C & V_0 B V_0^T + Y' \end{pmatrix}. \tag{5.8}$$

Therefore by Eq. (4.16) the positivity or otherwise of the matrix (5.7) is equivalent to the positivity or otherwise of either of the two following 4×4 matrices at the level of Γ:

$$\begin{aligned} \Omega &= \Gamma + V_0^T Y' V_0 \,, \\ V_0 \Omega V_0^T &= V_0 \Gamma V_0^T + Y'. \end{aligned} \tag{5.9}$$

Nonpositivity of either Ω or $V_0 \Omega V_0^T$ is proof of NPT entanglement of the output state $\hat{\rho}'^{(ab)}_{D}$. It is interesting to see the precise quantitative manner in which this test goes beyond examination of $M^{(2)}(\hat{\rho}'^{(ab)}_D)$ or Γ' alone.

A. An illustrative example

To see how the general procedure developed above works, we study a family of states which is analytically quite simple. We begin with the family of two-mode pure states of infinite Schmidt rank,

$$|\mu\rangle = e^{-\frac{1}{2}|\mu|^2} \sum_{n=0}^{\infty} \frac{\mu^n}{\sqrt{n!}} |n, n\rangle, \;\; \mu \in \mathcal{C}, \tag{5.10}$$

form the density matrix $\hat{\rho}^{(ab)} = |\mu\rangle\langle\mu|$, and pass to $\hat{\rho}_D^{(ab)}$ via Eq. (3.3):

$$\hat{\rho}_D^{(ab)} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n,n\rangle\langle n,n|,\ \ \lambda = |\mu|^2 \geq 0. \tag{5.11}$$

This is clearly separable though not of product form. For the Mandel matrix analysis, $|\mu\rangle\langle\mu|$ and $\hat{\rho}_D^{(ab)}$ are equivalent.

The matrices C, $C^\dagger$, B, Γ involved in $M^{(2)}(\hat{\rho}_D^{(ab)})$ are easy to calculate since

$$\langle \hat{a}^\dagger \hat{a}\rangle = \langle \hat{b}^\dagger \hat{b}\rangle = \lambda,\ \ \langle \hat{a}^{\dagger 2} \hat{a}^2\rangle = \langle \hat{b}^{\dagger 2} \hat{b}^2\rangle = \lambda^2. \tag{5.12}$$

We thus have:

$$C = \begin{pmatrix} \lambda \\ 0 \\ 0 \\ \lambda \end{pmatrix},\quad B = \begin{pmatrix} \lambda^2 & 0 & 0 & \lambda^2+\lambda \\ 0 & \lambda^2+\lambda & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda & 0 \\ \lambda^2+\lambda & 0 & 0 & \lambda^2 \end{pmatrix};$$

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & \lambda \\ 0 & \lambda^2+\lambda & 0 & 0 \\ 0 & 0 & \lambda^2+\lambda & 0 \\ \lambda & 0 & 0 & 0 \end{pmatrix}. \tag{5.13}$$

The eigenvalues of Γ being $\lambda(\lambda+1)$, $\lambda(\lambda+1)$, λ, $-\lambda$, the state $\hat{\rho}_D^{(ab)}$ in (5.11) is QO-noncl. To find its type we compute

$$\psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta) = 2|\alpha|^2|\beta|^2\lambda(\lambda+2) \geq 0, \tag{5.14}$$

so these states display hidden or Type II sub-PS.

In passing we note that the state $\hat{\rho}_D^{(a)}$ of mode a obtained from Eq. (5.11) by tracing over b alone is

$$\hat{\rho}_D^{(a)} = \mathrm{Tr}_b \hat{\rho}_D^{(ab)} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} |n\rangle_{aa}\langle n|, \tag{5.15}$$

for which the diagonal weight $P(I_a)$ is

$$P(I_a) = \delta(I_a - \lambda). \tag{5.16}$$

Partial trace over a gives exactly similar results for mode b. Thus both $\hat{\rho}_D^{(a)}$ and $\hat{\rho}_D^{(b)}$ are QO-cl, with their PND's coinciding exactly with that of a coherent state.

Now we pass the two-mode state $\hat{\rho}_D^{(ab)}$ of Eq. (5.11) through the BS $\hat{U}_0$ of Eq. (5.1); the resulting $\hat{\rho}'^{(ab)}_D$ is

$$\begin{aligned}\hat{\rho}'^{(ab)}_D &= \hat{U}_0 \hat{\rho}_D^{(ab)} \hat{U}_0^{-1} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \frac{1}{n!^3} (\hat{a}^{\dagger 2} - \hat{b}^{\dagger 2})^n |0,0\rangle\langle 0,0| (\hat{a}^2 - \hat{b}^2)^n. \end{aligned} \tag{5.17}$$

To apply the NPT entanglement test based on Eq. (5.9) it is convenient to examine $V_0 \Omega V_0^T$. Combining Eqs. (5.4, 5.13) we find the matrices Γ', Y' associated with $\hat{\rho}'^{(ab)}_D$ to be

$$\begin{aligned}\Gamma' = V_0 \Gamma V_0^T &= \begin{pmatrix} \frac{1}{2}\lambda^2 + \lambda & 0 & 0 & -\frac{1}{2}\lambda^2 \\ 0 & \frac{1}{2}\lambda^2 & -\frac{1}{2}\lambda^2 - \lambda & 0 \\ 0 & -\frac{1}{2}\lambda^2 - \lambda & \frac{1}{2}\lambda^2 & 0 \\ -\frac{1}{2}\lambda^2 & 0 & 0 & \frac{1}{2}\lambda^2 + \lambda \end{pmatrix}, \\ Y' &= \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda + 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}. \end{aligned} \tag{5.18}$$

Therefore according to Eq. (5.9) we have to test the positivity or otherwise of

$$\begin{aligned} V_0 \Omega V_0^T &= V_0 \Gamma V_0^T + Y' \\ &= \begin{pmatrix} \frac{1}{2}\lambda^2 + 2\lambda & 0 & 0 & -\frac{1}{2}\lambda^2 \\ 0 & \frac{1}{2}\lambda^2 & -\frac{1}{2}\lambda^2 - \lambda & 0 \\ 0 & -\frac{1}{2}\lambda^2 - \lambda & \frac{1}{2}\lambda^2 + 2\lambda + 1 & 0 \\ -\frac{1}{2}\lambda^2 & 0 & 0 & \frac{1}{2}\lambda^2 + 2\lambda \end{pmatrix} \end{aligned} \tag{5.19}$$

The $(2,3)$ submatrix here is indefinite as it has determinant $-\frac{1}{2}\lambda^2$. *This establishes that $\hat{\rho}'^{(ab)}_D$ of Eq. (5.17) is NPT entangled.* Qualitatively speaking, even though Y' in Eq. (5.18) is nonnegative, the total matrix $V_0 \Omega V_0^T$ in Eq. (5.19) is indefinite, with Γ' dominating Y'. *The emphasis here has been to show that the NPT entanglement produced by BS action can indeed be witnessed by the test based on Eq. (5.9), which goes beyond the examination of the Mandel matrix in a precise manner.*

We can characterize the NPT entanglement we have proved in this example further. From the expression in Eq. (5.17), the terms for $n = 0$ and $n = 1$ are respectively:

$$e^{-\lambda}|0,0\rangle\langle 0,0|, \quad \frac{\lambda}{2} e^{-\lambda}(|2,0\rangle - |0,2\rangle)(\langle 2,0| - \langle 0,2|), \tag{5.20}$$

giving the matrix elements

$$(\hat{\rho}'^{(ab)}_{D})_{00,00} = e^{-\lambda};$$
$$(\hat{\rho}'^{(ab)}_{D})_{20,20} = (\hat{\rho}'^{(ab)}_{D})_{02,02} = -(\hat{\rho}'^{(ab)}_{D})_{20,02} = -(\hat{\rho}'^{(ab)}_{D})_{02,20} = \frac{\lambda}{2}e^{-\lambda}. \tag{5.21}$$

One also obtains from the $n = 2$ term in Eq. (5.17) the matrix element

$$(\hat{\rho}'^{(ab)}_{D})_{22,22} = \frac{\lambda^2}{8}e^{-\lambda}. \tag{5.22}$$

To demonstrate distillability we follow the recipe of Ref.[40], and project $\hat{\rho}'^{(ab)}_{D}$ into the 2×2 bipartite subspace spanned by $|0\rangle$, $|2\rangle$ of the a-mode and $|0\rangle$, $|2\rangle$ for the b-mode. The resulting 2×2 state written in the basis $|0,0\rangle$, $|0,2\rangle$, $|2,0\rangle$, $|2,2\rangle$ reads

$$\hat{\rho}'^{(ab)}_{D} \rightarrow \left(\hat{\rho}'^{(ab)}_{D}\right)_{2\times 2} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{2} & -\frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} & \frac{\lambda}{2} & 0 \\ 0 & 0 & 0 & \frac{\lambda^2}{8} \end{pmatrix}. \tag{5.23}$$

That $\left(\hat{\rho}'^{(ab)}_{D}\right)_{2\times 2}$ is inseparable is manifest, for its partial transpose has negative determinant. Indeed, the partial transpose has an eigenvector of the form $\alpha|0,0\rangle + \beta|2,2\rangle$ with a negative eigenvalue. *This demonstrates that the NPT entanglement of $\hat{\rho}'^{(ab)}_{D}$ in Eq. (5.17) is in fact distillable.*

B. The two-mode squeezed vacuum

This state defined in Eq. (4.23) is both pure and of product form, i.e., of Schmidt rank one. It was studied in the previous Section via its Mandel matrix, which showed it to be QO-noncl with Type-II (i.e., hidden) sub-PS. In principle, after passing this state through the BS U_0, we can apply the test based on Eq. (5.9) to see if the output state is NPT entangled. However, this involves some amount of algebra. Fortunately, the fact that the output state of the BS is entangled can be seen in this case simply by inspection and without any calculations:

$$\hat{U}_0|\psi^{(ab)}(\omega,\omega')\rangle = \hat{U}_0 \exp\{\frac{1}{4}(\xi\hat{a}^{\dagger 2} - \xi^*\hat{a}^2) + \frac{1}{4}(\xi'\hat{b}^{\dagger 2} - \xi'^*\hat{b}^2)\}\hat{U}_0^{-1}|0,0\rangle$$
$$= \exp\{\frac{1}{8}(\xi(\hat{a}^\dagger - \hat{b}^\dagger)^2 - \xi^*(\hat{a} - \hat{b})^2) + \frac{1}{8}(\xi'(\hat{a}^\dagger + \hat{b}^\dagger)^2 - \xi'^*(\hat{a} + \hat{b})^2)\}|0,0\rangle. \tag{5.24}$$

This is because the final unitary operator acting on $|0,0\rangle$ is clearly not the tensor product of individual unitary operators acting separately on the two vacua. It is of course important that at least one of the factors $|\psi^{(a)}(\omega)\rangle$, $|\psi^{(b)}(\omega')\rangle$ in the initial product (4.23) **be** QO-noncl. A two-mode pure product QO-cl state is necessarily a product of single-mode coherent states, and this product structure is maintained by BS.

VI. TWO-MODE NONCLASSICALITY TO THREE-MODE ENTANGLEMENT

In the preceding Section we studied the possibility of a $U(2)$ BS converting a two-mode QO-noncl separable state into an entangled one since both nonclassicality and entanglement are meaningful concepts for such systems. The entire discussion was within the framework of the space of states of a two-mode system.

Now we present a treatment of two-mode states analogous to that in Section II for single-mode systems. That is, we couple a given two-mode state $\hat{\rho}_D^{(ab)}$ to a third ancilla mode in vacuum, pass such an input state $\hat{\rho}_{\text{in}}^{(abc)}$ through a '$U(3)$ beamsplitter' (a classicality preserving passive system), and obtain a three-mode output state $\hat{\rho}_{\text{out}}^{(abc)}$. We then test whether this shows NPT entanglement as a consequence of (Mandel level) QO nonclassicality assumed to be present initially in $\hat{\rho}_D^{(ab)}$; the PT operation is applied to the c-mode. The motivation is to explore the algebraic expressions and form of the test one is led to, apart from carrying the physical process described in Section II to the next higher level.

We begin with $\hat{\rho}_D^{(ab)}$ for which $M^{(2)}(\hat{\rho}_D^{(ab)})$ shows QO nonclassicality. With the ancilla c-mode in vacuum we have an input three-mode state

$$\hat{\rho}_{\text{in}}^{(abc)} = \hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|, \tag{6.1}$$

strictly analogous to Eq. (2.15). To a general matrix $u \in U(3)$ we associate a passive 'beamsplitter' which unitarily mixes the annihilation operators of the three modes in a manner analogous to Eq. (2.13), now conserving $\hat{N}_a + \hat{N}_b + \hat{N}_c$. In the three-mode Hilbert space this BS u acts through a unitary operator $\hat{U}$, and we have [19, 20]

$$u = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix} \in U(3) \ \rightarrow \ \hat{U} : \ \hat{U}^\dagger \hat{U} = \hat{U}\hat{U}^\dagger = 1,$$

$$\hat{U}\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}\hat{U}^{-1} = u^\dagger\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}, \quad \hat{U}\begin{pmatrix}\hat{a}^\dagger\\ \hat{b}^\dagger\\ \hat{c}^\dagger\end{pmatrix}\hat{U}^{-1} = u^T\begin{pmatrix}\hat{a}^\dagger\\ \hat{b}^\dagger\\ \hat{c}^\dagger\end{pmatrix},$$

$$\hat{U}^{-1}\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}\hat{U} = u\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}, \quad \hat{U}^{-1}\begin{pmatrix}\hat{a}^\dagger\\ \hat{b}^\dagger\\ \hat{c}^\dagger\end{pmatrix}\hat{U} = u^*\begin{pmatrix}\hat{a}^\dagger\\ \hat{b}^\dagger\\ \hat{c}^\dagger\end{pmatrix},$$

$$\hat{U}(\hat{N}_a + \hat{N}_b + \hat{N}_c) = (\hat{N}_a + \hat{N}_b + \hat{N}_c)\hat{U}. \tag{6.2}$$

Therefore upon passage through this BS the state in Eq. (6.1) changes to

$$\hat{\rho}_{\text{out}}^{(abc)} = \hat{U}\hat{\rho}_{\text{in}}^{(abc)}\hat{U}^{-1} = \hat{U}\{\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|\}\hat{U}^{-1}. \tag{6.3}$$

To test this output state for NPT entanglement, we apply the PT operation to the c-mode and then evaluate the 'expectation value' of a suitably chosen hermitian nonnegative operator:

$$\begin{aligned} A &= \alpha_0 + \alpha_1\hat{a}\hat{c} + \alpha_2\hat{b}\hat{c} + \alpha_3\hat{a}^\dagger\hat{c}^\dagger + \alpha_4\hat{b}^\dagger\hat{c}^\dagger : \\ &\quad \text{Tr}(\hat{\rho}_{\text{out}}^{(abc)PT}A^\dagger A) = \alpha^\dagger X\alpha, \\ X &= \text{Tr}(\hat{\rho}_{\text{out}}^{(abc)PT}\begin{pmatrix}1\\ \hat{a}^\dagger\hat{c}^\dagger\\ \hat{b}^\dagger\hat{c}^\dagger\\ \hat{a}\hat{c}\\ \hat{b}\hat{c}\end{pmatrix}\begin{pmatrix}1 & \hat{a}\hat{c} & \hat{b}\hat{c} & \hat{a}^\dagger\hat{c}^\dagger & \hat{b}^\dagger\hat{c}^\dagger\end{pmatrix}); \quad \alpha = \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}. \end{aligned} \tag{6.4}$$

In spirit the approach here is similar to that used in the preceding Section leading to Eq. (5.7), though here we are dealing with the extension from two to three modes. The 5×5 hermitian matrix X, constructed by taking entrywise expectation values as defined, is designed to be related to the input Mandel matrix $M^{(2)}(\hat{\rho}_D^{(ab)})$ but going beyond it in a well defined way. Developing it we find:

$$\left\{\begin{pmatrix}1\\ \hat{a}^\dagger\hat{c}^\dagger\\ \hat{b}^\dagger\hat{c}^\dagger\\ \hat{a}\hat{c}\\ \hat{b}\hat{c}\end{pmatrix}\begin{pmatrix}1 & \hat{a}\hat{c} & \hat{b}\hat{c} & \hat{a}^\dagger\hat{c}^\dagger & \hat{b}^\dagger\hat{c}^\dagger\end{pmatrix}\right\}^{PT} =$$

$$
:\begin{pmatrix} 1 \\ \hat a^\dagger \hat c \\ \hat b^\dagger \hat c \\ \hat c^\dagger \hat a \\ \hat c^\dagger \hat b \end{pmatrix} \begin{pmatrix} 1 & \hat c^\dagger \hat a & \hat c^\dagger \hat b & \hat a^\dagger \hat c & \hat b^\dagger \hat c \end{pmatrix} : + \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & & \\ 0 & 0 & 0 & \vdots & \multicolumn{2}{c}{\hat Z} \end{pmatrix},
$$

$$
\begin{aligned}
\hat Z &= \begin{pmatrix} \hat a^\dagger \hat a + \hat c^\dagger \hat c + 1 & \hat b^\dagger \hat a \\ \hat a^\dagger \hat b & \hat b^\dagger \hat b + \hat c^\dagger \hat c + 1 \end{pmatrix} \\
&= : \begin{pmatrix} \hat a \\ \hat b \end{pmatrix} \begin{pmatrix} \hat a^\dagger & \hat b^\dagger \end{pmatrix} : + (1 + \hat c^\dagger \hat c) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{6.5}
$$

Use this in Eq. (6.4), implement the conjugation $\hat U^{-1}(\cdots)\hat U$ and use the fact that the c-mode is initially in vacuum to get:

$$
\begin{aligned}
X &= \mathrm{Tr}(\hat\rho_D^{(ab)} : \begin{pmatrix} 1 \\ \hat a'^\dagger \hat c' \\ \hat b'^\dagger \hat c' \\ \hat c'^\dagger \hat a' \\ \hat c'^\dagger \hat b' \end{pmatrix} \begin{pmatrix} 1 & \hat c'^\dagger \hat a' & \hat c'^\dagger \hat b' & \hat a'^\dagger \hat c' & \hat b'^\dagger \hat c' \end{pmatrix} : \\
&\quad + \mathrm{Tr}(\hat\rho_D^{(ab)} \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & & \\ 0 & 0 & 0 & \vdots & \multicolumn{2}{c}{\hat Z'} \end{pmatrix}), \\
\hat Z' &= : \begin{pmatrix} \hat a' \\ \hat b' \end{pmatrix} \begin{pmatrix} \hat a'^\dagger & \hat b'^\dagger \end{pmatrix} : + (1 + \hat c'^\dagger \hat c') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\begin{pmatrix} \hat a' \\ \hat b' \\ \hat c' \end{pmatrix} &= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{pmatrix} \begin{pmatrix} \hat a \\ \hat b \end{pmatrix}.
\end{aligned} \tag{6.6}
$$

The appearance of the extra terms $\hat Z$, $\hat Z'$ is a result of normal ordering similar to the appearance of $\hat Y$ in Eq. (5.6). One can now disentangle the u- dependences and express the result in terms of $M^{(2)}(\hat{\rho}_D^{(ab)})$ and an additional piece involving $C = \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{C})$:

$$
X \;=\; W(u)M^{(2)}(\hat{\rho}_D^{(ab)})W(u)^\dagger \;+\; \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & \multicolumn{2}{c}{\multirow{2}{*}{Z'}} \\ 0 & 0 & 0 & \vdots & & \end{pmatrix},
$$

$$
W(u) \;=\; \begin{pmatrix} 1 & \vdots & 0 & 0 & \vdots & 0 & 0 \\ \cdots & & \cdots\;\cdots & \cdots\;\cdots & & \cdots\;\cdots & \cdots\;\cdots \\ 0 & \vdots & \begin{pmatrix} u_{11}^* \\ u_{21}^* \end{pmatrix} & \begin{pmatrix} u_{31} & u_{32} \end{pmatrix} & \vdots & \begin{pmatrix} u_{12}^* \\ u_{22}^* \end{pmatrix} & \begin{pmatrix} u_{31} & u_{32} \end{pmatrix} \\ 0 & \vdots & & & \vdots & & \\ \cdots & & \cdots\;\cdots & \cdots\;\cdots & & \cdots\;\cdots & \cdots\;\cdots \\ 0 & \vdots & \multicolumn{2}{c}{u_{31}^* \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}} & \vdots & \multicolumn{2}{c}{u_{32}^* \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}} \\ 0 & \vdots & & & \vdots & & \end{pmatrix},
$$

$$
\begin{aligned}
Z' \;&=\; \mathrm{Tr}(\hat{\rho}_D^{(ab)}\hat{Z}'), \\
\hat{Z}' \;&=\; \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} : \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \begin{pmatrix} \hat{a}^\dagger & \hat{b}^\dagger \end{pmatrix} : \begin{pmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{pmatrix} \\
&\quad + (1 + (u_{31}^*\hat{a}^\dagger + u_{32}^*\hat{b}^\dagger)(u_{31}\hat{a} + u_{32}\hat{b})) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{6.7}
$$

If this matrix X, dependent on $\hat{\rho}_D^{(ab)}$ and $u \in U(3)$, is indefinite, $\hat{\rho}_{\mathrm{out}}^{(abc)}$ of Eq. (6.3) is NPT entangled. For this to happen, as we have assumed, $\hat{\rho}_D^{(ab)}$ must be QO-noncl, since the BS $\hat{U}$ would map any QO-cl input into similar output, the ancilla being in a QO-classical state (vacuum).

A. Two illustrative examples

The first is a two-mode state with only a finite number of photons, so that its QO nonclassicality is a foregone conclusion:

$$
\begin{aligned}
\hat{\rho}_D^{(ab)} \;&=\; p|2,0\rangle\langle 2,0| + q|1,1\rangle\langle 1,1| + r|0,2\rangle\langle 0,2|, \\
&\quad p, q, r \geq 0 \qquad p + q + r = 1.
\end{aligned} \tag{6.8}
$$

This is separable, though not a product state. The only non vanishing expectation values needed to construct the Mandel matrix are

$$\begin{aligned}\langle\hat{a}^\dagger\hat{a}\rangle &= 2p+q,\ \langle\hat{b}^\dagger\hat{b}\rangle = 2r+q,\ \langle\hat{a}^{\dagger 2}\hat{a}^2\rangle = 2p,\\ \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle &= q,\ \langle\hat{b}^{\dagger 2}\hat{b}^2\rangle = 2r.\end{aligned} \tag{6.9}$$

Therefore the Mandel matrix is

$$M^{(2)}(\hat{\rho}_D^{(ab)}) = \begin{pmatrix} 1 & q+2p & 0 & 0 & q+2r \\ q+2p & 2p & 0 & 0 & q \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ q+2r & q & 0 & 0 & 2r \end{pmatrix}. \tag{6.10}$$

The determinants of various nontrivial 2×2 submatrices, the one nontrivial 3×3 submatrix, and finally of $M^{(2)}(\hat{\rho}_D^{(ab)})$ itself, are (indicating the submatrices by the relevant rows and columns):

$$\begin{aligned}&(1,2):\ 2p-(q+2p)^2;\ (1,5):\ 2r-(q+2r)^2;\ (2,5):\ 4pr-q^2;\\ &(1,2,5):\ q^2-4pr;\ \det M^{(2)}(\hat{\rho}_D^{(ab)}) =\ q^2(q^2-4pr).\end{aligned} \tag{6.11}$$

One can easily visualize situations for which the $(1,2)$ and $(1,5)$ submatrices become indefinite, for instance q close to unity and p,r close to zero. In any case, since the $(2,5)$ subdeterminant is opposite to the $(1,2,5)$ subdeterminant in sign and also to the full determinant, the state in Eq. (6.8) is always QO-noncl at the Mandel matrix level.

The type of sub-PS can be easily determined. From Eq. (6.10) we find the 4×4 matrix Γ to be:

$$\begin{aligned}\Gamma &= \begin{pmatrix} \delta_a & 0 & 0 & q-(q+2p)(q+2r) \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ q-(q+2p)(q+2r) & 0 & 0 & \delta_b \end{pmatrix},\\ \delta_a &= 2p-(q+2p)^2\,, \qquad \delta_b = 2r-(q+2r)^2.\end{aligned} \tag{6.12}$$

Therefore also

$$\begin{aligned}\psi_0(\alpha,\beta)^\dagger\Gamma\psi_0(\alpha,\beta) &= 2p|\alpha|^4+4q|\alpha|^2|\beta|^2+2r|\beta|^4\\ &\quad -((q+2p)|\alpha|^2+(q+2r)|\beta|^2)^2.\end{aligned} \tag{6.13}$$

For $\alpha = 1$, $\beta = 0$ this is δ_a; for $\alpha = 0$, $\beta = 1$ it is δ_b. We now consider p running over its range $[0, 1]$ in successive portions and draw corresponding conclusions:

$$\begin{aligned}
p = 0: &\quad q = 0 \;\Rightarrow\; \delta_b = -2;\; q > 0 \;\Rightarrow\; \delta_a < 0; \\
0 < p < \frac{1}{2}: &\quad \delta_a > 0 \;\Rightarrow\; 2p - (p - r + 1)^2 > 0 \;\Rightarrow \\
&\quad (p-r)^2 + 1 - 2r < 0 \;\Rightarrow\; 2r > 1 \;\Rightarrow\; \delta_b < 0; \\
&\quad \delta_a = 0 \;\Rightarrow\; (p-r)^2 + 1 - 2r = 0 \;\Rightarrow\; p \neq r, \\
&\quad 2r > 1 \;\Rightarrow\; \delta_b < 0; \\
p = \frac{1}{2}: &\quad q = 0 \;\Rightarrow\; p = r = \frac{1}{2},\; \delta_a = \delta_b = 0; \\
&\quad q > 0 \;\Rightarrow\; \delta_a < 0; \\
\frac{1}{2} < p \leq 1: &\quad 2p > 1 \;\Rightarrow\; \delta_a < 0.
\end{aligned} \tag{6.14}$$

Thus in every situation except $p = r = \frac{1}{2}$, $q = 0$, either δ_a or δ_b is negative. In this one exceptional case we find from Eq. (6.13):

$$p = r = \frac{1}{2},\; q = 0: \;\; \psi_0(\alpha, \beta)^\dagger \Gamma \psi_0(\alpha, \beta) = -2|\alpha|^2|\beta|^2, \tag{6.15}$$

which is negative for $\alpha, \beta \neq 0$. This establishes that the state (6.8) is of Type I sub-PS.

Now we couple this state to the third c-mode in vacuum, and pass it through a particular $U(3)$ BS, namely a $50 : 50$ BS acting on the b and c modes alone. The output state is calculated using Eq. (6.3), and to test whether it is NPT entangled we need to calculate the matrix X of Eq. (6.7) involving the Mandel matrix term and the added Z' term. The choice of $u \in U(3)$, the resulting $W(u)$, and the two parts of X are as follows:

$$u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \in U(3);$$

$$W(u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & -1/2 \end{pmatrix};$$

$$
\begin{aligned}
W(u)M^{(2)}(\hat{\rho}_D^{(ab)})W(u)^\dagger &= \begin{pmatrix} 1 & 0 & -r-q/2 & 0 & -r-q/2 \\ 0 & q/2 & 0 & 0 & 0 \\ -r-q/2 & 0 & r/2 & 0 & r/2 \\ & & & \cdot\ \cdots & \cdots \\ 0 & 0 & 0 & \vdots\ q/2 & 0 \\ -r-q/2 & 0 & r/2 & \vdots\ 0 & r/2 \end{pmatrix}; \\
Z' &= \begin{pmatrix} 2p+3q/2+r+1 & 0 \\ 0 & q+2r+1 \end{pmatrix}.
\end{aligned}
\tag{6.16}
$$

The dotted lines indicate where the 2×2 block Z' has to be inserted. Leaving out the trivial second and fourth rows and columns as they do not couple to any others, the determinants of the various 2×2 submatrices and the 3×3 submatrix in X are:

$$
\begin{aligned}
&(1,3):\ r/2-(q/2+r)^2;\ \ (1,5):\ 5r/2+q+1-(q/2+r)^2; \\
&(3,5):\ r(q+2r+1)/2;\ \ (1,3,5):\ (q+2r+1)(r/2-(q/2+r)^2).
\end{aligned}
\tag{6.17}
$$

Comparing these with Eqs. (6.11) we see: whenever the QO nonclassicality of $\hat{\rho}_D^{(ab)}$ manifests itself in the $(1,5)$ submatrix of $M^{(2)}(\hat{\rho}_D^{(ab)})$ being indefinite, simultaneously the 3-mode state $\hat{\rho}_{\text{out}}^{(abc)}$ displays NPT entanglement. If on the other hand the $(1,2)$ submatrix of $M^{(2)}(\hat{\rho}_D^{(ab)})$ were indefinite, then by suitably altering the $U(3)$ element u in Eq. (6.16) we can again achieve NPT entanglement of $\hat{\rho}_{\text{out}}^{(abc)}$. In either event, we see how a $U(3)$ BS can produce NPT entanglement starting from a two-mode nonclassical state (6.8), and how the signatures go beyond the indefiniteness of $M^{(2)}(\hat{\rho}_D^{(ab)})$ in a precise manner.

The second example to illustrate the ideas of this Section is similar in structure to the example (5.11) of the preceding Section, but differs in certain details. For a real nonnegative parameter η we define the separable state

$$
\hat{\rho}_D^{(ab)} = \frac{1}{C}\sum_{n=0}^{\infty}\frac{\eta^{2n}}{(2n)!}|n,n\rangle\langle n,n|, \tag{6.18}
$$

where $C=\text{Cosh}\,\eta$, and later $S=\text{Sinh}\,\eta$ and $t=\tanh\eta$. The case $\eta=0$ corresponds to the two-mode vacuum, and so we take $0<\eta<\infty$. Using the elementary sums

$$
\sum_{n=0}^{\infty}(n \text{ or } n^2)\frac{\eta^{2n}}{(2n)!} = \frac{\eta}{2}S \text{ or } \frac{\eta}{4}(S+\eta C), \tag{6.19}
$$

the nonzero expectation values needed for the Mandel matrix are:

$$\begin{aligned}
\langle\hat{a}^\dagger\hat{a}\rangle &= \langle\hat{b}^\dagger\hat{b}\rangle = \frac{\eta}{2}t;\\
\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle &= \langle\hat{b}^{\dagger 2}\hat{b}^2\rangle = \frac{\eta}{4}(\eta-t);\\
\langle\hat{a}^\dagger\hat{b}^\dagger\hat{a}\hat{b}\rangle &= \frac{\eta}{4}(\eta+t).
\end{aligned} \tag{6.20}$$

Therefore we find:

$$M^2(\hat{\rho}_D^{ab}) = \begin{pmatrix} 1 & \frac{\eta t}{2} & 0 & 0 & \frac{\eta t}{2} \\ \frac{\eta t}{2} & \frac{\eta}{4}(\eta-t) & 0 & 0 & \frac{\eta}{4}(\eta+t) \\ 0 & 0 & \frac{\eta}{4}(\eta+t) & 0 & 0 \\ 0 & 0 & 0 & \frac{\eta}{4}(\eta+t) & 0 \\ \frac{\eta t}{2} & \frac{\eta}{4}(\eta+t) & 0 & 0 & \frac{\eta}{4}(\eta-t) \end{pmatrix}. \tag{6.21}$$

Leaving out the third and fourth rows and columns, the remaining 2×2 subdeterminants are:

$$(1,2) \text{ and } (1,5):\ \frac{\eta}{4}(\frac{\eta}{C^2}-t);\ \ (2,5):\ -\frac{\eta^3 t}{4}. \tag{6.22}$$

The function $\frac{\eta}{C^2}-t$ decreases monotonically from 0 to -1 as η runs from zero to infinity. We see that the state (6.18) is QO-noncl for all $\eta>0$. To determine its Type we compute Γ and its 'expectation value' in $\psi_0(\alpha,\beta)$:

$$\begin{aligned}
\Gamma \ &= \ \frac{\eta}{4}\begin{pmatrix} \frac{\eta}{C^2}-t & 0 & 0 & \frac{\eta}{C^2}+t \\ 0 & \eta+t & 0 & 0 \\ 0 & 0 & \eta+t & 0 \\ \frac{\eta}{C^2}+t & 0 & 0 & \frac{\eta}{C^2}-t, \end{pmatrix},\\
\psi_0(\alpha,\beta)^\dagger\Gamma\psi_0(\alpha,\beta) \ &= \ \frac{\eta}{4}\{\frac{\eta}{C^2}-t+2|\alpha|^2|\beta|^2(\eta+3t)\}.
\end{aligned} \tag{6.23}$$

At both $\alpha=1$, $\beta=0$ and $\alpha=0$, $\beta=1$ the last expression is negative, so the state (6.18) is QO-noncl Type I sub-PS. In this context we note that the single-mode state $\hat{\rho}^{(a)}$ obtained from (6.18) by tracing over b is

$$\hat{\rho}_D^{(a)} \ = \ \frac{1}{C}\sum_{n=0}^{\infty}\frac{\eta^{2n}}{(2n)!}|n\rangle_{aa}\langle n|, \tag{6.24}$$

and this has the Mandel matrix and determinant

$$\begin{aligned}
M^{(1)}(\hat{\rho}_D^{(a)}) \ &= \ \begin{pmatrix} 1 & \frac{\eta t}{2} \\ \frac{\eta t}{2} & \frac{\eta}{4}(\eta-t) \end{pmatrix},\\
\det M^{(1)}(\hat{\rho}_D^{(a)}) \ &= \ \frac{\eta}{4}(\frac{\eta}{C^2}-t)<0.
\end{aligned} \tag{6.25}$$

The properties of $\hat{\rho}_D^{(b)}$ are identical. So in contrast to the state (5.11), now both $\hat{\rho}_D^{(a)}$ and $\hat{\rho}_D^{(b)}$ are QO-noncl, accompanying the Type I nature of $\hat{\rho}_D^{(ab)}$.

We now apply the NPT entanglement test outlined in Eqs. (6.4, 6.6, 6.7). The necessary expressions are:

$$W(u)M^{(2)}(\hat{\rho}^{(ab)})W(u)^\dagger = \begin{pmatrix} 1 & 0 & -\frac{\eta t}{4} & 0 & -\frac{\eta t}{4} \\ 0 & \frac{\eta}{8}(\eta+t) & 0 & 0 & 0 \\ -\frac{\eta t}{4} & 0 & \frac{\eta}{16}(\eta-t) & 0 & \frac{\eta}{16}(\eta-t) \\ 0 & 0 & 0 & \frac{\eta}{8}(\eta+t) & 0 \\ -\frac{\eta t}{4} & 0 & \frac{\eta}{16}(\eta-t) & 0 & \frac{\eta}{16}(\eta-t) \end{pmatrix},$$

$$Z' = \begin{pmatrix} 1+\frac{3\eta t}{4} & 0 \\ 0 & 1+\frac{\eta t}{2} \end{pmatrix}. \tag{6.26}$$

The 2×2 matrix Z', which is positive definite, has to be 'added' at the lower right hand corner of the 5×5 matrix, leading to X of Eq. (6.7). Then the positivity or otherwise of X has to be examined. However, even without taking account of Z', and unaffected by Z', the $(1,3)$ subdeterminant of X is $\frac{\eta}{16}(\frac{\eta}{C^2} - t)$, which is negative. This establishes the NPT entanglement of $\hat{\rho}_{\text{out}}^{(abc)}$ in this example.

The considerations of this Section show that the scheme described in Section II, elevating single mode QO-noncl states to the two-mode level and then allowing BS action to produce NPT entanglement, generalizes to the next higher level. The method of Mandel matrices is a practical way to see these processes in action.

VII. GENUINE TRIPARTITE ENTANGLEMENT

Now that the main methods of our approach-signatures of nonclassicality at the Mandel matrix level, their nontrivial extensions to signatures of (NPT) entanglement created by BS action – have been applied to several examples, we go on to consider some more subtle features of entanglement. We will show via an example that in the three-mode case the BS action on an initial two-mode nonclassical state can lead to genuine residual tripartite entanglement. This is in the sense of [43], whereby the output is a tripartite state similar to the GHZ state [44]: there is no bipartite entanglement when any one of the three modes is traced away.

We consider the state (5.11) studied in Section V, and subject it to the treatment of Section VI. As we have seen, this (separable) state shows Type II sub-PS. With $\hat{\rho}_D^{(ab)}$ as in (5.11), we pass the state (6.1),

$$\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0| = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}|n,n\rangle_{ab\,ab}\langle n,n| \otimes |0\rangle_{cc}\langle 0|, \tag{7.1}$$

through a 50:50 $b-c$ BS, a special $U(3)$ element, whose action on the mode operators $\hat{b}$ and $\hat{c}$ is

$$\hat{U}\begin{pmatrix}\hat{b}\\ \hat{c}\end{pmatrix}\hat{U}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\hat{b}\\ \hat{c}\end{pmatrix}. \tag{7.2}$$

The resulting state is

$$\begin{aligned}\hat{\rho}_{\rm out}^{(abc)} &= \hat{U}(\hat{\rho}_D^{(ab)} \otimes |0\rangle_{cc}\langle 0|)\hat{U}^{-1}\\ &= e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{2^n n!}|n\rangle_{aa}\langle n| \otimes (\hat{b}^\dagger+\hat{c}^\dagger)^n|0,0\rangle_{bc\,bc}\langle 0,0|(\hat{b}+\hat{c})^n\\ &= e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}|n\rangle_{aa}\langle n| \otimes \sum_{r,s=0}^{n}\frac{|r,n-r\rangle_{bc\,bc}\langle s,n-s|}{\sqrt{r!(n-r)!s!(n-s)!}}.\end{aligned} \tag{7.3}$$

Clearly this is separable in the a/bc cut. However it is entangled in both the c/ab and b/ac cuts as we show below. As a test for NPT entanglement in the c/ab cut, we evaluate the 'expectation value' of a suitably chosen positive operator on the partially transposed output $\hat{\rho}_{\rm out}^{(abc)PT}$, the partial transpose being applied on the c mode. For the choice of operator $\hat{A}^\dagger\hat{A}$ where

$$\hat{A} = \alpha_0 + \alpha_1\hat{b}\hat{c} + \alpha_2\hat{a}^\dagger\hat{a}, \tag{7.4}$$

a test for NPT entanglement would be to check for violation of positivity of

$$\begin{aligned}{\rm Tr}(\hat{\rho}_{\rm out}^{(abc)PT}\hat{A}^\dagger\hat{A}) &= {\rm Tr}(\hat{\rho}_{\rm out}^{(abc)}(\hat{A}^\dagger\hat{A})^{PT})\\ &= \begin{pmatrix}\alpha_0^* & \alpha_1^* & \alpha_2^*\end{pmatrix} X \begin{pmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{pmatrix},\\ X &= {\rm Tr}(\hat{\rho}_{\rm out}^{(abc)}\begin{pmatrix}1 & \hat{b}\hat{c}^\dagger & \hat{a}^\dagger\hat{a}\\ \hat{b}^\dagger\hat{c} & \hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c} & \hat{b}^\dagger\hat{c}\hat{a}^\dagger\hat{a}\\ \hat{a}^\dagger\hat{a} & \hat{a}^\dagger\hat{a}\hat{b}\hat{c}^\dagger & \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\end{pmatrix}).\end{aligned} \tag{7.5}$$

Using the fact that initially the c-mode is in the vacuum, we find:

$$X = \begin{pmatrix} 1 & \lambda/2 & \lambda \\ \lambda/2 & \lambda^2/4 & \lambda(\lambda+1)/2 \\ \lambda & \lambda(\lambda+1)/2 & \lambda(\lambda+1) \end{pmatrix}. \tag{7.6}$$

As the (2,3) submatrix of X has negative determinant, $\hat{\rho}_{\rm out}^{(abc)}$ is NPT entangled across the c/ab cut. It is easy to see that a similar test with the same choice of $\hat{A}$, except that now the PT operation is applied to the b mode, yields the conclusion that $\hat{\rho}_{\rm out}^{(abc)}$ is NPT entangled across the b/ac cut. So we find in this example bipartite entanglement in a tripartite setup, as a result of BS action.

Now to show that the entanglement is genuine tripartite, 'residual' in the sense of [43], we have the following:

$$\begin{aligned}
\hat{\rho}_{\rm out}^{(ab)} &= {\rm Tr}_c(\hat{\rho}_{\rm out}^{(abc)}) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}|n\rangle_{a\,a}\langle n|\sum_{r=0}^{n}\frac{|r\rangle_{b\,b}\langle r|}{r!(n-r)!},\\
\hat{\rho}_{\rm out}^{(ac)} &= {\rm Tr}_b(\hat{\rho}_{\rm out}^{(abc)}) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}|n\rangle_{a\,a}\langle n|\sum_{r=0}^{n}\frac{|r\rangle_{c\,c}\langle r|}{r!(n-r)!},\\
\hat{\rho}_{\rm out}^{(bc)} &= {\rm Tr}_a(\hat{\rho}_{\rm out}^{(abc)}) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n n!}{2^n}\sum_{r,s=0}^{n}\frac{|r,n-r\rangle_{bc\,bc}\langle s,n-s|}{\sqrt{r!(n-r)!s!(n-s)!}}.
\end{aligned} \tag{7.7}$$

The first two are manifestly separable. It may not be obvious at first glance that the third is also separable but a closer look shows that it can be written in the form

$$\hat{\rho}_{\rm out}^{(bc)} = e^{-\lambda}\hat{U}(\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}|n\rangle_{b\,b}\langle n|\otimes|0\rangle_{cc}\langle 0|)\hat{U}^{-1}, \tag{7.8}$$

where U is the 50:50 $b-c$ BS (7.2). Note that $\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}|n\rangle_{bb}\langle n|$, the state of the b- mode at the input, is simply the phase averaged (coarse grained) version of the coherent state $|\sqrt{\lambda}\rangle$. Thus $\hat{\rho}_{\rm out}^{(bc)}$ is the outcome of a classical state passed through a BS, so it is classical and hence separable.

It is interesting that the feature of genuine tripartite entanglement is reminiscent of Type II sub-PS for a two-mode state at the Mandel level, where the nonclassicality never shows up at any single mode level. An interesting question in this context is the possibility of extension of monogamy relations to this non-Gaussian case [45, 46].

VIII. FURTHER PROPERTIES OF MANDEL PARAMETERS AND BEAM-SPLITTERS

For one and two mode field states, we have used the 2×2 and 5×5 Mandel matrices respectively to classify the states in a physically useful manner. It is convenient to also have suitably normalized single parameter – 'scalar' – measures of nonclassicality defined in terms of the Mandel matrices. In the two-mode case a useful requirement would be invariance of such measures under BS action.

We begin with the single mode case. Here the Mandel Q parameter was defined **in [8]** as

$$\begin{aligned} Q &= \frac{(\Delta \hat{N}_a)^2 - \langle \hat{N}_a \rangle}{\langle \hat{N}_a \rangle} \\ &= \frac{\langle \hat{a}^{\dagger 2} \hat{a}^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2}{\langle \hat{a}^\dagger \hat{a} \rangle} \\ &= \frac{\det M^{(1)}(\hat{\rho}_D^{(a)})}{\langle \hat{a}^\dagger \hat{a} \rangle}. \end{aligned} \tag{8.1}$$

Here all expectation values are in the state $\hat{\rho}_D^{(a)}$, and Eq. (4.2) has been used. From the classification (4.4), $Q > 0$ and $Q < 0$ correspond respectively to super-PS and nonclassical sub-PS cases. This parameter is bounded below by -1, which is a convenient normalization. For $Q > 0$ there is no upper bound.

In attempting to generalize to two modes, as a first step we show that the separation of QO-noncl states into Types I and II is BS action invariant. For any $u \in U(2)$, from Eq. (2.13) and the direct product structure of $\hat{C}$ in Eq. (4.8) we easily obtain:

$$\begin{aligned} \hat{U}^{-1} \hat{C} \hat{U} &= V \hat{C}, \quad \hat{U}^{-1} \hat{C}^\dagger \hat{U} = \hat{C}^\dagger V^\dagger, \\ \hat{U}^{-1} \hat{B} \hat{U} &= V \hat{B} V^\dagger, \\ V &= u^* \otimes u. \end{aligned} \tag{8.2}$$

Therefore, with C and B defined as in Eq. (4.15), under general BS action we have:

$$\begin{aligned} \hat{\rho}_D^{\prime(ab)} &= \hat{U} \hat{\rho}_D^{(ab)} \hat{U}^{-1} \Rightarrow C' = VC, \quad B' = VBV^\dagger, \\ \Gamma' &= V \Gamma V^\dagger. \end{aligned} \tag{8.3}$$

Now in the QO-noncl family of states, corresponding to $M^{(2)}(\hat{\rho}_D^{(ab)}) \ngeq 0$ or equivalently to $\Gamma \ngeq 0$, the further separation into Types I and II is given in Eq. (4.18). Here it is the 'expectation values' of Γ in four component column vectors $\psi_0(\alpha,\beta)$ that are relevant. However these vectors too have a direct product structure (4.12), so they are mapped into similar vectors under the above changes:

$$
\begin{aligned}
V^\dagger \psi_0(\alpha,\beta) &= (u^T \otimes u^\dagger) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix} = \psi_0(\alpha',\beta'), \\
\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} &= u^T \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
\end{aligned} \tag{8.4}
$$

This proves that the separation into Types I and II is preserved under BS action. More specifically, if a BS converts a QO-noncl separable state of a definite Type into an NPT entangled state, this change occurs within the subfamily of that Type.

Now we generalize (8.1) to the two mode case. Keeping the requirement of BS action invariance in mind, we define the two-mode Mandel parameter as

$$
Q' = \frac{\mathrm{Tr}(\Gamma) - ||\Gamma||}{2(\langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{b}^\dagger \hat{b} \rangle)}, \tag{8.5}
$$

Here $||\Gamma||$ is the trace norm of Γ, which for hermitian Γ is the sum of the absolute values of its eigenvalues. Therefore Q' is simply the sum of the negative eigenvalues of Γ divided by the expectation value of the total number operator $\hat{N}_a + \hat{N}_b$. From Eq. (8.3), the invariance of Q' under BS action is obvious.

The parameter Q' vanishes for QO-cl states (as defined via the Mandel matrix), and is strictly negative for QO-noncl states. For two-mode product Fock states, for instance, $Q' = -1$. To follow the distinction between the two Types, we combine Eqs. (4.12, 8.1) to define the variable single mode Mandel parameter

$$
\begin{aligned}
Q(\alpha,\beta) &= \frac{\langle \hat{A}^{\dagger 2} \hat{A}^2 \rangle - \langle \hat{A}^\dagger \hat{A} \rangle^2}{\langle \hat{A}^\dagger \hat{A} \rangle}, \\
\hat{A} &= \alpha \hat{a} + \beta \hat{b}.
\end{aligned} \tag{8.6}
$$

The minimum value of $Q(\alpha,\beta)$ as α, β vary is also useful:

$$
Q^{\min} = \min_{|\alpha|^2 + |\beta|^2 = 1} Q(\alpha,\beta). \tag{8.7}
$$

From all the previous discussions we draw up a table of results characterizing various two-mode states (always at the Mandel level) in Table I. For some of these, we have examples

Category	Definition	Description
(i)	$Q^{\min} \geq 0,\ Q' = 0$	QO-cl
(iia)	$Q^{\min} \leq Q' < 0$	QO-noncl Type I
(iib)	$Q' < Q^{\min} < 0$	QO-noncl Type I
(iii)	$Q^{\min} \geq 0,\ Q' < 0$	QO-noncl Type II

TABLE I:

from previous Sections. All QO-cl states, including the states (4.20) for $\det M^{(1)}(\hat{\rho}_D^{(a)}) \geq 0$, come under category (i). On the other hand, the states (4.20) for $\det M^{(1)}(\hat{\rho}_D^{(a)}) < 0$ belong to category (ii). The subclassification into (iia) and (iib) is subtle, but simple examples of each can be provided. For category (iia) we consider the product of a Fock state at the a mode and a coherent state at the b mode:

$$|\psi\rangle = |n\rangle_a \otimes |z\rangle_b. \tag{8.8}$$

This QO-noncl separable. The matrix Γ is diagonal,

$$\Gamma = \mathrm{diag}(-n, n|z|^2, n|z|^2, 0), \tag{8.9}$$

leading to

$$Q_2 = -n/(n + |z|^2) \geq -1. \tag{8.10}$$

On the other hand, $Q(1,0) = -1$, so $Q^{\min} \leq Q' < 0$ which falls under (iia). For category (iib) we can take the states (6.18) which are QO-noncl and separable. Using Eqs. (6.20, 6.23) we find:

$$\begin{aligned} Q' &= -1/2 \ : \\ Q(\alpha,\beta) &= \psi_0(\alpha,\beta)^\dagger \Gamma \psi_0(\alpha,\beta)/\langle \hat{A}^\dagger \hat{A} \rangle \\ &= \frac{1}{2t}\{\frac{\eta}{C^2} - t + 2|\alpha|^2|\beta|^2(\eta + 3t)\}. \end{aligned} \tag{8.11}$$

The minimum of $Q(\alpha,\beta)$ is reached when either α or β is zero:

$$Q^{\min} = \frac{1}{2t}\{\frac{\eta}{C^2} - t\} > -\frac{1}{2}. \tag{8.12}$$

As $0 > Q^{\min} > Q'$, this falls under category (iib).

Another interesting example for (iib) is the class of pure states obtained as an equal in-phase superposition of product Fock states with given total occupation number n :

$$|\psi_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{r=0}^{n} |r, n-r\rangle. \tag{8.13}$$

For the cases $n = 1, 2, 3, 4$ the numerically computed values of Q' are -1, -1.085, -1.123, -1.143 respectively. Since in any case $Q(\alpha, \beta)$ and Q^{min} are bounded below by -1, we have $Q' < Q^{\text{min}}$. We may also note that these entangled states cannot be produced by BS action on product Fock states, except when $n = 1$.

Turning finally to category (iii), we have examples from Sections 4 and 5. For the two-mode squeezed vacuum state (4.23), we have using Eq. (4.26) and the eigenvalue spectrum of Γ stated after Eq. (4.25) :

$$\begin{aligned} Q' &= SS'(SS' - CC')/(S^2 + S'^2) < 0, \\ Q^{\text{min}} &> 0. \end{aligned} \tag{8.14}$$

For the family of states (5.11), from Eqs. (5.12, 5.13) we find :

$$Q' = -1/2, \quad Q^{\text{min}} > 0. \tag{8.15}$$

So in both cases we have category (iii) states.

IX. CONCLUSIONS

In this work we have investigated the relationships between quantum optical nonclassicality of the phase insensitive type and entanglement in multimode radiation fields In particular we have examined the possibilities of converting nonclassicality in such fields into entanglement through the use of classicality preserving passive devices such as beamsplitters. For the case of a single mode, after giving a complete characterisation of the quantum optical nonclassicality at the level of phase insensitive quantities, we have shown that such states through a beamsplitter action with vacuum or more generally a coherent state at the other port, always give rise to an NPT entangled state. For the case of two mode radiation fields we have presented a test which simultaneously witnesses both nonclassicality and entanglement in such states and have also developed a scheme based on Mandel matrices for characterising and classifying nonclassicality in one and two mode states. In particular, it is shown that in the two mode case, the characterisation at the level of Mandel matrices permits us to divide two mode states into two categories–Type I where the nonclassicality manifests itself already at the single mode level and Type II where the nonclassicality is intrinsically two mode in character with no Mandel type signatures of nonclassicality at the single mode level. We have also examined in detail the possibility of a $U(2)$ beamsplitter converting Mandel level nonclassicality in a two mode separable state into an NPT entangled state and have given tests for NPT entanglement in the state resulting from such an action. Distillability of the state so produced is demonstrated in one case. Further, in a similar spirit as for the case of a single mode, we have also analysed the action of a $U(3)$ beamsplitter on a two mode Mandel level nonclassical state and have derived conditions under which the two mode nonclassicality manifests itself in NPT entanglement in the resulting three mode states. In this context, we have also shown, via an example, how such an action can lead to a genuine tripartite entangled state in the sense that there is no bipartite entanglement when any one of the three modes is traced away. With a view to ease in categorisation of nonclassicality in two mode states, by appealing to invariance under beamsplitter action we have suggested analogues of the Mandel Q-parameter originally introduced in the context of single mode radiation states.

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