Asset pricing puzzles explained by incomplete Brownian equilibria[a]

September 20, 2010

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[a]We wish to thank Jerome Detemple, Darrell Duffie, Costis Skiadas, Semyon Malamud, and Steve Shreve for constructive comments.

Abstract: We examine a class of Brownian based models which produce tractable incomplete equilibria. The models are based on finitely many investors with heterogeneous exponential utilities over intermediate consumption who receive partially unspanned income. The investors can trade continuously on a finite time interval in a money market account as well as a risky security. Besides establishing the existence of an equilibrium, our main result shows that the resulting equilibrium can display a lower risk-free rate and a higher risk premium relative to the usual Pareto efficient equilibrium in complete markets. Consequently, our model can simultaneously help explaining the risk-free rate and equity premium puzzles.

Keywords: Incomplete markets · equity premium puzzle · risk-free rate puzzle · non-Pareto efficiency · stochastic volatility · stochastic interest rates

1 Introduction and notation

We present incomplete Brownian based models allowing us to explicitly quantify the impact that unspanned income and preference heterogeneity can have on the resulting equilibrium interest rate and risk premium. The finite number of investors can trade continuously on a finite time horizon and they maximize expected exponential utility of intermediate consumption. We show that if the investors cannot consume continuously over time, unspanned income can lower the risk-free rate and raise the risk premium when compared to the standard complete Pareto efficient equilibrium. Subsequently, we consider the limiting case where investors can consume continuously over time and in a model-free manner we show that unspanned income can affect the equilibrium risk-free rate but can never affect the equilibrium instantaneous risk premium relative to the complete Pareto efficient equilibrium. However, if risk premia are measured over finite time-intervals (as in empirical studies of asset pricing puzzles), our model with unspanned income and stochastic volatility can raise the equilibrium risk premium (and lower the equilibrium risk-free rate) relative to the Pareto efficient analogue.

The questions of existence and characterization of complete equilibria in continuous time and state models are well-studied, and we refer to Chapter 4 in [KS98] and Chapter 10 in [Duf01] for a literature overview. More recent references on complete equilibria include [Žit06], [CJMN09] and [HMT09]. The most common technique is based on the martingale method from [KLS87] and [CH89] which in complete markets settings provides an explicit characterization of the investor's optimizer. By using the so-called representative agent method the search for a complete market equilibrium can be reduced to a finite dimensional fixed point problem. To the best of our knowledge, only [CH94] and [Žit10] consider the abstract existence of a non-Pareto efficient equilibrium in a continuous trading setting. We provide tractable incomplete models for which the equilibrium price processes can be computed explicitly and, consequently, we can quantify the impact of market incompleteness.

To obtain incompleteness effects on the equilibrium risk premium we incorporate a stochastic volatility v à la Heston's model into the equilibrium stock price dynamics. In Heston's original model [Hes93] the stock's relative volatility is v whereas in this paper v will be the stock's absolute volatility. We explicitly derive expressions for equilibrium risk-free rate and the risk premium in terms of the individual income dynamics as well as the absolute risk aversion coefficients. The resulting type of equilibrium equity premium has been widely used in various optimal investment models, see e.g., [CV05] and [Kra05], whereas the resulting type of equilibrium interest rate is similar to the celebrated CIR-term structure model.

Translation invariant models (such as the exponential model we consider) allow consumption to be negative, see e.g., the discussion in the textbook [Ski09]. [SS05] show that this class of models is fairly tractable even when income is unspanned. We first conjecture the equilibrium form of the market price of risk process and then use the idea in [CH94] to re-write the individual investor's problem as a problem with spanned income and heterogeneous beliefs. In certain affine settings with a deterministic interest rate the exponential investor's value function is available in closed form, see e.g., [Hen05], [Wan04], [Wan06], and [CLM10]. However, the incorporation of stochastic volatility produces a stochastic equilibrium interest rate preventing the corresponding HJB-equation to have the usual exponential affine form. Therefore, the individual investor's value function is not available in closed form in our setting, however, by using martingale methods we obtain tractable expressions for the individual optimal consumption policies which turn out to be sufficient to produce the incomplete equilibrium.

In a discrete infinite time horizon model with a continuum of agents, [Wan03] illustrates the negative effect unhedgeable income risk can have on the risk-free interest rate. [CLM10] present a continuous model with a finite number of agents exhibiting the same risk-free rate phenomena. In a discrete setting [KL10] provide conditions for power preferences under which non-hedgeable idiosyncratic income risk will lower the risk-free rate, but not affect the risk premium. We extend these results by proving that as long as the income dynamics are continuous over time, any equilibrium based on exponential preferences produces the same instantaneous risk premium as the standard Pareto efficient analogue. Nevertheless, we also prove that risk premia measured over finite time-intervals can be higher due to non-hedgeable income risk components.

[CD96] produce similar conclusions but they perform a discrete trading analysis under various assumptions including bounds on aggregate endowments and requiring agents to have identical risk preferences. Our model does not rely on such assumptions, and it quantifies the role of preference heterogeneity.

$(\Omega, \mathcal{F}, \mathbb{P})$ denotes the probability space on which all stochastic quantities are defined. We consider a pure exchange economy with a single consumption good which we use as the numéraire. (W, Z) denotes an $I + 1$ dimensional Brownian motion where W is scalar valued and I is the (finite) number of investors. The standard Brownian filtration generated by (W, Z) is denoted by $\mathcal{F}_t$, $t \in [0, T]$, where T is the finite time horizon and we consider $\mathcal{F} := \mathcal{F}_T$. We will write $\mathbb{E}_t[\cdot]$ instead of $\mathbb{E}^{\mathbb{P}}[\cdot|\mathcal{F}_t]$. As usual $\mathcal{L}^p$ denotes the space of measurable and adapted processes f such that

$$\int_0^T |f_u|^p du < \infty, \quad \mathbb{P}\text{-almost surely}, \quad p \in \{1, 2\}.$$

Finally, Λ denotes the set of bounded real-valued functions on $[0, T]$.

2 The financial market

We start by defining the exogenous Feller process

$$dv_t := \mu_v dt + \sigma_v \sqrt{v_t} dW_t, \quad t \in [0, T], \quad v_0 > 0. \tag{2.1}$$

The constants (μ_v, σ_v) satisfy Feller's condition $\mu_v \geq \sigma_v^2/2$ which ensures that v remains strictly positive on $[0, T]$. We conjecture (and confirm) that there exists an interest rate process $r \in \mathcal{L}^1$ such that the equilibrium dynamics of the money market account $S^{(0)}$ are given by

$$dS_t^{(0)} = S_t^{(0)} r_t dt, \quad t \in [0, T], \quad S_0^{(0)} > 0, \tag{2.2}$$

for some process r adapted to the filtration generated by v, i.e., r_t is $\mathcal{F}_t^v := \sigma(v_u)_{u\in[0,t]}$-measurable for $t \in [0, T]$. We conjecture (and confirm) that there exists $\mu_S \in \Lambda$ such that the equilibrium stock price dynamics are given by

$$dS_t = \Big(r_t S_t + \mu_S(t) v_t\Big) dt + \sqrt{v_t} dW_t, \quad t \in [0, T], \quad S_0 > 0. \tag{2.3}$$

The idiosyncratic Brownian motions Z_i do not appear directly in the stock price dynamics (2.3) nor in the spot rate dynamics. However, as we shall see, the presence of $(Z_i)_{i=1}^I$ has a significant impact on these quantities. Since $d\langle S, v\rangle_t = v_t \sigma_v dt$, the parameter σ_v controls the cross variation between the stock and the volatility. In what follows σ_v plays an important role and to be consistent with empirical observations this parameter should be negative, see, e.g., the discussions in Chapter 5 in [Gat06].

A strictly positive progressively measurable process ξ is a state-price deflator if both $\xi_t S_t^{(0)}$ and $\xi_t S_t$ are driftless (under $\mathbb{P}$), see, e.g., Chapter 6 in [Duf01]. An important ingredient will be the minimal state-price density defined by

$$d\xi_t^{\min} := -\xi_t^{\min}\Big(r_t dt + \mu_S(t)\sqrt{v_t} dW_t\Big), \quad \xi_0^{\min} > 0. \tag{2.4}$$

The corresponding minimal martingale measure $\mathbb{Q}^{\min}$ is defined via the Radon-Nikodym derivative on $\mathcal{F}_T$ as

$$\frac{d\mathbb{Q}^{\min}}{d\mathbb{P}} := \frac{\xi_T}{\xi_0} \exp\left(\int_0^T r_u du\right) > 0.$$

Since v is a Feller process and $\mu_S \in \Lambda$, we see that Novikov's condition is satisfied[2]. Consequently, Girsanov's theorem ensures that

$$dW_t^{\min} := dW_t + \mu_S(t)\sqrt{v_t}dt, \quad W_0^{\min} := 0,$$

is a Brownian motion under $\mathbb{Q}^{\min}$ which is independent of $(Z_1, ..., Z_I)$.

3 The individual investor's problem

In this section we first set up the investors' problems and subsequently present partial solutions.

3.1 Problem formulation

Investor i, $i = 1, ..., I$, receives income determined by the process

$$dY_{it} := \sqrt{v_t}\Big(\sigma_{Y_i} dW_t + \beta_{Y_i} dZ_{it}\Big), \quad t \in [0,T], \quad Y_{i0} \geq 0. \tag{3.1}$$

Here σ_{Y_i} and β_{Y_i}, $i = 1, ..., I$, are constants. The W-Brownian motion affects all investors whereas Z_i models investor i's idiosyncratic risk. The Feller process v affects each investor's individual income process Y_i and acts as a common stochastic volatility. The investor's cumulative income at time t is defined by $\int_0^t Y_{iu}\Gamma(du)$ where Γ is a finite measure on $[0,T]$. It is straightforward to allow Γ to be investor specific in what follows. We will need two specifications of Γ: For $N \in \mathbb{N}$ we first define Γ_N by

$$\int_0^t Y_{iu}\Gamma_N(du) := \Delta \sum_{n\,:\,t_n \leq t} Y_{t_n}, \quad \Delta := \frac{T}{N}, \quad t_n := n\Delta, \quad n = 0, 1, ..., N.$$

[2]More specifically, since v_t is non-centrally χ^2-distributed Novikov's condition is satisfied for $T > 0$ sufficiently small. We can then use a localization argument (see, e.g., Section 6.2 in [LS77]) to obtain the global martingale property. We will use this observation without mentioning multiple times in what follows.

Secondly, we need the continuous case where Γ_∞ is the Lebesgue measure, i.e., the cumulative income up to time t is given by

$$\int_0^t Y_{iu}\Gamma_\infty(du) := \int_0^t Y_{iu}du, \quad t \in [0,T].$$

The following analysis remains valid if we allow v and $(Y_i)_{i=1}^I$ defined by (2.1) and (3.1) to have an affine drift of v. On the other hand, it is not immediate how to adjust our approach to cover the mean-reverting income models used in [Wan04] and [Wan06]. These optimal investment models are based on deterministic interest rates, however, the corresponding equilibrium interest rate cannot be deterministic in these affine settings. Unlike the power investor, stochastic interest rates complicate the exponential investor's optimal investment problem tremendously. As we shall see, the income processes (3.1) produce an equilibrium stochastic interest rate for which the individual exponential investor's optimal investment problem remains partially tractable.

The investor chooses a trading strategy θ as well as a cumulative consumption process C (in excess of income) and (as we shall see) we will only need to consider right-continuous processes of the form

$$C_t := \int_0^t c_u \Gamma(du), \quad t \in [0,T],$$

for some Γ-rate process c. $X_t^{\theta,c}$ denotes the time t value of the investor's financial wealth, i.e., $X_t^{\theta,c} := \theta_t S_t + \theta_t^{(0)} S_t^{(0)}$ where θ denotes the number of units held of the risky security S and $\theta^{(0)}$ denotes the number of units of the money market account $S^{(0)}$ held. The corresponding self-financing wealth dynamics read

$$dX_t^{\theta,c} = r_t X_t^{\theta,c}dt + \theta_t\Big(\mu_S(t)v_t dt + \sqrt{v_t}dW_t\Big) - dC_t, \quad X_0^{\theta,c} := 0.$$

In the market $(S^{(0)}, S)$ all European claims written on the stock, i.e., claims paying out $g(S_T)$ at time T for some payoff function g, are replicable. However, the individual investor's endowment process Y_i cannot be fully hedged due to the presence of Z_i in the dynamics of Y_i. Therefore, $(S^{(0)}, S)$ is an incomplete market.

In order to ensure that the wealth dynamics are well-defined, we require that the processes (c, θ) are progressively measurable and satisfy the integrability requirements

$$\int_0^T |c_u|\Gamma(du) < \infty, \quad \int_0^T \theta_t^2 v_t dt < \infty, \quad \mathbb{P}\text{-almost surely.} \tag{3.2}$$

We deem measurable and adapted processes (θ, c) admissible if in addition to (3.2) the budget constraint (recall that the initial wealths are zero)

$$\mathbb{E}\left[\int_0^T \xi_u^{\min} c_{iu}\Gamma(du)\right] \leq 0, \tag{3.3}$$

is satisfied in which case we write $(c, \theta) \in \mathcal{A} = \mathcal{A}(\xi^{\min})$. This condition also ensures that there are no arbitrage opportunities in the set $\mathcal{A}$.

Investor i's preferences are modeled by the negative exponential utility function

$$U_i(x) := -e^{-a_i x}, \quad x \in \mathbb{R}, \quad i = 1, ..., I,$$

where $a_i > 0$ denotes the investor's absolute risk aversion coefficient.

The investor maximizes time-additive expected utility stemming from running consumption in addition to the investor's income, i.e., the investor seeks $(\hat{c}_i, \hat{\theta}_i) \in \mathcal{A}$ such that

$$\sup_{(c,\theta)\in\mathcal{A}} \mathbb{E}\left[\int_0^T U_i(c_u + Y_{iu})\Gamma(du)\right] = \mathbb{E}\left[\int_0^T U_i(\hat{c}_{iu} + Y_{iu})\Gamma(du)\right]. \tag{3.4}$$

To simplify the presentation we have assumed that all (finitely many) investors receive income at the same time points. Because the individual consumption c is allowed to be negative it is mathematically tractable to allow for consumption at time points where aggregate income is zero. The above problem formulation explicitly restricts the individual investors to only consume at time points when aggregate income is available.

3.2 Partial solution

We next partially solve (3.4). Because the interest rate r is stochastic the PDE produced by the HJB-approach does not have the usual exponential affine solution that [Hen05] and [CLM10] rely on. Inspired by [CH94], we instead convert the problem into an equivalent problem with spanned income but heterogeneous beliefs. We define the $\mathbb{P}$-equivalent probability measure $\mathbb{P}_i$, $i = 1, ..., I$, via the Radon-Nikodym derivative $\frac{d\mathbb{P}_i}{d\mathbb{P}} := \pi_{iT} > 0$ on $\mathcal{F}_T$ where

$$\pi_{it} := \exp\left(-a_i\beta_{Y_i}\int_0^t \sqrt{v_u}dZ_{iu} - \frac{1}{2}a_i^2\beta_{Y_i}^2\int_0^t v_u du\right), \quad t \in [0,T].$$

By Novikov's condition and Girsanov's theorem we know that W remains a Brownian motion under each $\mathbb{P}_i$, $i = 1, ..., I$. Therefore, S and $X^{c,\theta}$ also have the same dynamics under $\mathbb{P}_i$ as under $\mathbb{P}$. Problem (3.4) then becomes

$$\sup_{(c,\theta)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}_i}\left[\int_0^T U_i\left(c_u + \tilde{Y}_{iu}\right)\Gamma(du)\right], \tag{3.5}$$

which can be seen as a standard complete optimization problem with the spanned income Γ-rate process

$$\tilde{Y}_{it} := Y_{i0} + \sigma_{Y_i}\int_0^t \sqrt{v_u}dW_u - \frac{1}{2}a_i\beta_{Y_i}^2\int_0^t v_u du, \quad t \in [0,T]. \tag{3.6}$$

Theorem 3.1. *Let $r \in \mathcal{L}^1$ be adapted to $\mathcal{F}_t^v := \sigma(v_u)_{u\in[0,t]}$ and let $\mu_S \in \Lambda$.*

1. *Discrete case Γ_N: Define recursively the consumption strategy*

$$\hat{c}_{i0} := -\frac{1}{a_i}\log\left(\frac{\alpha_i \xi_0}{a_i}\right) - Y_{i0}, \tag{3.7}$$

$$\hat{c}_{it_n} := \hat{c}_{it_{n-1}} + \int_{t_{n-1}}^{t_n} \frac{1}{a_i}\Big\{r_u + \frac{1}{2}\Big(\mu_S(u)^2 + a_i^2\beta_{Y_i}^2\Big)v_u\Big\}du + \int_{t_{n-1}}^{t_n}\Big(\frac{1}{a_i}\mu_S(u) - \sigma_{Y_i}\Big)\sqrt{v_u}dW_u, \tag{3.8}$$

where the constant $\alpha_i > 0$ is determined by the budget constraint

$$0 = \mathbb{E}\left[\int_0^T \xi_u^{min}\hat{c}_{iu}\Gamma(du)\right]. \tag{3.9}$$

Then there exists an investment strategy $\hat{\theta}_i$ such that the pair $(\hat{c}_i, \hat{\theta}_i) \in \mathcal{A}$ is optimal for investor i, $i = 1, ..., I$.

2. *Continuous case Γ_∞: Let $\hat{c}_{i0}$ be defined by (3.7) and define the dynamics*

$$d\hat{c}_{it} := \frac{1}{a_i}\Big\{r_t + \frac{1}{2}\Big(\mu_S(t)^2 + \frac{1}{2}a_i^2\beta_{Y_i}^2\Big)v_t\Big\}dt + \Big(\frac{\mu_S(t)}{a_i} - \sigma_{Y_i}\Big)\sqrt{v_t}dW_t, \tag{3.10}$$

where α_i is determined by (3.9). Then there exists an investment strategy $\hat{\theta}_i$ such that the pair $(\hat{c}_i, \hat{\theta}_i) \in \mathcal{A}$ is optimal for investor i, $i = 1, ..., I$.

Because problem (3.5) corresponds to a complete market with spanned income processes $(\tilde{Y}_i)_{i=1}^I$ the result follows immediately from the standard martingale method. Therefore, the optimal investment strategy $\hat{\theta}_i$ is produced by the martingale representation theorem via the relation

$$X_t^{\hat{\theta}_i,\hat{c}_i} = \mathbb{E}_t^{\mathbb{Q}^{\min}}\left[\int_t^T \exp\left(-\int_t^u r_s ds\right)\hat{c}_{iu}\Gamma(du)\right], \quad t \in [0,T]. \tag{3.11}$$

However, a tractable expression for the optimal investment strategy $\hat{\theta}_i$ is not available because the interest rate r is stochastic. Fortunately, we will have no need of such an expression in what follows.

From (3.11) we see that in the continuous case (Γ_∞) we have $X_T^{\hat{c}_i,\hat{\theta}_i} = 0$, $\mathbb{P}$-a.s., meaning that the investors optimally leave no wealth behind after maturity. On the other hand, (3.11) shows that in the discrete case (Γ_N) we have $X_T^{\hat{c}_i,\hat{\theta}_i} = \hat{c}_{iT}\Delta$, $\mathbb{P}$-a.s., which means that at maturity each investor consumes whatever remaining wealth there is.

4 Equilibrium

The money market account is assumed to be in zero net supply and we follow the part of the literature which assumes a zero net stock supply too, see, e.g., the framework used in Chapter 4 in [KS98]. This means that an investor holding one unit of the risky security at time T receives S_T units of the consumption good after the market closes at time T. The alternative assumption of a positive net stock supply allows an additional inflow of exogenous dividends into the economy. Such a model is in contrast to our setting where the dividends paid by the stock at time T are determined endogenously. In our model, the stock only serves as a device for investors to trade their non-idiosyncratic risk parts between themselves. However, as we explain in Section 5.2 incorporating exogenous dividends into our setting is straightforward and will not add any additional insight.

Definition 4.1. An equilibrium is a minimal state-price deflator $\xi^{\min}$, see (2.4), characterized by $(\xi_0^{\min}, r, \mu_S)$ such that all markets clear, i.e.,

$$\sum_{i=1}^I \hat{c}_{it} = 0, \quad \mathbb{P}\otimes\Gamma\text{-a.e.}, \quad \sum_{i=1}^I \hat{\theta}_{it}^{(0)} = 0, \quad \sum_{i=1}^I \hat{\theta}_{it} = 0, \quad \mathbb{P}\otimes\text{Leb-a.e.},$$

and such that given $(\xi_0^{\min}, r, \mu_S)$ the processes $(\hat{c}_i, \hat{\theta}_i) \in \mathcal{A}$ are optimal for investor i, $i = 1, 2..., I$. ♢

It is important to note that clearing in the good's market, i.e., $\sum_{i=1}^I \hat{c}_{it} = 0$, $\mathbb{P}\otimes\Gamma$-a.e., ensures clearing in both the stock market as well as in the money market. To see this we sum over $i = 1, ..., I$ in (3.11) to get $\sum_{i=1}^I X_t^{\hat{\theta}_i,\hat{c}_i} = 0$, $\mathbb{P}\otimes$Leb-a.e., and in particular we have $\mathbb{P}\otimes$Leb-a.e. that $\sum_{i=1}^I e^{-\int_0^t r_u du} X_t^{\hat{\theta}_i,\hat{c}_i} = 0$. Computing the dynamics gives us

$$0 = \sum_{i=1}^I d\Big(e^{-\int_0^t r_u du} X_t^{\hat{\theta}_i,\hat{c}_i}\Big) = \sum_{i=1}^I \hat{\theta}_{it} e^{-\int_0^t r_u du} \sqrt{v_t} dW_t^{\min},$$

which combined with the uniqueness of Itô-dynamics produces clearing in the stock market. To obtain clearing in the money market, the numéraire invariance property of self-financing strategies produces the money market investment strategy

$$\theta_{it}^{(0)} := \int_0^t e^{-\int_0^s r_u du}\Big(\hat{\theta}_{is}\sqrt{v_t} dW_s^{\min} - \hat{c}_{is}\Gamma(ds)\Big) - e^{-\int_0^t r_u du}\hat{\theta}_{it} S_t, \tag{4.1}$$

since initial wealths are assumed to be zero. By summing over $i = 1, ..., I$ we obtain the money market clearing requirement.

In the next sections we will use the positive constants

$$\tau_\Sigma := \sum_{i=1}^I \frac{1}{a_i}, \quad \sigma_\mathcal{E} := \sum_{i=1}^I \sigma_{Y_i}, \quad \sigma_\beta := \sum_{i=1}^I a_i \beta_{Y_i}^2.$$

4.1 Main existence result

We start by defining the functions $\lambda_n(t)$ for $t \in [t_{n-1}, t_n]$ via the Riccati equations

$$\lambda_n'(t) := \frac{1}{2}\sigma_v\left(\lambda_n(t)^2 + \frac{1}{\tau_\Sigma}\sigma_\beta\right), \quad \lambda_n(t_n) := \frac{\sigma_\mathcal{E}}{\tau_\Sigma}, \tag{4.2}$$

where $\Delta := T/N$ and $t_n := \Delta n$, $n = 0, 1, ..., N$, for some $N \in \mathbb{N}$. We will assume that for $n = 1, ..., N$ there exists a (unique) solution λ_n of (4.2). If

$\sigma_v < 0$, this is trivially ensured and, otherwise, $\Delta > 0$ being sufficiently small produces this feature. For a complete description of this issue, we refer to the appendix in [KO96]. Furthermore, we define the two sequences of constants

$$r_n^0 := \frac{\mu_v}{\sigma_v}\left(\frac{1}{\Delta}\int_{t_{n-1}}^{t_n}\lambda_n(u)du - \lambda_n(t_n)\right), \quad n = 1, ..., N, \tag{4.3}$$

$$r_n^v := \frac{1}{\sigma_v\Delta}\Big(\lambda_n(t_{n-1}) - \lambda(t_n)\Big), \quad n = 1, ..., N. \tag{4.4}$$

Our main existence result is the next theorem.

Theorem 4.2. *Assume* (4.2) *have solutions in* Λ *for* $n \in \{1, ..., N\}$, $N \in \mathbb{N}$.

1. *Discrete case* Γ_N*: Assume that* $\Delta := T/N > 0$ *is such that* λ_n *defined by* (4.2) *is well-defined for* $n = 1, ..., N$*. Then* $r \in \mathcal{L}^1$ *and* $\mu_S \in \Lambda$ *defined by*

$$r_t := r_n^0 + r_n^v v_{t_{n-1}}, \quad \mu_S(t) := \lambda_n(t), \quad \textit{for} \quad t \in (t_{n-1}, t_n],$$

constitute an equilibrium interest rate and market price of risk.

2. *Continuous case* Γ_∞*: An equilibrium interest rate* $r \in \mathcal{L}^1$ *and market price of risk* $\mu_S\sqrt{v_t}$, $\mu_S \in \Lambda$*, are given by*

$$\mu_S(t) := \frac{\sigma_{\mathcal{E}}}{\tau_\Sigma}, \quad r_t := -\frac{1}{2\tau_\Sigma}\Big(\sigma_\beta + \frac{\sigma_{\mathcal{E}}^2}{\tau_\Sigma}\Big)v_t.$$

The constants defined by (4.3) and (4.4) satisfy

$$\lim_{t_n \downarrow t_{n-1}}\Big(r_n^0 + r_n^v v_{t_{n-1}}\Big) = -\frac{1}{\sigma_v}\lambda_n'(t_{n-1})v_{t_{n-1}} = -\frac{1}{2}\frac{1}{\tau_\Sigma}\Big(\frac{\sigma_{\mathcal{E}}^2}{\tau_\Sigma} + \sigma_\beta\Big)v_{t_{n-1}},$$

which indeed agrees with the second part of the above theorem. From the proof (see Section 6) we see that the equilibrium interest rate defined by (4.3) and (4.4) is not unique in the discrete case Γ_N.

Finally we mention that this result is robust to various model variations, see Section 5, some of which are needed to obtain more realistic equilibrium predictions. E.g., having drift in the individual income processes would produce aggregate consumption growth leading to a positive interest rate r_t. Such extensions are straightforward and will not add any additional insights.

4.2 Risk-free rate and equity premium puzzles

In this section we illustrate how the incomplete equilibrium established in Theorem 4.2 can be used to simultaneously explain the risk-free interest rate puzzle and the equity premium puzzle. We first consider discrete income and, subsequently, the continuous analogue is presented.

4.2.1 Discrete income

For simplicity, we consider the case of income/consumption at $t = 0$ and $t = T$ (the general discrete case is similar). The standard representative agent is modeled by the utility function

$$U_{\text{rep}}(x;\gamma) := \sup_{\sum_{i=1}^I x_i = x} \sum_{i=1}^I \gamma_i U_i(x_i), \quad \gamma \in \mathbb{R}_+^I, \quad x \in \mathbb{R},$$

where γ is a weight vector. Given that each investor is modeled by a negative exponential utility function, the representative agent's utility function becomes (see, e.g., Section 5.26 in [HL88])

$$U_{\text{rep}}(x;\gamma) = -e^{-\frac{1}{\tau_\Sigma}x} \prod_{i=1}^I (\gamma_i a_i)^{\frac{1}{a_i \tau_\Sigma}}, \quad x \in \mathbb{R}.$$

This expression shows that the weight γ does not matter for the representative agent's equilibrium (Gorman aggregation).

We define the aggregate endowment process $\mathcal{E}_t := \sum_{i=1}^I Y_{it}$ with dynamics

$$d\mathcal{E}_t = \sqrt{v_t}\Big(\sigma_{\mathcal{E}} dW_t + \sum_{i=1}^I \beta_{Y_i} dZ_{it}\Big), \quad t \in [0,T].$$

If the equilibrium is given by the representative agent, the martingale method produces the proportionality relation

$$e^{-\frac{1}{\tau_\Sigma}\mathcal{E}_t} \propto \xi_t^{\mathrm{rep}}, \quad t \in \{0,T\}, \tag{4.5}$$

where ξ^{rep} denotes the unique state-price deflator prevailing in the representative agent's economy.

Lemma 4.3. *For $\Delta := T > 0$ sufficiently small*[3] *the Riccati equation*

$$\lambda'_{rep}(t) = \frac{1}{2}\sigma_v \left(\lambda_{rep}(t)^2 + \frac{1}{\tau_\Sigma^2}\sum_{i=1}^I \beta_{Y_i}^2\right), \quad \lambda_{rep}(T) = \frac{\sigma_{\mathcal{E}}}{\tau_\Sigma}, \tag{4.6}$$

has a well-defined solution. The representative agent's market price of risk process is given by $\lambda_{rep}(t)\sqrt{v_t}$ whereas the corresponding risk-free rate reads

$$r_{rep} = \frac{\mu_v}{\sigma_v}\left(\frac{1}{T}\int_0^T \lambda_{rep}(u)du - \lambda_{rep}(T)\right) + \frac{v_0}{\sigma_v}\frac{1}{T}\Big(\lambda_{rep}(0) - \lambda_{rep}(T)\Big). \tag{4.7}$$

Provided that the ODEs for λ_{rep} and λ have well-defined solutions, see (4.2) and (4.6), we see that the equilibrium market price of risk process $\lambda\sqrt{v_t}$ coincides with the representative agent's market price of risk process $\lambda_{\mathrm{rep}}\sqrt{v_t}$ if, and only if, we have

$$\sum_{i=1}^I a_i \beta_{Y_i}^2 = \frac{1}{\tau_\Sigma}\sum_{i=1}^I \beta_{Y_i}^2. \tag{4.8}$$

[3]Specifically, $T > 0$ needs to be so small that the Ricatti-ODE for B_{rep} in the below Lemma 6.1 has a (unique) solution.

In this case we also have $r = r_{\text{rep}}$. This indeed holds true in the complete market's case in which $\beta_{Y_i} := 0$ for $i = 1, ..., I$. For nonzero $(\beta_{Y_i})_{i=1}^I$ the left-hand-side dominates the right-hand-side of (4.8) since $\tau_\Sigma \geq 1/a_i$ for all $i = 1, ..., I$. So for $\sigma_v < 0$ comparing (4.2) and (4.6) shows that $\lambda_{\text{rep}} < \lambda$ and, consequently, by the first part of Theorem 4.2 and (4.7) we also have that $r_{\text{rep}} > r$. We note that $\sigma_v := 0$ produces a deterministic volatility, and the model becomes similar to the model presented in [CLM10] which does not produce any incompleteness effects on the equity risk premium.

To put the above discussion into a different perspective, let us conclude this section by re-considering the heterogenous formulation (3.5). It follows from Bayes' rule that since W remains a Brownian motion under each $\mathbb{P}_i$ and since the adjusted income processes $(\tilde{Y}_i)_{i=1}^I$ defined by (3.6) as well as the wealth dynamics $dX_t^{\theta,c}$ are driven solely by W we have

$$\mathbb{E}^{\mathbb{P}_i}\left[\int_0^T U_i\left(c_u + \tilde{Y}_{iu}\right)\Gamma(du)\right] = \mathbb{E}^{\mathbb{P}_1}\left[\int_0^T U_i\left(c_u + \tilde{Y}_{iu}\right)\Gamma(du)\right],$$

for $i = 1, 2..., I$ and $(\theta, c) \in \mathcal{A}$. Consequently, if we define $\tilde{\mathcal{E}}_t := \sum_{i=1}^I \tilde{Y}_{it}$ with the dynamics

$$d\tilde{\mathcal{E}}_t = \sqrt{v_t}\sigma_{\mathcal{E}} dW_t - \frac{1}{2}\sigma_\beta v_t dt,$$

as the economy's "aggregate endowment" we can reduce the search for an equilibrium to a complete market equilibrium with a modified aggregate endowment. In other words, by replacing (4.5) with the following adjusted first-order condition in the representative agent's problem

$$e^{-\frac{1}{\tau_\Sigma}\tilde{\mathcal{E}}_t} \propto \xi_t^{\text{rep}}, \quad t \in \{0, T\}, \tag{4.9}$$

the proof of Lemma 4.3 can be used to recover the actual incomplete equilibrium derived in Theorem 4.2 (provided that $T := \Delta > 0$ is small enough).

4.2.2 Continuous income

In the case of continuous income processes (Γ_∞), the resulting equilibrium risk-free rate is affected by the income incompleteness. The first order condition (4.5) for $t \in [0,T]$ produces the representative agent's interest rate

$$r_t^{\text{rep}} := -\frac{1}{2\tau_\Sigma^2}\Big(\sum_{i=1}^I \beta_{Y_i}^2 + \sigma_\mathcal{E}^2\Big)v_t, \quad t \in [0,T].$$

This combined with the second part of Theorem 4.2 produces the interest rate reduction

$$r_t^{\text{rep}} - r_t = \frac{1}{2}\frac{1}{\tau_\Sigma}\left(\sum_{i=1}^I a_i\beta_{Y_i}^2 - \frac{1}{\tau_\Sigma}\sum_{i=1}^I \beta_{Y_i}^2\right)v_t > 0, \tag{4.10}$$

which is an analogue of the result presented in [CLM10]. On the other hand, the equity premium based on the representative agent agrees with $\mu_S = \sigma_\mathcal{E}/\tau_\Sigma$. Our next result shows that any family of continuous income processes necessarily produces an (instantaneous) equilibrium equity premium identical to that of the representative agent.

Let the money market account $S^{(0)}$ given by (2.2) and let

$$dS_t := \Big(r_tS_t + \lambda_t'\sigma_{St}\Big)dt + \sigma_{St}dB_t, \quad t \in [0,T], \quad S_0 > 0, \tag{4.11}$$

constitute an equilibrium for some $r \in \mathcal{L}^1$, $(\sigma_S, \lambda') \in \mathcal{L}^2$, $\sigma_S \neq 0$, and some Brownian motion B. In this setting a pair (c,θ) is admissible if we have

$$\int_0^T \Big(|c_u| + \theta_u^2\sigma_{Su}^2\Big)du < \infty, \text{ } \mathbb{P}\text{-a.s., and } \quad \mathbb{E}\left[\int_0^T \xi_u c_u du\right] \leq 0,$$

for all state-price densities ξ (recall that ξ is a state-price density if $\xi_t S_t^{(0)}$ and $\xi_t S_t$ are local martingales).

Theorem 4.4. *For $i = 1, ..., I$ we let the income process Y'_i be an arbitrary Itô-process with dynamics*

$$dY'_{it} = \mu'_{Y_i t}dt + \sigma'_{Y_i t}dB_t + \beta'_{Y_i t}dB^{\perp}_{it}, \quad t \in [0,T], \quad Y'_{i0} \geq 0.$$

Here $B^{\perp}_i$ denotes a Brownian motion independent of B, $\mu_{Y_i} \in \mathcal{L}^1$, and $(\sigma'_{Y_i}, \beta'_{Y_i}) \in \mathcal{L}^2$. Assume that (2.2) and (4.11) constitute an equilibrium. Then the equilibrium market price of risk process λ' satisfies $\lambda'_t = \frac{1}{\tau_\Sigma}\sum_{i=1}^I \sigma'_{Y_i t}$.

In the setting of this theorem, we let $\mathcal{E}'_t := \sum_{i=1}^I Y'_{it}$ denote aggregate endowment. By computing the dynamics of the representative agent's state-price density (proportional to $e^{-\frac{1}{\tau_\Sigma}\mathcal{E}'_t}$), we see that the market price of risk process based on the representative agent agrees with λ' stated in the above theorem.

Finally, we note there is no loss of generality in assuming the above form for $(Y'_i)_{i=1}^I$ and S. Indeed, by assuming that an equilibrium stock price S exists, we can use Lévy's characterization for Brownian motion as well as the martingale representation theorem for $\mathcal{F}_t := \sigma(W_u, Z_{u1}, ..., Z_{uI})_{u\in[0,t]}$ to write the martingale component of dS as $\sigma_{St}dB_t$ for some Brownian motion B. Subsequently, we can decompose the martingale part of Y'_i into its projection onto B and some residual orthogonal martingale component (possibly depending on i) which produces the above form for dY'_i, $i = 1, ..., I$.

4.2.3 Risk premia over finite time-intervals

The preceding analysis has established that unspanned income risk affects the equilibrium risk-free rate, but also (in a model-free manner) that the equilibrium instantaneous equity premium is affected only if income/consumption is discrete. The latter impossibility result may seem to imply that the equity premium puzzle cannot be explained by unspanned income risk using a model based on exponential investors with continuous income streams, however, as we now explain this is not exactly true. The risk-free rate and equity premium puzzles are results of empirical studies of asset pricing properties relying on risk premia measured over finite time-intervals (given by the sampling frequency). In this section, we derive the relation between instantaneous equity premia and the equity premia measured over finite time-intervals. We show that unspanned income risk with stochastic volatility can raise equity premia over finite time-intervals even with continuous income streams and, thus, simultaneously explain the risk-free rate and equity premium puzzles.

The relation between instantaneous equity premia and equity premia measured over finite time-intervals is given by the relation between the minimal spot martingale measure $\mathbb{Q}^{\min}$ and the associated minimal forward measure for the finite time-interval over which returns are measured. For simplicity we only consider the time-interval $[0, T]$; the general case being completely similar. If there is only income/consumption at $t \in \{0, T\}$ we have seen that the equilibrium interest rate is deterministic, hence, there is no difference between spot and forward measures. On the other hand, in the setting with continuous income/consumption the equilibrium interest rates are stochastic and, consequently, the minimal spot and forward measures differ. In what follows we consider the continuous case Γ_∞ where $\mu_S^{\text{rep}}(t) = \mu_S(t) = \sigma_{\mathcal{E}}/\tau_\Sigma$.

In order to calculate the zero-coupon bond prices, we need the dynamics of v under the minimal martingale measure $\mathbb{Q}^{\min}$

$$dv_t = \Big(\mu_v - \mu_S(t)\sigma_v v_t\Big)dt + \sigma_v\sqrt{v_t}dW_t^{\min}.$$

We can then compute the CIR-type zero-coupon bond prices to be

$$B(t,T) := \mathbb{E}^{\mathbb{Q}^{\min}}\left[\exp\left(-\int_t^T r_u du\right)\right] = \exp\Big(a(T-t) + b(T-t)v_t\Big).$$

In this expression a and b are deterministic functions characterized by the following coupled set of ODEs for $s \in [0,T]$

$$a'(s) = b(s)\mu_v, \quad a(0) = 0,$$
$$b'(s) = -b(s)\mu_S(t)\sigma_v + \frac{1}{2}b(s)^2\sigma_v^2 + \frac{1}{2\tau_\Sigma}\Big(\sum_{i=1}^{I} a_i \beta_{Y_i}^2 + \frac{\sigma_\mathcal{E}^2}{\tau_\Sigma}\Big), \quad b(0) = 0,$$

which have a (unique) solution provided that $T > 0$ is small enough. The corresponding minimal forward measure $\mathbb{Q}^T$ is defined by the Radon-Nikodym derivative on $\mathcal{F}_T$ as

$$\frac{d\mathbb{Q}^T}{d\mathbb{Q}^{\min}} := \frac{\exp\Big(-\int_0^T r_u du\Big)}{B(0,T)}.$$

Novikov's condition and Girsanov's theorem ensure that

$$dW_t^{\mathbb{Q}^T} := dW_t^{\min} - b(T-t)\sigma_v\sqrt{v_t}dt = dW_t + \Big(\mu_S(t) - b(T-t)\sigma_v\Big)\sqrt{v_t}dt,$$

is a $\mathbb{Q}^T$-Brownian motion.

We can perform exactly the same calculation for minimal forward measure $\mathbb{Q}^T_{\text{rep}}$ corresponding to the representative agent's equilibrium and we find

$$dW_t^{\mathbb{Q}^T_{\text{rep}}} := dW_t^{\min} - b_{\text{rep}}(T-t)\sigma_v\sqrt{v_t}dt = dW_t + \Big(\mu_S(t) - b_{\text{rep}}(T-t)\sigma_v\Big)\sqrt{v_t}dt.$$

In this expression b_{rep} is given by (again, provided that $T > 0$ is small enough)

$$b'_{\text{rep}}(s) = -b_{\text{rep}}(s)\mu_S(t)\sigma_v + \frac{1}{2}b_{\text{rep}}(s)^2\sigma_v^2 + \frac{1}{2\tau_\Sigma^2}\Big(\sum_{i=1}^{I} \beta_{Y_i}^2 + \sigma_\mathcal{E}^2\Big), \quad b_{\text{rep}}(0) = 0,$$

where we are using that $\mu_S^{\text{rep}}(t) = \mu_S(t)$. By comparing the coefficients for the two Riccati equations for b and b_{rep} show that b dominates b_{rep}, see the discussion following Lemma 4.3. In conclusion, the equilibrium equity premium measured over finite time-intervals is raised relative to the Pareto efficient analogue provided as before that $\sigma_v < 0$.

5 Model variations

The model used for the income processes (3.1) is chosen for its mathematical simplicity and in this section we will briefly mention some variations. First of all, it is straightforward to allow v defined by (2.1) to contain a mean-reversion component and also to allow Y_i's drift to be an arbitrary affine function of v, see (3.1). Furthermore, these constants can be replaced by suitable deterministic functions of time. Such components are naturally required for model calibration, however, the analysis is completely similar but produces more cumbersome expressions for the equilibrium quantities (r, μ_S).

5.1 Gaussian models

We can modify our setting to produce an equilibrium in which the absolute volatility process follows a Gaussian process. This is inspired by Stein and Stein's stochastic volatility model [SS91] where the relative volatility process is the Gaussian process

$$dv_t := (\mu_v + \kappa_v v_t)dt + \sigma_v dW_t, \quad t \in [0,T], \quad v_0 > 0,$$

for $(\mu_v, \kappa_v, \sigma_v) \in \mathbb{R}$. Instead of the income dynamics (3.1), we consider

$$dY_{it} := \mu_{Y_i} dt + v_t\Big(\sigma_{Y_i} dW_t + \beta_{Y_i} dZ_{it}\Big), \quad t \in [0,T], \quad Y_{i0} \geq 0.$$

In this setting we can find deterministic functions $(\mu_S, \mu_S^0) \in \Lambda$ such that the equilibrium stock price dynamics are given by

$$dS_t = \Big(\mu_S(t) v_t + \mu_S^0(t)\Big)dt + dW_t, \quad t \in [0,T].$$

In other words, the market price of risk process is the Gaussian process $\mu_S^0(t) + \mu_S(t)v_t$ which can be seen as a generalization of the Gaussian model developed in [KO96]. Gaussian based market price of risk models have been widely used in the finance literature, see e.g., [Wac02], [MS04] and [BK05].

5.2 Dividends

One can interpret S as a derivative security written on an underlying non-traded dividend paying stock. If investor i is endowed with units of this underlying stock, the processes $(Y_i)_{i=1}^I$ model the investors' aggregate payments, i.e., dividend plus income payments. Alternatively, we can incorporate dividend payments made by the risky security S directly into the setting. To do so, we let the risky security S be in unit positive supply, i.e., $\sum_{i=1}^I \hat{\theta}_{it} = 1$. $D_t := \int_0^t \delta_u \Gamma(du)$ denotes the aggregate dividend payments made by the risky security S up to time t for the dividend Γ-rate process

$$d\delta_t := \sigma_\delta \sqrt{v_t} dW_t, \quad t \in [0, T],$$

where $\sigma_D > 0$ is some constant. In this case, the equilibrium stock price dynamics are given by

$$dS_t := \Big(r_t S_t + \mu_S(t) \sqrt{v_t} \sigma_{St} \Big) dt - dD_t + \sigma_{St} dW_t,$$

for some $\sigma_S \in \mathcal{L}^2$. Since D and σ_S do not matter for the individual investor's problem Theorem 3.1 remains valid. The clearing condition for the good's market becomes $\delta_t = \sum_{i=1}^I \hat{c}_{it}$ almost everywhere with respect to $\mathbb{P} \otimes \Gamma$. Consequently, Theorem 4.2 remains valid with $\sigma_{\mathcal{E}} := \sigma_D + \sum_{i=1}^I \sigma_{Y_i}$ and produces (r, μ_S). Finally, the relation

$$S_t = \mathbb{E}_t^{\mathbb{Q}^{\min}} \left[\int_t^T \exp\left(- \int_t^u r_s ds \right) \delta_u \Gamma(du) \right], \quad t \in [0, T],$$

produces the volatility process $\sigma_S \in \mathcal{L}^2$ via the martingale representation theorem.

6 Proofs

Proof of Theorem 4.2: For the discrete case we start by introducing the martingale M for $t \in [t_{n-1}, t_n]$ defined by

$$M_t := \frac{1}{2}\mathbb{E}_t\left[\int_{t_{n-1}}^{t_n} \Big(\tau_\Sigma \lambda_n(u)^2 + \sigma_\beta\Big) v_u du\right] + \int_{t_{n-1}}^{t} \Big(\tau_\Sigma \lambda_n(u) - \sigma_\mathcal{E}\Big)\sqrt{v_u} dW_u.$$

Fubini's Theorem and Leibnitz's rule produce the dynamics of M to be

$$dM_t = \frac{1}{2}\int_t^{t_n} \Big(\tau_\Sigma \lambda_n(u)^2 + \sigma_\beta\Big) du \sigma_v \sqrt{v_t} dW_t + \sqrt{v_t}\Big(\tau_\Sigma \lambda_n(t) - \sigma_\mathcal{E}\Big) dW_t.$$

Since λ_n satisfies (4.2) we have $dM_t = 0$ for all $t \in [t_{n-1}, t_n]$. Consequently, we have

$$\begin{aligned} M_{t_n} = M_{t_n - 1} &= \frac{1}{2}\int_{t_{n-1}}^{t_n} \Big(\tau_\Sigma \lambda_n(u)^2 + \sigma_\beta\Big)\mathbb{E}_{t_{n-1}}[v_u] du \\ &= \frac{\tau_\Sigma}{\sigma_v}\int_{t_{n-1}}^{t_n} \lambda_n'(u)\Big(v_{t_{n-1}} + \mu_v(u - t_{n-1})\Big) du. \end{aligned}$$

Inserting this expression into (3.8) and using the definition of r produce for $n = 1, ..., N$ the clearing requirement

$$\begin{aligned} \sum_{i=1}^{I}\Big(\hat{c}_{it_n} - \hat{c}_{it_{n-1}}\Big) &= \int_{t_{n-1}}^{t_n} \Big\{\tau_\Sigma r_u + \frac{1}{2}\Big(\tau_\Sigma \mu_S(u)^2 + \sigma_\beta\Big) v_u\Big\} du \\ &+ \int_{t_{n-1}}^{t_n} \Big(\tau_\Sigma \mu_S(u) - \sigma_\mathcal{E}\Big)\sqrt{v_u} dW_u \\ &= \Delta \tau_\Sigma (r_n^0 + r_n^v v_{t_{n-1}}) \\ &+ \frac{\tau_\Sigma}{\sigma_v}\int_{t_{n-1}}^{t_n} \lambda_n'(u)\Big(v_{t_{n-1}} + \mu_v(u - t_{n-1})\Big) du = 0. \end{aligned}$$

It remains to define $\xi_0^{\min}$ such that

$$0 = \sum_{i=1}^{I} \hat{c}_{i0} = -\sum_{i=1}^{I} \left(\frac{1}{a_i} \log \left(\frac{\alpha_i \xi_0^{\min}}{a_i} \right) + Y_{i0} \right).$$

For the continuous case we use the second part of Theorem 3.1. By summing up the expressions for $\hat{c}_{it}$ and equating the sum to zero we find the stated expressions for r and μ_S.

♢

Lemma 6.1. *For $T > 0$ sufficiently small the coupled system of ODEs*

$$B'_{rep}(t) = \frac{1}{2} B^2_{rep}(t)\sigma_v^2 + \frac{1}{2}\frac{1}{\tau_\Sigma^2}\Big(\sigma_\mathcal{E}^2 + \sum_{i=1}^{I} \beta_{iY}^2\Big) + B_{rep}(t)\frac{1}{\tau_\Sigma}\sigma_\mathcal{E}\sigma_v, \quad B_{rep}(T) = 0,$$
$$A'_{rep}(t) = -B_{rep}(t)\mu_v, \quad A_{rep}(T) = 0,$$

has well-defined solutions in Λ and we have the following representation

$$\mathbb{E}_t\left[e^{-\frac{1}{\tau_\Sigma}\mathcal{E}_T}\right] = \exp\Big(- A_{rep}(t) - B_{rep}(t)v_t - \frac{1}{\tau_\Sigma}\mathcal{E}_t\Big), \quad t \in [0,T]. \tag{6.1}$$

Proof. We define M_t to be the right-hand-side of (6.1). The dynamics of M follows from Itô's lemma and are given by

$$dM_t = -M_t\sqrt{v_t}\Big\{B_{\text{rep}}(t)\sigma_v dW_t + \frac{1}{\tau_\Sigma}\Big(\sigma_\mathcal{E} dW_t + \sum_{i=1}^{I}\beta_{iY} dZ_{it}\Big)\Big\}, \quad t \in [0,T].$$

Since v is a Feller process, Novikov's condition ensures that M is a martingale and, hence, the terminal conditions for A_{rep} and B_{rep} produce the claim.

♢

Proof of Lemma 4.3: The above lemma allows us to read off the representative agent based market price of risk process related to W, i.e., the process

$\lambda_{\text{rep}}(t)\sqrt{v_t}$, to be

$$\lambda_{\text{rep}}(t) := B_{\text{rep}}(t)\sigma_v + \frac{\sigma_{\mathcal{E}}}{\tau_\Sigma}, \quad t \in [0,T].$$

By computing the derivative and using the Riccati equation for B_{rep} provided in the previous lemma we find the ODE that λ_{rep} satisfies to be (4.6). To obtain the interest rate (4.7) we use the relation

$$\begin{aligned} e^{r_{\text{rep}}T}\frac{1}{\tau_\Sigma}\mathbb{E}[e^{-\frac{1}{\tau_\Sigma}\mathcal{E}_T}] &= e^{r_{\text{rep}}T}\mathbb{E}[U'_{\text{rep}}(\mathcal{E}_T)] \\ &= e^{r_{\text{rep}}T}\mathbb{E}[\xi_T^{\text{rep}}] = \xi_0^{\text{rep}} = U'_{\text{rep}}(\mathcal{E}_0) = \frac{1}{\tau_\Sigma}e^{-\frac{1}{\tau_\Sigma}\mathcal{E}_0}. \end{aligned}$$

The previous lemma then shows

$$r_{\text{rep}} := \frac{1}{T}\Big(A_{\text{rep}}(0) + B_{\text{rep}}(0)v_0\Big).$$

Inserting the expressions for A_{rep} and B_{rep} produces (4.7).

$\diamondsuit$

Proof of Theorem 4.4: The first-order condition for the individual investor produces the proportionality requirement

$$U_i'(\hat{c}_t + Y'_{it}) \propto \hat{\xi}_{it}. \tag{6.2}$$

The individual investor's specific optimal state-price density $\hat{\xi}_i$ has the form

$$d\hat{\xi}_{it} = -\hat{\xi}_{it}\Big(r_t dt + \lambda_t' dB_t + dM_{it}^\perp\Big), \quad i = 1, ..., I,$$

for some local martingale $M_i^\perp$ orthogonal to B, i.e., $\langle B, M_i^\perp\rangle_t = 0$ for all $t \in [0,T]$. This representation of state-price densities can be found [KS98]. Computing the dynamics of both sides of (6.2) gives us the relation

$$d\hat{c}_{it} = ...dt + \Big(a_i\lambda_t' - \sigma_{Y_i't}\Big)dB_t + ...dB_{it}^\perp + ...dM_{it}^\perp.$$

It follows by summing over investors and matching the B-components that the equilibrium market price of risk process satisfies

$$\lambda_t' = \frac{1}{\tau_\Sigma} \sum_{i=1}^{I} \sigma_{Y_i' t}, \quad t \in [0, T].$$

$\diamondsuit$

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