Phase Space Quantum Mechanics

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Abstract

The paper review and develop the alternative formulation of quantum mechanics known as the phase space quantum mechanics or deformation quantization. It is shown that the quantization naturally arises as an appropriate deformation of the classical Hamiltonian mechanics. More precisely, the deformation of the point-wise product of observables to an appropriate noncommutative $\star$-product and the deformation of the Poisson bracket to an appropriate Lie bracket is the key element in introducing the quantization of classical Hamiltonian systems.

The formalism of the phase space quantum mechanics is presented in a very systematic way for the case of Hamiltonian systems without any constrains and for a very wide class of deformations. The considered class of deformations and the corresponding $\star$-products contains all deformations which can be found in the literature devoted to the subject of the phase space quantum mechanics.

Fundamental properties of $\star$-products of observables, associated with the considered deformations are presented as well. Moreover, a space of states containing all admissible states is introduced, where the admissible states are appropriate pseudo-probability distributions defined on the phase space. It is proved that the space of states is endowed with a structure of a Hilbert algebra with respect to the $\star$-multiplication.

The most important result of the paper shows that developed formalism is more fundamental then the axiomatic ordinary quantum mechanics which appears in the presented approach as the intrinsic element of the general formalism. In addition, examples of a free particle and a simple harmonic oscillator illustrating the formalism of the deformation quantization and its classical limit are given.

Keywords: quantum mechanics, deformation quantization, Hilbert space, quantum distribution function

Contents

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Introduction

Since early works of Weyl [1], Wigner [2], Groenewold [3] and Moyal [4] many authors started to think about an alternative formulation of quantum mechanics, nowadays referred to as the *phase space quantum mechanics* or *deformation quantization*. In the past years many efforts have been made in order to develop the phase space quantum mechanics [5]-[18]. At first this formulation were treated as an interesting mathematical fact but quickly was realized that it have many applications. Nowadays phase space quantum mechanics have wide range of applications in many fields of research including quantum optics, field theory [19, 20], theory of dynamical systems and M-theory [21]-[24]. The deformation quantization formalism can also be adopted in formulation of noncommutative differential geometry which have applications in some quantum gravity theories.

The ordinary description of quantum mechanics is given by a set of axioms from which the connection with classical mechanics is not evident. The natural formulation of quantum mechanics seems to be a result of a generalization of classical Hamiltonian mechanics, in such a way that the new formulation of quantum mechanics should smoothly reduce to the formulation of classical mechanics as the Planck constant $\hbar$ goes to 0. The phase space quantum mechanics is such a natural formulation of quantum theory.

The idea behind the phase space quantum mechanics relies on a deformation, with respect to some parameter $\hbar$ (the Planck constant), of a classical algebra of observables $\mathcal{A}_C$ to some noncommutative algebra $\mathcal{A}_Q$, which plays the role of a quantum algebra of observables. By the deformation of $\mathcal{A}_C$ is meant the deformation of the point-wise product between observables to some noncommutative product, commonly denoted by $\star$, and also the deformation of the Poisson bracket $\{\cdot\,,\,\cdot\}$ to a Lie bracket $[\![\cdot\,,\,\cdot]\!] = \frac{1}{i\hbar}[\,\cdot\,,\,\cdot\,]$, where $[\,\cdot\,,\,\cdot\,]$ denotes a $\star$-commutator. The noncommutativity of observables is the source of a quantization of classical Hamiltonian systems. In particular, noncommutativity of observables results in the Heisenberg uncertainty principle.

In addition to the deformation of the product between observables also the definition of states have to be changed slightly. The admissible states of the classical Hamiltonian system are defined as probability distributions on the phase space. In the phase space quantum mechanics states have to be defined as *pseudo-probability distributions* on the phase space (the distributions do not have to have values in the range $[0, 1]$). This reflects the quantum character of the states. It is possible to introduce a Hilbert space $\mathcal{H}$ which contains all admissible states. Moreover, the $\star$-product can be extended to a noncommutative product between functions from $\mathcal{H}$ creating from $\mathcal{H}$ a Hilbert algebra. The action of some observable from $\mathcal{A}_Q$ on some state from $\mathcal{H}$ can also be expressed by the $\star$-product. Furthermore, the expectation values of observables and the time evolution equation are defined like in the classical case, except the fact that the point-wise product and the Poisson bracket are replaced by the $\star$-product and the Lie bracket $[\![\cdot\,,\,\cdot]\!]$.

In the paper the case of the deformation quantization of the Hamiltonian system without any constrains, which $\star$-product is related to the canonical Poisson bracket is considered. Even in this particular case one can introduce infinitely many $\star$-products inducing proper deformations of a classical Hamiltonian system. In the majority of papers only the special case of the $\star$-product is considered, namely the *Moyal product*. In some papers also other particular $\star$-products, *gauge equivalent* to the Moyal product, are discussed. Moreover, quite often (especially in quantum optics) the deformation quantization in *holomorphic coordinates* is also considered. In the paper the very general three parameter family of $\star$-products, gauge equivalent to the Moyal product, is constructed. This family of $\star$-products contains all examples of particular $\star$-products which can be found in the variety of papers devoted to the phase space quantum mechanics.

The formulation of quantum mechanics on the phase space is equivalent to the ordinary formulation of quantum mechanics. In the majority of papers this is proved with the help of the so called Wigner map and its inverse. Actually, it is a morphism between the algebra $\mathcal{A}_Q$ with noncommutative star multiplication and the algebra of linear operators in a Hilbert space with the multiplication being simple composition.

Another approach to construction of the phase space quantum mechanics is to extend ordinary Schrödinger equation to so called "Schrödinger equation in phase space" with related eigenfunctions being appropriate distributions [25]-[27]. As was shown recently it is equivalent to an eigenvalue problem of classical Hamiltonian and an appropriate star multiplication [28].

These points of view are however a little bit miss leading since one could thought that both descriptions of quantum mechanics are completely different and that the phase space quantum mechanics is just some representation of the "more fundamental" ordinary quantum mechanics. The paper presents an alternative point of view on the relation between the phase space quantum mechanics and the ordinary description of quantum mechanics. It is shown that the ordinary description of quantum mechanics appears as a natural consequence of the phase space quantum mechanics. Moreover, from the presented construction it is evident that the phase space quantum mechanics is the most fundamental formulation of quantum mechanics.

To summarize, the aim of this paper is to present in a systematic way the phase space quantum mechanics in a canonical regime as a natural deformation of classical Hamiltonian mechanics for a very general class of gauge equivalent deformations. Moreover, it is shown that from the phase space quantum mechanics naturally appears, at least in the canonical regime, the ordinary description of quantum mechanics. In addition, the physical equivalence of the presented family of deformations is discussed.

The paper is organized as follows. In Section 2 the classical Hamiltonian mechanics is reviewed. In this section the basic concepts of the Hamiltonian mechanics are given including the definitions of a phase space, observables and Hamiltonian systems. Moreover, the definitions of pure and mixed states and expectation values of observables are given together with their basic properties. Also the time evolution of pure and mixed states and observables is presented. In Section 3 the formulation of the phase space quantum mechanics is presented. This section starts with the introduction of some basics of the deformation quantization of general Hamiltonian systems followed by the full description of the deformation quantization procedure of classical Hamiltonian systems without any constrains and in canonical regime. In particular, there are presented fundamental properties of canonical $\star$-products together with the systematic construction of the space of states. Also the definitions of pure and mixed states and expectation values of observables are given together with their basic properties. Moreover, the time evolution equations of states and observables are presented. In Section 4 the equivalence of the ordinary formulation and the formulation on the phase space of quantum mechanics is presented. This section contains also proofs of some properties from former sections. In Section 5 examples of a free particle and a simple harmonic oscillator are presented illustrating the formalism of the deformation quantization. In the example of a free particle the time evolution of a free particle initially in some given state is derived. In the example of a harmonic oscillator stationary states and coherent states of the harmonic oscillator are derived. Moreover, it is proved that the presented examples of quantum states converge to appropriate classical states as Planck constant goes to zero. Section 6 contains final comments and conclusions together with a discussion on the physical equivalence of the presented family of deformations. The notation and conventions used in the paper as well as longer technical proof of some theorems from a main text can be found in Appendix A.1-A.8.

2

Classical Hamiltonian mechanics

2.1 Classical Hamiltonian systems

For further use during the quantization procedure, in what follows, complex tensor fields and functions on a manifold will be considered. Nevertheless, all below considerations can be made using only real tensor fields and functions.

In the classical Hamiltonian mechanics pure states are represented by points in a *phase space*, which in turn is represented by a *Poisson manifold*. The *Poisson manifold* is a smooth manifold M endowed with a two times contravariant antisymmetric (real) tensor field $\mathcal{P}$ satisfying the below relation

$$\mathcal{L}_{\zeta_f}\mathcal{P} = 0, \tag{2.1}$$

($\mathcal{L}_{\zeta_f}$ denotes a Lie derivative in the direction ζ_f) for every vector fields ζ_f defined as

$$\zeta_f := \mathcal{P}\mathrm{d}f\,, \quad f \in C^\infty(M).$$

The tensor field $\mathcal{P}$ is called a *Poisson tensor* and the vector fields ζ_f are called *Hamiltonian fields*. The space of all Hamiltonian fields on M will be denoted by $\mathrm{Ham}(M)$. In the rest of the paper it will be assumed that the Poisson tensor $\mathcal{P}$ is non-degenerate. In this case it can be proved that the Poisson manifold M is even-dimensional.

Using the Poisson tensor $\mathcal{P}$ a structure of a Lie algebra can be added to the space $C^\infty(M)$ of all (complex valued) smooth functions on M, namely a Lie bracket can be defined as

$$\{f, g\}_{\mathcal{P}} := \mathcal{P}(\mathrm{d}f\,, \mathrm{d}g\,), \quad f, g \in C^\infty(M). \tag{2.2}$$

It is obviously antisymmetric since $\mathcal{P}$ is antisymmetric and the Jacobi's identity follows from relation (2.1). In fact there holds

$$\begin{aligned} \{f,g\} &= -\{g,f\} && \text{(antisymmetry)}, \\ \{f,gh\} &= \{f,g\}h + g\{f,h\} && \text{(Leibniz's rule)}, \\ 0 &= \{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} && \text{(Jacobi's identity)}. \end{aligned}$$

The bracket (2.2) is called a *Poisson bracket* and the algebra $C^\infty(M)$ endowed with a Poisson bracket is called a *Poisson algebra*.

To describe a physical system besides a phase space also an *algebra of observables* is needed. Lets introduce a notation $\mathcal{A}_C = C^\infty(M)$. Usually in classical mechanics as the algebra of observables a real subalgebra $\mathcal{O}_C \subset \mathcal{A}_C$ of all real valued smooth functions on M is taken, but for further use during the introduction of the ordinary description of quantum mechanics it will be better to define the algebra of observables in a different way. First, lets define an algebra $\hat{\mathcal{A}}_C$ of all operators defined on the space $C^\infty(M)$ of the form $\hat{A} = A\,\cdot$, where $A \in \mathcal{A}_C$ and $\cdot$ denotes an ordinary point-wise product of functions from $C^\infty(M)$. The algebra $\hat{\mathcal{A}}_C$ have a structure of a Lie algebra with a Lie bracket defined by the formula

$$\{\hat{A},\hat{B}\} := \{A,B\}\,\cdot\,, \quad \hat{A},\hat{B} \in \hat{\mathcal{A}}_C.$$

Now, the algebra of observables can be defined as a real subalgebra $\hat{\mathcal{O}}_C$ of all operators from $\hat{\mathcal{A}}_C$ induced by real valued smooth functions on M. One of the admissible observables from $\hat{\mathcal{O}}_C$ have a special purpose, namely a *Hamiltonian* $\hat{H}$. This observable corresponds to the total energy of the system and it governs the time evolution of the system. A triple $(M,\mathcal{P},\hat{H})$ is then called a *classical Hamiltonian system*.

Local coordinates q^i, p_i $(i = 1,\dots,N)$ in which a Poisson tensor $\mathcal{P}$ have (locally) a form

$$\mathcal{P} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i} \quad \text{i.e.} \quad \mathcal{P}^{ij} = \begin{pmatrix} 0_N & \mathbb{I}_N \\ -\mathbb{I}_N & 0_N \end{pmatrix}$$

are called *canonical coordinates*. Furthermore, in the canonical coordinates a Hamiltonian field ζ_f and a Poisson bracket take a form

$$\zeta_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{2.3}$$

$$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}. \tag{2.4}$$

It can be proved that for every Poisson manifold canonical coordinates always exist, at least locally.

2.2 Pure states, mixed states and expectation values of observables

In the rest of the section for simplicity it will be assumed that the Hamiltonian system do not have any constrains, i.e. the phase space M is of the form $M = \mathbb{R}^{2N}$. The further quantization procedure will focus only on such special case of Hamiltonian systems.

As mentioned earlier *pure states* of a classical Hamiltonian system are points in a phase space M. They represent generalized positions and momenta of a phase space particle. Values of generalized positions and momenta of the particle can be extracted from a point in M (a pure state) by writing this point in canonical coordinates q^i, p_i. Then, q^i are the values of generalized positions and p_i are the values of generalized momenta of the particle.

When one does not know the exact positions and momenta of the phase space particle, but only a probability distribution that the system is in some point of the phase space then there is a need to extend the concept of a state. It is natural to generalize the states to probability distribution functions defined on the phase space M, i.e. to smooth functions ρ on M satisfying

1. $0 \leq \rho(\xi) \leq 1$,
2. $\int_M \rho(\xi) \mathrm{d}\xi = 1$.

Such generalized states are called *mixed states.* The probability distribution functions ρ on M will be also called a *classical distribution functions.* Pure states $\xi_0 \in M$ can be then identified with Dirac delta distributions $\delta(\xi - \xi_0)$.

An expectation value $\langle \hat{A} \rangle_\rho$ of an observable $\hat{A} \in \hat{\mathcal{A}}_C$ in a state ρ is defined by

$$\langle \hat{A} \rangle_\rho := \int_M (\hat{A}\rho)(\xi) \mathrm{d}\xi = \int_M A(\xi) \cdot \rho(\xi) \mathrm{d}\xi \,. \tag{2.5}$$

Note that an expectation value of the observable $\hat{A}$ in a pure state $\delta(\xi - \xi_0)$ is just equal $A(\xi_0)$.

2.3 Time evolution of classical Hamiltonian systems

For a given Hamiltonian system $(M, \mathcal{P}, \hat{H})$ the Hamiltonian $\hat{H}$ governs the time evolution of the system. Namely, the Hamiltonian $\hat{H}$ generates a Hamiltonian field ζ_H. The flow Φ_t^H (called the *phase flow* or the *Hamiltonian flow*) of this Hamiltonian field moves the points of M, which is interpreted as the time development of pure states ($\xi(t) = \Phi_t^H(\xi(0))$). A trajectory of a point $\xi \in M$ (a pure state) can be then calculated from the equation

$$\dot{\xi} = \zeta_H. \tag{2.6}$$

In canonical coordinates q^i, p_i, using formula (2.3), equation (2.6) takes a form of the following system of differential equations called the *canonical Hamilton equations*

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.7}$$

The equation of motion of mixed states can be derived from the probability conservation law. From this law follows that a probability $P(E)$ of finding a system in a given domain $E \subset M$ of the phase space M do not change during the time evolution, i.e.

$$P_t(E(t)) = P_{t+\Delta t}(E(t + \Delta t)), \tag{2.8}$$

where

$$P_t(E(t)) = \int_{E(t)} \rho(\xi, t) \mathrm{d}\xi\,. \tag{2.9}$$

Points of M evolve according to the Hamilton equations, hence

$$E(t + \Delta t) = \Phi_{\Delta t}^H(E(t)). \tag{2.10}$$

From equations (2.8), (2.9) and (2.10) it follows that

$$\int_{E(t)} \rho(\xi, t) \mathrm{d}\xi = \int_{\Phi_{\Delta t}^H(E(t))} \rho(\xi, t + \Delta t) \mathrm{d}\xi = \int_{E(t)} (\Phi_{\Delta t}^H)^* \rho(\xi, t + \Delta t) \mathrm{d}\xi\,. \tag{2.11}$$

Since the domain $E(t)$ is arbitrary, equation (2.11) implies that

$$\rho(t) = (\Phi_{\Delta t}^H)^* \rho(t + \Delta t).$$

From above equation it follows that

$$\begin{aligned} L(H, \rho)(t) &:= \lim_{\Delta t \to 0} \frac{(\Phi_{\Delta t}^H)^* \rho(t + \Delta t) - \rho(t)}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}s} (\Phi_s^H)^* \rho(t + s) \bigg|_{s=0} \\ &= \frac{\partial \rho}{\partial t} + \mathcal{L}_{\zeta_H} \rho = \frac{\partial \rho}{\partial t} + \zeta_H \rho = 0, \end{aligned}$$

which implies that

$$L(H,\rho) = \frac{\partial\rho}{\partial t} - \{H,\rho\} = 0. \tag{2.12}$$

Equation (2.12) is called the *Liouville equation* and it describes the time development of an arbitrary state ρ.

Lets check if for a pure state $\delta(\xi - \xi(t))$ the Liouville equation (2.12) is equivalent to the Hamilton equations (2.7). From (2.12), for the canonical coordinates q^i, p_i, it follows that

$$\left(\dot{\delta}(q-q(t)) + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i}\delta(q-q(t))\right)\delta(p-p(t)) +$$
$$+\left(\dot{\delta}(p-p(t)) - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}\delta(p-p(t))\right)\delta(q-q(t)) = 0.$$

Multiplying above equation by $x^j(t)$ and integrating over the phase space M one gets the first part of the Hamilton equations, and multiplying above equation by $p_j(t)$ and integrating over M one gets the second part of the Hamilton equations.

From (2.12) it follows that a time dependent expectation value of an observable $\hat{A} \in \hat{\mathcal{A}}_C$ in a state $\rho(t)$, i.e. $\langle\hat{A}\rangle_{\rho(t)}$, fulfills the following equation of motion

$$\langle\hat{A}\rangle_{L(H,\rho)} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle_{\rho(t)} - \langle\{\hat{A},\hat{H}\}\rangle_{\rho(t)} = 0. \tag{2.13}$$

Indeed

$$\begin{aligned}
\int_M A(\xi)\cdot\frac{\partial\rho}{\partial t}(\xi,t)\mathrm{d}\xi &= \frac{\mathrm{d}}{\mathrm{d}t}\int_M A(\xi)\cdot\rho(\xi,t)\mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{A}\rangle_{\rho(t)},\\
\int_M A(\xi)\cdot\{H,\rho\}(\xi)\mathrm{d}\xi &= \int_M \{H,A\rho\}(\xi)\mathrm{d}\xi - \int_M \{H,A\}(\xi)\cdot\rho(\xi)\mathrm{d}\xi\\
&= -\int_M \{H,A\}(\xi)\cdot\rho(\xi)\mathrm{d}\xi = \langle\{\hat{A},\hat{H}\}\rangle_{\rho(t)}.
\end{aligned}$$

Until now the states undergo the time development whereas the observables do not. This is called a *Schrödinger-like picture* of the time evolution. There is also a dual point of view (which, in turn, is referred to as a *Heisenberg-like picture*), in which states remain still whereas the observables undergo a time development. A pull-back of the Hamiltonian flow $(\Phi_t^H)^* = e^{t\mathcal{L}_{\zeta_H}}$ induces, for every t, an automorphism U_t^H of the algebra $\hat{\mathcal{A}}_C$ (a one-to-one map preserving the linear structure as well as the dot-product and the Lie bracket) given by the equation

$$U_t^H\hat{A} := (\Phi_t^H)^* A\,\cdot\,, \quad \hat{A}\in\hat{\mathcal{A}}_C.$$

Its action on an arbitrary observable $\hat{A} \in \hat{\mathcal{A}}_C$ is interpreted as the time development of $\hat{A}$

$$\hat{A}(t) = U_t^H \hat{A}(0) = e^{t\mathcal{L}_{\zeta_H}} A(0) \cdot \; = e^{t\zeta_H} A(0) \cdot . \tag{2.14}$$

Differentiating equation (2.14) with respect to t one receives

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}(t) = \zeta_H A(t) \cdot .$$

From above equation the following time evolution equation for an observable $\hat{A}$ follows

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}(t) - \{\hat{A}(t), \hat{H}\} = 0. \tag{2.15}$$

Both presented approaches to the time development yield equal predictions concerning the results of measurements, since

$$\begin{aligned}\langle \hat{A}(0) \rangle_{\rho(t)} &= \int_M A(\xi, 0)\rho(\xi, t)\mathrm{d}\xi \; = \int_M A(\xi, 0)(\Phi_{-t}^H)^* \rho(\xi, 0)\mathrm{d}\xi \\ &= \int_M (\Phi_t^H)^* A(\xi, 0)\rho(\xi, 0)\mathrm{d}\xi \; = \int_M A(\xi, t)\rho(\xi, 0)\mathrm{d}\xi \; = \langle \hat{A}(t) \rangle_{\rho(0)}.\end{aligned}$$

3

Quantization procedure on a phase space

3.1 Basics of deformation quantization

Let $(M, \mathcal{P})$ be a $2N$-dimensional phase space (i.e. a smooth Poisson manifold) and $\mathcal{A}_C = C^\infty(M)$ be the algebra of all smooth complex valued functions on M with respect to the standard point-wise product and a Poisson bracket associated to $\mathcal{P}$. The idea of quantization of the classical algebra of observables $\hat{\mathcal{O}}_C$ relays on a deformation, with respect to some parameter $\hbar$, of the algebra $\mathcal{A}_C$ to some noncommutative algebra $\mathcal{A}_Q$. The deformation parameter $\hbar$ for physical systems is the ordinary Planck constant. The deformed noncommutative multiplication on $\mathcal{A}_Q$ will be denoted by $\star$. The Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ on $\mathcal{A}_C$ should be deformed to some Lie bracket $[\![\,\cdot\,,\,\cdot\,]\!]$. It is natural to expect that the Lie bracket $[\![\,\cdot\,,\,\cdot\,]\!]$ should be expressed by the $\star$-commutator

$$[f, g] := f \star g - g \star f, \quad f, g \in \mathcal{A}_Q.$$

In fact, the Lie bracket $[\![\,\cdot\,,\,\cdot\,]\!]$ is defined by the formula

$$[\![f, g]\!] := \frac{1}{i\hbar}[f, g], \quad f, g \in \mathcal{A}_Q. \tag{3.1}$$

In general, to avoid problems with convergence of infinite series, not the whole algebra $\mathcal{A}_C$ will be deformed but only some subalgebra. Hence $\mathcal{A}_Q$, as a vector space, will be a subspace of $\mathcal{A}_C$. Moreover, a space $\mathcal{O}_Q$ of all real valued functions from $\mathcal{A}_Q$, will be a subspace of $\mathcal{O}_C$. Note that in general $\mathcal{O}_Q$ do not need to constitute an algebra with respect to the $\star$-product. As a space of admissible quantum observables the space $\mathcal{O}_Q$ can be taken, but for the same reason as in Section 2 it will be better to define the space of quantum observables in a similar way as in Section 2. First, lets introduce an algebra $\hat{\mathcal{A}}_Q$ of all operators defined on the space $C^\infty(M)$ of the form $\hat{A} = A\,\star$, where $A \in \mathcal{A}_Q$. The algebra $\hat{\mathcal{A}}_Q$ have a structure of a Lie algebra with a Lie bracket defined by the formula

$$[|\hat{A}, \hat{B}|] := [|A, B|]\,\star\, = \frac{1}{i\hbar}(\hat{A}\hat{B} - \hat{B}\hat{A}), \quad \hat{A}, \hat{B} \in \hat{\mathcal{A}}_Q.$$

All operators from $\hat{\mathcal{A}}_Q$ induced by real valued function are defined as the admissible quantum observables. The space of all admissible quantum observables will be denoted by $\hat{\mathcal{O}}_Q$. Note, that the algebra $\hat{\mathcal{A}}_Q$ is a deformation of the algebra $\hat{\mathcal{A}}_C$.

The deformed noncommutative multiplication on $\mathcal{A}_Q$ should satisfy such natural conditions

1. $f \star (g \star h) = (f \star g) \star h$ (associativity),
2. $f \star g = fg + o(\hbar)$,
3. $[|f, g|] = \{f, g\} + o(\hbar)$,
4. $f \star 1 = 1 \star f = f$,

for $f, g, h \in \mathcal{A}_Q$. Moreover, it is assumed that the $\star$-product can be expanded in the following infinite series with respect to the parameter $\hbar$

$$f \star g = \sum_{k=0}^{\infty} \hbar^k B_k(f, g), \quad f, g \in \mathcal{A}_Q, \tag{3.2}$$

where $B_k\colon \mathcal{A}_Q \times \mathcal{A}_Q \to \mathcal{A}_Q$ are bilinear operators. In order to avoid problems with convergence of the above series it will be assumed that for every pair of functions $f, g \in \mathcal{A}_Q$ only finite number of $B_k(f, g)$ is nonzero. From the construction of the $\star$-product it can be immediately seen that in the limit $\hbar \to 0$ the quantized algebra $\hat{\mathcal{A}}_Q$ reduces to the classical algebra $\hat{\mathcal{A}}_C$. From the associativity of the $\star$-product it follows that the bracket (3.1) is a well-defined Lie bracket. In fact, it satisfies the following relations

$$[|f, g|] = -[|g, f|] \qquad \text{(antisymmetry)},$$

$$[|f, g \star h|] = [|f, g|] \star h + g \star [|f, h|] \qquad \text{(Leibniz's rule)},$$

$$0 = [|f, [|g, h|]|] + [|h, [|f, g|]|] + [|g, [|h, f|]|] \qquad \text{(Jacobi's identity)}.$$

From the definition of the $\star$-product it follows that

$$B_0(f,g) = fg$$

and

$$B_1(f,g) - B_1(g,f) = \{f,g\}.$$

The associativity of the $\star$-product implies that the bilinear maps B_k satisfy the equations

$$\sum_{s=0}^{k}(B_s(B_{k-s}(f,g),h) - B_s(f,B_{k-s}(g,h))) = 0 \quad \text{for } k = 1,2,\ldots.$$

Hence, in particular B_1 satisfies the equation

$$B_1(f,g)h - fB_1(g,h) + B_1(fg,h) - B_1(f,gh) = 0.$$

Let $S\colon \mathcal{A}_Q \to \mathcal{A}_Q$ be a vector space automorphism, such that

$$Sf = \sum_{k=0}^{\infty} \hbar^k S_k f, \quad S_0 = 1, \tag{3.3}$$

where S_k are linear operators. As earlier, assume that for every $f \in \mathcal{A}_Q$ only finite number of $S_k f$ is nonzero. Such an automorphism produces a new $\star'$ in $\mathcal{A}_Q$ in the following way

$$f \star' g := S(S^{-1}f \star S^{-1}g). \tag{3.4}$$

Indeed, the associativity of the new $\star'$ follows from the associativity of the old $\star$-product, as

$$\begin{aligned} f \star' (g \star' h) &= f \star' S(S^{-1}g \star S^{-1}h) = S(S^{-1}f \star (S^{-1}g \star S^{-1}h)) \\ &= S((S^{-1}f \star S^{-1}g) \star S^{-1}h) = S(S^{-1}f \star S^{-1}g) \star' h \\ &= (f \star' g) \star' h. \end{aligned}$$

Using formula (3.3) one finds the following expression

$$B'_1(f,g) = B_1(f,g) - fS_1(g) - S_1(f)g + S_1(fg).$$

Then

$$B'_1(f,g) - B'_1(g,f) = B_1(f,g) - B_1(g,f) = \{f,g\}$$

and

$$\lim_{\hbar\to 0}[|f,g|]' = \lim_{\hbar\to 0}[|f,g|] = \{f,g\}.$$

Hence, the new $\star'$ is the second well-defined $\star$-product.

Two $\star$-products: $\star$ and $\star'$ are called *gauge equivalent* or simply *equivalent* if there exists a vector space automorphism $S\colon \mathcal{A}_Q \to \mathcal{A}_Q$ of the form (3.3) such that (3.4) holds. Note that such automorphism S is an isomorphism of the algebra $(\mathcal{A}_Q,\star)$ onto the algebra $(\mathcal{A}_Q,\star')$. It becomes evident that there can be many equivalent quantizations of a given algebra of observables. Even though, all this quantizations are mathematically equivalent they do not need to be physically equivalent.

In the rest of the paper as the phase space M the manifold $\mathbb{R}^{2N}$ will be chosen, i.e. in what follows, only the quantization of a Hamiltonian system without any constrains will be presented. It is well known that in this case an arbitrary Poisson tensor $\mathcal{P}$ can be presented in the following form

$$\mathcal{P} = \sum_{i=1}^{N} X_i \wedge Y_i = \sum_{i=1}^{N}(X_i \otimes Y_i - Y_i \otimes X_i),$$

where X_i, Y_i $(i = 1,\ldots,N)$ are some pair-wise commuting (real) vector fields on M. The corresponding Poisson bracket takes the form

$$\begin{aligned}\{f,g\}_{\mathcal{P}} &= f\left(\sum_{i=1}^{N}(X_i \otimes Y_i - Y_i \otimes X_i)\right) g = f\left(\sum_{i=1}^{N}(\overleftarrow{X}_i\overrightarrow{Y}_i - \overleftarrow{Y}_i\overrightarrow{X}_i)\right) g \\ &= \sum_{i=1}^{N}(X_i(f)Y_i(g) - Y_i(f)X_i(g)),\end{aligned} \tag{3.5}$$

where

$$f\left(\sum_{i=1}^{N}(X_i \otimes Y_i - Y_i \otimes X_i)\right) g$$

is to be understood as

$$\left(m \circ \sum_{i=1}^{N}(X_i \otimes Y_i - Y_i \otimes X_i)\right)(f \otimes g),$$

where m is a multiplication operator defined by $m(f \otimes g) = fg$ and the arrows over the vector fields X_i, Y_i denotes that a given vector field works only on a function standing on the left or on the right side of the vector field. The simplest natural deformation of the algebra $\mathcal{A}_C$ with the Poisson bracket (3.5) is given by such deformed $\star$-multiplication

$$f \star g = f \exp\left(-i\hbar \sum_{i=1}^{N} Y_i \otimes X_i\right) g = f \exp\left(-i\hbar \sum_{i=1}^{N} \overleftarrow{Y}_i \overrightarrow{X}_i\right) g$$

$$= \sum_{k \in \mathbb{N}^N} \frac{1}{k!} (-i\hbar)^{|k|} (Y^k f)(X^k g). \tag{3.6}$$

Theorem 3.1. *The product (3.6) is associative. Moreover, it is a well-defined $\star$-product.*

Proof. From one side one have

$$\begin{aligned} f \star (g \star h) &= \sum_{k \in \mathbb{N}^N} \frac{1}{k!} (-i\hbar)^{|k|} (Y^k f)(X^k (g \star h)) \\ &= \sum_{k \in \mathbb{N}^N} \sum_{s \in \mathbb{N}^N} \frac{1}{k! s!} (-i\hbar)^{|k|+|s|} (Y^k f)(X^k ((Y^s g)(X^s h))) \\ &= \sum_{k \in \mathbb{N}^N} \sum_{s \in \mathbb{N}^N} \sum_{r=0}^{k} \frac{1}{k! s!} \binom{k}{r} (-i\hbar)^{|k|+|s|} (Y^k f)(X^r Y^s g)(X^{s+k-r} h) \\ &= \sum_{n \in \mathbb{N}^N} \sum_{s=0}^{n} \sum_{r=0}^{n-s} \frac{1}{s! r! (n-s-r)!} (-i\hbar)^{|n|} (Y^{n-s} f)(X^r Y^s g)(X^{n-r} h), \end{aligned}$$

where in the last equality a summation over $k, s \in \mathbb{N}^N$ was replaced by a summation over $n \in \mathbb{N}^N$ and $s \in \{s \in \mathbb{N}^N : s_1 \leq n_1, \ldots, s_N \leq n_N\}$, where $n = k + s$. From the other side

$$\begin{aligned} (f \star g) \star h &= \sum_{n \in \mathbb{N}^N} \sum_{s=0}^{n} \sum_{r=0}^{n-s} \frac{1}{s! r! (n-s-r)!} (-i\hbar)^{|n|} (Y^{n-r} f)(X^s Y^r g)(X^{n-s} h) \\ &= \sum_{n \in \mathbb{N}^N} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{1}{s! r! (n-s-r)!} (-i\hbar)^{|n|} (Y^{n-r} f)(X^s Y^r g)(X^{n-s} h), \end{aligned}$$

which shows that $f \star (g \star h) = (f \star g) \star h$. The rest of the properties of the $\star$-product is obvious. ∎

Lets define the vector space automorphism (3.3) $S_{\sigma,\alpha,\beta} \colon \mathcal{A}_Q \to \mathcal{A}_Q$ by

$$S_{\sigma,\alpha,\beta} = \exp\left(i\hbar\sigma^{ij} X_i Y_j + \tfrac{1}{2}\hbar\alpha^{ij} X_i X_j + \tfrac{1}{2}\hbar\beta^{ij} Y_i Y_j\right), \tag{3.7}$$

where σ, α, β are some real matrices. Without loosing the generality it can be assumed that matrices α, β are symmetric, because the terms with anti-symmetric parts of matrices α, β will vanish. By relation (3.4) the automorphism $S_{\sigma,\alpha,\beta}$, acting on the $\star$-product

(3.6), can be used to define (σ,α,β)-parameter family of well-defined $\star$-products gauge equivalent to the $\star$-product (3.6)

$$f \star_{\sigma,\alpha,\beta} g := S_{\sigma,\alpha,\beta}(S^{-1}_{\sigma,\alpha,\beta} f \star S^{-1}_{\sigma,\alpha,\beta} g). \tag{3.8}$$

Theorem 3.2. *The product $\star_{\sigma,\alpha,\beta}$ is given by the formulae*

$$f \star_{\sigma,\alpha,\beta} g = f \exp\left(i\hbar\sigma^{ij}\overleftarrow{X}_i\overrightarrow{Y}_j - i\hbar\bar{\sigma}^{ij}\overleftarrow{Y}_i\overrightarrow{X}_j + \hbar\alpha^{ij}\overleftarrow{X}_i\overrightarrow{X}_j + \hbar\beta^{ij}\overleftarrow{Y}_i\overrightarrow{Y}_j \right) g \tag{3.9a}$$

$$= \sum_{\substack{n,m,r,s\\ \in M_N(\mathbb{N})}} (-1)^{|m|}(i\hbar)^{|n|+|m|}\hbar^{|r|+|s|}\frac{\sigma^n\bar{\sigma}^m\alpha^r\beta^s}{n!m!r!s!}(X^{n+r}Y^{m+s}f)(X^{m^T+r^T}Y^{n^T+s^T}g) \tag{3.9b}$$

$$= \sum_{\substack{k,r,s\\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\alpha^r\beta^s}{k!r!s!}\sum_{m\in M_N(k)}\binom{k}{m}\sigma^{k-m}(-\bar{\sigma})^m(X^{k-m+r}Y^{m+s}f)$$

$$\cdot (X^{m^T+r^T}Y^{k^T-m^T+s^T}g), \tag{3.9c}$$

where $\bar{\sigma}^{ij} = \delta^{ij} - \sigma^{ij}$ and $M_N(k) = \{n \in M_N(\mathbb{N})\colon n_{ij} \leq k_{ij}, \quad i,j = 1,\dots,N\}$.

The proof is given in A.2.

Note that the automorphism (3.7) induces an algebra isomorphism between algebras $(\mathcal{A}_Q, \star_{\sigma,\alpha,\beta})$ and $(\mathcal{A}_Q, \star_{\sigma',\alpha',\beta'})$

$$S_{\sigma'-\sigma,\alpha'-\alpha,\beta'-\beta}\colon (\mathcal{A}_Q, \star_{\sigma,\alpha,\beta}) \to (\mathcal{A}_Q, \star_{\sigma',\alpha',\beta'}),$$

hence, all algebras $(\mathcal{A}_Q, \star_{\sigma,\alpha,\beta})$ are gauge equivalent.

3.2 Weyl operator calculus

In quantization procedure it is often needed to assign to a function of commuting variables respective operators, i.e. admissible functions of noncommuting variables. In this section there will be reviewed and discussed a way of assigning operators to functions, often referred to as the *Weyl operator calculus* [31]-[33].

There is a natural way of assigning an operator to a function in the case when the function is a polynomial. Lets consider some Hilbert space $\mathcal{H}$ and distinguish on it operators of the position and momentum, i.e. hermitian operators $\hat{q}$, $\hat{p}$ for which $[\hat{q},\hat{p}] = i\hbar$. Assume for example that the function on the phase space $\mathbb{R}^2$ is of the form

$A(x,p)=x^2+p^2$. The following operator could be assigned to it: $A(\hat{q},\hat{p})=\hat{q}^2+\hat{p}^2$. Arises here a problem of the ordering of the operators. Assume that the function A is of the form $A(x,p)=xp$. The operator $A(\hat{q},\hat{p})=\hat{q}\hat{p}$ could be assigned to it, but equally good are operators $A(\hat{q},\hat{p})=\hat{p}\hat{q}$ or $A(\hat{q},\hat{p})=\frac{1}{2}\hat{q}\hat{p}+\frac{1}{2}\hat{p}\hat{q}$ since $A(x,p)=xp=px=\frac{1}{2}xp+\frac{1}{2}px$. This simple example shows that there are many ways of assigning an operator to a function depending on the chosen ordering.

Now a question arises how to assign an operator to a general function. For the most functions, which are interesting from the physical point of view, there hold the following identity

$$A(x,p)=\frac{1}{2\pi\hbar}\iint \mathcal{F}A(\xi,\eta)e^{\frac{i}{\hbar}(\xi x-\eta p)}\mathrm{d}\xi\,\mathrm{d}\eta\,,$$

which is just the inverse Fourier transform from the Fourier transform from the function A. Now according to the scheme shown earlier replacing x and p with the operators $\hat{q}$ and $\hat{p}$ would create an operator from the function A. But as was seen earlier there are many ways of replacing x and p with the operators $\hat{q}$ and $\hat{p}$ depending on the chosen ordering ($e^{\frac{i}{\hbar}\xi x}e^{-\frac{i}{\hbar}\eta p}=e^{\frac{i}{\hbar}(\xi x-\eta p)}$ but $e^{\frac{i}{\hbar}\xi\hat{q}}e^{-\frac{i}{\hbar}\eta\hat{p}}\neq e^{\frac{i}{\hbar}(\xi\hat{q}-\eta\hat{p})}$). It is desired to parametrize different orderings with three numbers $\sigma,\alpha,\beta\in\mathbb{R}$ in such a way that for an arbitrary monomial $x^n p^m$ the operator assigned to it, for the case $\sigma=\alpha=\beta=0$, will be *standard ordered* [34], i.e. operator should have all position operators on the left and all momentum operators on the right. For the case $\sigma=1,\alpha=\beta=0$ the operator assigned to the monomial should be *anti-standard ordered*, i.e. it should have all position operators on the right and all momentum operators on the left. For the case $\sigma=\frac{1}{2},\alpha=\beta=0$ the operator assigned to the monomial should be *symmetrically ordered* (*Weyl ordered*) [2], i.e. operator should be arithmetic average of all possible permutations of position and momentum operators. There are also known another orderings like *normal ordering*: $\sigma=\frac{1}{2},\alpha=-\frac{1}{2}\omega^{-1},\beta=-\frac{1}{2}\omega$ [35], [36] and *anti-normal ordering*: $\sigma=\frac{1}{2},\alpha=\frac{1}{2}\omega^{-1},\beta=\frac{1}{2}\omega$ [37], related to observables written in *holomorphic coordinates*. Actually, from definition, an *operator function* assigned to the function A is given by the equation

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}):=\frac{1}{2\pi\hbar}\iint \mathcal{F}A(\xi,\eta)e^{\frac{i}{\hbar}(\xi\hat{q}-\eta\hat{p})}e^{\frac{i}{\hbar}(\frac{1}{2}-\sigma)\xi\eta+\frac{i}{\hbar}(\frac{1}{2}\alpha\xi^2+\frac{1}{2}\beta\eta^2)}\mathrm{d}\xi\,\mathrm{d}\eta\,,$$

where $\sigma,\alpha,\beta\in\mathbb{R}$ are parameters describing different orderings.

Above definition of the operator function can be generalized to the case when the phase space is $2N$ dimensional. Now, there are needed N position operators $\hat{q}^1,\ldots,\hat{q}^N$

and N momentum operators $\hat{p}_1, \ldots, \hat{p}_N$ satisfying the following commutation relations

$$[\hat{q}^i, \hat{p}_j] = i\hbar\delta^i_j, \qquad [\hat{q}^i, \hat{q}^j] = [\hat{p}_i, \hat{p}_j] = 0, \qquad i, j = 1, \ldots, N.$$

The definition of the operator function now reads as follows

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) := \frac{1}{(2\pi\hbar)^N} \iint \mathcal{F}A(\xi, \eta) e^{\frac{i}{\hbar}(\xi_i \hat{q}^i - \eta^i \hat{p}_i)} e^{\frac{i}{\hbar}(\frac{1}{2}\delta^i_j - \sigma^i_j)\xi_i \eta^j + \frac{1}{\hbar}(\frac{1}{2}\alpha^{ij}\xi_i\xi_j + \frac{1}{2}\beta_{ij}\eta^i\eta^j)} \mathrm{d}\xi \, \mathrm{d}\eta \,, \tag{3.10}$$

where now σ, α, β are real matrices. Above equation, using the Baker-Campbell-Hausdorff formula (see Appendix A.4), can be rewritten in a form

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) := \frac{1}{(2\pi\hbar)^N} \iint \mathcal{F}A(\xi, \eta) e^{\frac{i}{\hbar}\xi_i \hat{q}^i} e^{-\frac{i}{\hbar}\eta^i \hat{p}_i} e^{-\frac{i}{\hbar}\sigma^i_j \xi_i \eta^j + \frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j + \frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j} \mathrm{d}\xi \, \mathrm{d}\eta \,. \tag{3.11}$$

In Section 3.3 it will be shown that quantum observables can be written as operator functions of appropriate operators $\hat{q}^1, \ldots, \hat{q}^N$, $\hat{p}_1, \ldots, \hat{p}_N$.

Equation (3.10) can be written in a different form.

Theorem 3.3. *Equation (3.10), defining an operator function, can be written in a differential form*

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) = A(-i\hbar\partial_\xi, i\hbar\partial_\eta) e^{\frac{i}{\hbar}(\xi_i \hat{q}^i - \eta^i \hat{p}_i + (\frac{1}{2}\delta^i_j - \sigma^i_j)\xi_i\eta^j) + \frac{1}{2\hbar}(\alpha^{ij}\xi_i\xi_j + \beta_{ij}\eta^i\eta^j)} \bigg|_{\xi=\eta=0} . \tag{3.12}$$

Proof. The proof of the theorem reduces to the proof of the following equality

$$f(-i\hbar\partial_x)g(x)\bigg|_{x=0} = \frac{1}{\sqrt{2\pi\hbar}} \int \mathcal{F}f(x)g(x)\mathrm{d}x \,,$$

for some complex functions f and g defined on $\mathbb{R}$. For $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$ there holds $\mathcal{F}f(y) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\mathcal{F}(x \to x^n)(y) = \sqrt{2\pi\hbar} \sum_{n=0}^{\infty} \frac{1}{n!}(i\hbar)^n f^{(n)}(0)\delta^{(n)}(y)$. Now, from one side there is

$$f(-i\hbar\partial_x)g(x)\bigg|_{x=0} = \sum_{n=0}^{\infty} \frac{1}{n!}(-i\hbar)^n f^{(n)}(0)g^{(n)}(0),$$

and from the other

$$\begin{aligned}
\frac{1}{\sqrt{2\pi\hbar}} \int \mathcal{F}f(y)g(y)\mathrm{d}y &= \sum_{n=0}^{\infty} \frac{1}{n!}(i\hbar)^n f^{(n)}(0) \int \delta^{(n)}(y)g(y)\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(-i\hbar)^n f^{(n)}(0)g^{(n)}(0).
\end{aligned}$$

□

Lets consider some examples of operator functions for the case of two dimensional phase space. Using (3.12) it can be easily calculated that for the function $A(x,p) = x^2 + p^2$ the operator function is equal

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = (\hat{q}^2 + \hat{p}^2)_{\sigma,\alpha,\beta} = \hat{q}^2 + \hat{p}^2 - \hbar\alpha - \hbar\beta = \hat{q}^2 + \hat{p}^2 + i\alpha[\hat{q},\hat{p}] + i\beta[\hat{q},\hat{p}]$$
$$= \hat{q}^2 + \hat{p}^2 + i(\alpha+\beta)\hat{q}\hat{p} - i(\alpha+\beta)\hat{p}\hat{q},$$

which is σ-independent. Analogically, for $A(x,p) = xp$ the operator function is equal

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = (\hat{q}\hat{p})_{\sigma,\alpha,\beta} = \hat{q}\hat{p} - i\hbar\sigma = \hat{q}\hat{p} - \sigma[\hat{q},\hat{p}] = \bar{\sigma}\hat{q}\hat{p} + \sigma\hat{p}\hat{q},$$

which is (α,β)-independent. In particular, the case when $\sigma = 0, \frac{1}{2}, 1$ gives standard, Weyl and anti-standard orderings, respectively. Finally, for $A(x,p) = xp^2$ the operator function is equal

$$\begin{aligned}A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) &= (\hat{q}\hat{p}^2)_{\sigma,\alpha,\beta} = \hat{q}\hat{p}^2 - 2i\hbar\sigma\hat{p} - \hbar\beta\hat{q} \\ &= \hat{q}\hat{p}^2 - 2\sigma\lambda[\hat{q},\hat{p}]\hat{p} - 2\sigma(1-\lambda)\hat{p}[\hat{q},\hat{p}] + i\beta(1-\gamma)[\hat{q},\hat{p}]\hat{q} + i\beta\gamma\hat{q}[\hat{q},\hat{p}] \\ &= (1-2\sigma\lambda)\hat{q}\hat{p}^2 + 2\sigma(2\lambda-1)\hat{p}\hat{q}\hat{p} + 2\sigma(1-\lambda)\hat{p}^2\hat{q} \\ &\quad + i\beta\gamma\hat{q}^2\hat{p} - i\beta(2\gamma-1)\hat{q}\hat{p}\hat{q} - i\beta(1-\gamma)\hat{p}\hat{q}^2.\end{aligned}$$

In this case the (λ,γ)-family of (σ,α,β)-orderings was received for $\lambda,\gamma \in \mathbb{R}$. The case when $\sigma = \frac{1}{2}$ and $\beta = 0$ gives

$$(\hat{q}\hat{p}^2)_{\sigma=\frac{1}{2},\alpha,\beta=0} = (1-\lambda)\hat{q}\hat{p}^2 + (2\lambda-1)\hat{p}\hat{q}\hat{p} + (1-\lambda)\hat{p}^2\hat{q}.$$

In particular

$$(\hat{q}\hat{p}^2)_{\sigma=\frac{1}{2},\alpha,\beta=0} = \frac{1}{3}\hat{q}\hat{p}^2 + \frac{1}{3}\hat{p}\hat{q}\hat{p} + \frac{1}{3}\hat{p}^2\hat{q} = \frac{1}{2}\hat{q}\hat{p}^2 + \frac{1}{2}\hat{p}^2\hat{q} = \hat{p}\hat{q}\hat{p},$$

for $\lambda = \frac{2}{3}, \frac{1}{2}, 1$.

Using equation (3.12) the formula for the operator function of a general monomial can be derived.

Theorem 3.4. *For diagonal matrices σ, α, β, i.e. when $\sigma_i^j = \sigma_i\delta_i^j$, $\alpha^{ij} = \alpha^i\delta^{ij}$, $\beta_{ij} = \beta_i\delta_{ij}$, the operator function of a general monomial $x^n p^m = (x^1)^{n_1}\ldots(x^N)^{n_N}p_1^{m_1}\ldots p_N^{m_N}$ reads*

$$(\hat{q}^n\hat{p}^m)_{\sigma,\alpha,\beta} = \prod_{j=1}^{N}\sum_{k_j=0}^{n_j}\sum_{r_j=0}^{[n_j-k_j/2]}\sum_{s_j=0}^{[m_j-k_j/2]}(-i\hbar\sigma_j)^{k_j}(-\hbar\alpha^j)^{r_j}(-\hbar\beta_j)^{s_j}k_j!(2r_j-1)!!(2s_j-1)!!$$
$$\cdot\binom{n_j}{k_j}\binom{m_j}{k_j}\binom{n_j-k_j}{2r_j}\binom{m_j-k_j}{2s_j}(\hat{q}^j)^{n_j-k_j-2r_j}\hat{p}_j^{m_j-k_j-2s_j},$$

where $[a]$ *denotes an integral part of* $a \in \mathbb{R}$.

The formal proof is given in A.3.

There is a useful property of operator functions. Namely, there holds

$$A^{\dagger}_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = A^{*}_{\bar{\sigma},\alpha,\beta}(\hat{q},\hat{p}) \tag{3.13}$$

Indeed

$$A^{\dagger}_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^N}\iint \mathcal{F}A^*(-\xi,-\eta)e^{-\frac{i}{\hbar}(\xi_i\hat{q}^i-\eta^i\hat{p}_i+(\frac{1}{2}\delta^i_j-\sigma^i_j)\xi_i\eta^j)+\frac{1}{2\hbar}(\alpha^{ij}\xi_i\xi_j+\beta_{ij}\eta^i\eta^j)}\mathrm{d}\xi\,\mathrm{d}\eta$$

and above equation after the change of coordinates: $\xi \to -\xi$, $\eta \to -\eta$ can be written in a form

$$\begin{aligned}A^{\dagger}_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) &= \frac{1}{(2\pi\hbar)^N}\iint \mathcal{F}A^*(\xi,\eta)e^{\frac{i}{\hbar}(\xi_i\hat{q}^i-\eta^i\hat{p}_i+(\frac{1}{2}\delta^i_j-\bar{\sigma}^i_j)\xi_i\eta^j)+\frac{1}{2\hbar}(\alpha^{ij}\xi_i\xi_j+\beta_{ij}\eta^i\eta^j)}\mathrm{d}\xi\,\mathrm{d}\eta \\ &= A^{*}_{\bar{\sigma},\alpha,\beta}(\hat{q},\hat{p}).\end{aligned}$$

For further use it will be useful to introduce an operator function from hermitian operators $\underline{\hat{q}}^1,\dots,\underline{\hat{q}}^N$, $\underline{\hat{p}}_1,\dots,\underline{\hat{p}}_N$ satisfying the following commutation relations

$$[\underline{\hat{q}}^i,\underline{\hat{p}}_j] = -i\hbar\delta^i_j, \qquad [\underline{\hat{q}}^i,\underline{\hat{q}}^j] = [\underline{\hat{p}}_i,\underline{\hat{p}}_j] = 0, \qquad i,j = 1,\dots,N.$$

This will be needed in Section 3.3 where it will be shown that operators of the form $\star A$ ($A \in \mathcal{A}_Q$) can be written as operator functions of appropriate operators $\underline{\hat{q}}^1,\dots,\underline{\hat{q}}^N$, $\underline{\hat{p}}_1,\dots,\underline{\hat{p}}_N$. For this purpose the same defining equation (equation (3.10)) can be used as in the previous definition of operator functions. Using the Baker-Campbell-Hausdorff formula equation (3.10) can be now rewritten in the form

$$A_{\sigma,\alpha,\beta}(\underline{\hat{q}},\underline{\hat{p}}) := \frac{1}{(2\pi\hbar)^N}\iint \mathcal{F}A(\xi,\eta)e^{\frac{i}{\hbar}\xi_i\underline{\hat{q}}^i}e^{-\frac{i}{\hbar}\eta^i\underline{\hat{p}}_i}e^{\frac{i}{\hbar}\bar{\sigma}^i_j\xi_i\eta^j+\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j+\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j}\mathrm{d}\xi\,\mathrm{d}\eta\,. \tag{3.14}$$

From above equation it can be seen that in this case the roles of σ and $\bar{\sigma}$ are reversed. For example an operator function is standard ordered for $\sigma = 1$ and anti-standard ordered for $\sigma = 0$. All properties and equations derived for the previous case can be rederived in a similar way for this case.

3.3 Space of states and properties of canonical $\star_{\sigma,\alpha,\beta}$-products

In the rest of the paper the case of $\star_{\sigma,\alpha,\beta}$-products related to the canonical Poisson tensor $\mathcal{P} = \partial_{x^i} \wedge \partial_{p_i}$ on a manifold $M = \mathbb{R}^{2N}$ will be considered. To avoid problems with convergence of infinite series as the algebra $\mathcal{A}_Q$ the algebra of all complex valued smooth functions on the phase space polynomial in momenta coordinates will be taken. The algebra $\hat{\mathcal{A}}_Q$ induced by $\mathcal{A}_Q$ contains all observables of physical interest. From the equations (3.9) by taking $X_i = \partial_{x^i}$ and $Y_i = \partial_{p_i}$ the $\star_{\sigma,\alpha,\beta}$-product takes the form

$$f \star_{\sigma,\alpha,\beta} g = f \exp\left(i\hbar\sigma^i_j \overleftarrow{\partial}_{x^i} \overrightarrow{\partial}_{p_j} - i\hbar\bar{\sigma}^j_i \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{x^j} + \hbar\alpha^{ij} \overleftarrow{\partial}_{x^i} \overrightarrow{\partial}_{x^j} + \hbar\beta_{ij} \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{p_j}\right) g \tag{3.15a}$$

$$= \sum_{\substack{n,m,r,s \\ \in M_N(\mathbb{N})}} (-1)^{|m|} (i\hbar)^{|n|+|m|} \hbar^{|r|+|s|} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} (\partial_x^{n+r} \partial_p^{m+s} f)(\partial_x^{m^T+r^T} \partial_p^{n^T+s^T} g) \tag{3.15b}$$

$$= \sum_{\substack{k,r,s \\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|} \hbar^{|r|+|s|} \frac{\alpha^r \beta^s}{k!r!s!} \sum_{m \in M_N(k)} \binom{k}{m} \sigma^{k-m} (-\bar{\sigma})^m (\partial_x^{k-m+r} \partial_p^{m+s} f)$$
$$\cdot (\partial_x^{m^T+r^T} \partial_p^{k^T-m^T+s^T} g), \tag{3.15c}$$

where $M_N(k) = \{n \in M_N(\mathbb{N}) \colon n_{ij} \leq k_{ij}, \quad i,j = 1,\dots,N\}$, σ is a real matrix, α, β are real symmetric matrices and $\bar{\sigma}^j_i = \delta^j_i - \sigma^j_i$. For simplicity it will be assumed that α, β are matrices which induced quadratic forms are positive define.

The well-known particular cases of the product (3.15) are

1. for $\sigma = \alpha = \beta = 0$, the multidimensional Kupershmidt-Manin product

$$f \star g = f \exp\left(-i\hbar \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{x^i}\right) g = f \exp\left(-i\hbar \partial_{p_i} \otimes \partial_{x^i}\right) g$$
$$= \sum_{k \in \mathbb{N}^N} \frac{1}{k!} (-i\hbar)^{|k|} (\partial_p^k f)(\partial_x^k g),$$

2. for $\sigma^{ij} = \frac{1}{2}\delta^{ij}, \alpha = \beta = 0$, the multidimensional Moyal (or Groenewold) product

$$f \star_{\frac{1}{2}} g = f \exp\left(\frac{1}{2} i\hbar(\overleftarrow{\partial}_{x^i}\overrightarrow{\partial}_{p_i} - \overleftarrow{\partial}_{p_i}\overrightarrow{\partial}_{x^i})\right) g = f \exp\left(\frac{1}{2} i\hbar \partial_{x^i} \wedge \partial_{p_i}\right) g$$
$$= \sum_{k \in M_N(\mathbb{N})} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^{|k|} \sum_{m \in M_N(k)} (-1)^{|m|}\binom{k}{m}(\partial_x^{k-m}\partial_p^m f)(\partial_x^{m^T}\partial_p^{k^T-m^T} g),$$

3. for $N = 1$, $\sigma = \frac{1}{2}$, $\alpha = \frac{2\lambda-1}{2\omega}$, $\beta = \omega^2\alpha$ where $\omega, \lambda \in \mathbb{R}$ and $\omega > 0$

$$f \star_\lambda g = f \exp\left(\hbar\lambda \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \hbar\bar{\lambda}\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a\right) g,$$

where the new coordinates $a(x,p) = (\omega x + ip)/\sqrt{2\omega}$, $\bar{a}(x,p) = (\omega x - ip)/\sqrt{2\omega}$ called *holomorphic coordinates* were used.

Useful in some applications can be an integral form of the $\star_{\sigma,\alpha,\beta}$-product. There holds

Theorem 3.5. *For appropriate $f, g \in \mathcal{A}_Q$ the following integral form of the $\star_{\sigma,\alpha,\beta}$-product is valid*

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^{2N}} \iiiint \mathcal{F}f(\xi', \eta')\mathcal{F}g(\xi'', \eta'') e^{\frac{i}{\hbar}\sum_i \xi''_i(x^i - \bar{\sigma}^i_j\eta'^j + i\alpha^{ij}\xi'_j)}$$
$$\cdot e^{-\frac{i}{\hbar}\sum_i \eta''^i(p_i - \sigma^j_i\xi'_j - i\beta_{ij}\eta'^j)} e^{\frac{i}{\hbar}(\xi'_i x^i - \eta'^i p_i)} \mathrm{d}\xi' \,\mathrm{d}\eta' \,\mathrm{d}\xi'' \,\mathrm{d}\eta''$$
$$\equiv \frac{1}{(2\pi\hbar)^N} \iint \mathcal{F}f(\xi, \eta) g(x - \bar{\sigma}\eta + i\alpha\xi, p - \sigma\xi - i\beta\eta) e^{\frac{i}{\hbar}(\xi_i x^i - \eta^i p_i)} \mathrm{d}\xi \,\mathrm{d}\eta$$
$$\equiv \frac{1}{(2\pi\hbar)^N} \iint f(x + \sigma\eta + i\alpha\xi, p + \bar{\sigma}\xi - i\beta\eta)\mathcal{F}g(\xi, \eta) e^{\frac{i}{\hbar}(\xi_i x^i - \eta^i p_i)} \mathrm{d}\xi \,\mathrm{d}\eta\,. \tag{3.16}$$

For some special cases it can be written differently. For $\alpha = \beta = 0$ and $\det\sigma \neq 0, \det\bar{\sigma} \neq 0$ one have

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^{2N}|\det(\sigma\bar{\sigma})|} \iiiint f(x', p')g(x'', p'') e^{\frac{i}{\hbar}\sum_{i,j}(\sigma^{-1})^j_i(x^i - x'^i)(p_j - p''_j)}$$
$$\cdot e^{-\frac{i}{\hbar}\sum_{i,j}(\bar{\sigma}^{-1})^i_j(p_i - p'_i)(x^j - x''^j)} \mathrm{d}x' \,\mathrm{d}p' \,\mathrm{d}x'' \,\mathrm{d}p''\,.$$

For $N = 1$ and $\alpha\beta \neq \sigma\bar{\sigma}$ one have

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^2|\alpha\beta - \sigma\bar{\sigma}|} \iiiint f(x', p')g(x'', p'') \exp\left(\frac{-i}{\hbar(\alpha\beta - \sigma\bar{\sigma})}\right.$$
$$\cdot \Big((\bar{\sigma}(p'' - p) - i\beta(x'' - x))(x' - x) - (\sigma(x'' - x)$$
$$\left. + i\alpha(p'' - p))(p' - p)\Big)\right) \mathrm{d}x' \,\mathrm{d}p' \,\mathrm{d}x'' \,\mathrm{d}p''\,.$$

The formal proof is given in A.5.

Using equations (3.15b) and (3.15c) a useful property of the $\star_{\sigma,\alpha,\beta}$-product can be derived. Namely, there holds

Theorem 3.6. *Let $f, g \in \mathcal{A}_Q$ be such that $f \star_{\sigma,\alpha,\beta} g$ and $g \star_{\sigma,\alpha,\beta} f$ are integrable functions. Then there holds*

$$\iint (f \star_{\sigma,\alpha,\beta} g)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint (g \star_{\sigma,\alpha,\beta} f)(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

Moreover, for the Moyal $\star$-product (the case of $\sigma^i_j = \frac{1}{2}\delta^i_j$ and $\alpha = \beta = 0$) there holds

$$\iint (f \star_{\frac{1}{2}} g)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

The proof is given in A.6.

Note that from the proof it follows that in general the $\star_{\sigma,\alpha,\beta}$-product do not change into the point-wise product under the integral sign. Only for $\sigma^i_j = \frac{1}{2}\delta^i_j$ and $\alpha = \beta = 0$ it happens that all terms in the sum in the definition of the $\star_{\sigma,\alpha,\beta}$-product will cancel out each other except the first term.

From equation (3.15b) immediately follows another two interesting properties of the $\star_{\sigma,\alpha,\beta}$-product. Namely, there holds

Theorem 3.7. *For $f, g \in \mathcal{A}_Q$ there holds*

$$\begin{aligned}
(f \star_{\sigma,\alpha,\beta} g)^* &= g^* \star_{\bar{\sigma},\alpha,\beta} f^*, \\
\partial_{x^i}(f \star_{\sigma,\alpha,\beta} g) &= (\partial_{x^i} f) \star_{\sigma,\alpha,\beta} g + f \star_{\sigma,\alpha,\beta} (\partial_{x^i} g), \\
\partial_{p_i}(f \star_{\sigma,\alpha,\beta} g) &= (\partial_{p_i} f) \star_{\sigma,\alpha,\beta} g + f \star_{\sigma,\alpha,\beta} (\partial_{p_i} g).
\end{aligned}$$

In particular, for the case of $\sigma^i_j = \frac{1}{2}\delta^i_j$ the complex conjugation of functions is an involution of the algebra $\mathcal{A}_Q$.

In what follows the problem of defining a space of states will be discussed. In analogy with the classical Hamiltonian mechanics one could try to define admissible states of the quantum Hamiltonian system as probabilistic distributions on the phase space. After doing this one would quickly realize that it is necessary to extend the space of admissible states to *pseudo-probabilistic distributions*, i.e. complex valued functions on the phase space which are normalized but need not to have the values in the range $[0,1]$. Hence, it is postulated that the space, which contains all admissible states, for the case $\alpha = \beta = 0$ is the space $L^2(M)$ of all square integrable functions on the phase space $M = \mathbb{R}^{2N}$ with respect to the Lebesgue measure.

It is possible to introduce the $\star_\sigma$-product between functions from $L^2(M)$, as to make from $L^2(M)$ an algebra with respect to the $\star_\sigma$-multiplication [38]. First, note that, by using the integral form (3.16) of the $\star_{\sigma,\alpha,\beta}$-product, the $\star_\sigma$-product of two Schwartz functions can be defined. Moreover, the $\star_\sigma$-product of two Schwartz functions is again a Schwartz function, hence the Schwartz space $\mathcal{S}(M)$ is an algebra with respect to the $\star_\sigma$-multiplication. Indeed, the Schwartz space $\mathcal{S}(M)$ is the space of all smooth functions $f \in C^\infty(M)$ such that $\|x^n p^m \partial_x^r \partial_p^s f\|_\infty = \sup_{(x,p)\in M} |x^n p^m \partial_x^r \partial_p^s f(x,p)| < \infty$ for every $n, m, r, s \in \mathbb{N}^N$. For $f, g \in \mathcal{S}(M)$ from (3.16) it immediately follows that

$$
\begin{aligned}
\partial_{x^i}(f \star_\sigma g) &= (\partial_{x^i} f) \star_\sigma g + f \star_\sigma (\partial_{x^i} g), \\
\partial_{p_i}(f \star_\sigma g) &= (\partial_{p_i} f) \star_\sigma g + f \star_\sigma (\partial_{p_i} g), \\
x^i(f \star_\sigma g) &= f \star_\sigma (x^i g) + i\hbar\bar{\sigma}(\partial_{p_i} f) \star_\sigma g = (x^i f) \star_\sigma g - i\hbar\sigma f \star_\sigma (\partial_{p_i} g), \\
p_i(f \star_\sigma g) &= f \star_\sigma (p_i g) - i\hbar\sigma(\partial_{x^i} f) \star_\sigma g = (p_i f) \star_\sigma g + i\hbar\bar{\sigma} f \star_\sigma (\partial_{x^i} g).
\end{aligned}
$$

By induction on these formulas, $f \star_\sigma g \in C^\infty(M)$ and $\|x^n p^m \partial_x^r \partial_p^s (f \star_\sigma g)\|_\infty < \infty$ for every $n, m, r, s \in \mathbb{N}^N$.

The below theorem says about the possibility of extension of the $\star_\sigma$-product to the whole space $L^2(M)$.

Theorem 3.8. *For $\Psi, \Phi \in \mathcal{S}(M)$ there holds*

$$
\|\Psi \star_\sigma \Phi\|_{L^2} \leq \frac{1}{(2\pi\hbar)^{N/2}} \|\Psi\|_{L^2} \|\Phi\|_{L^2}. \tag{3.17}
$$

There exists an unique extension of the $\star_\sigma$-product from the space $\mathcal{S}(M)$ to the whole space $L^2(M)$, such that relation (3.17) holds.

Proof. First, lets prove that the $\star_\sigma$-product on $\mathcal{S}(M)$ is separately continuous in the L^2-norm (i.e. the $\star_\sigma$-product as a map $\mathcal{S}(M) \times \mathcal{S}(M) \to \mathcal{S}(M)$ is continuous with respect to the first and second argument separately). Note that from Jensen's inequality for $f, g \in \mathcal{S}(M)$ there holds (see Appendix A.7)

$$
\left| \iint f(x,p) g(x,p) \mathrm{d}x\, \mathrm{d}p \right|^2 \leq \iint |g(x,p)| \mathrm{d}x\, \mathrm{d}p \iint |f(x,p)|^2 |g(x,p)| \mathrm{d}x\, \mathrm{d}p\,.
$$

For $\Psi, \Phi \in \mathcal{S}(M)$ from the above equation and the integral form (3.16) of the $\star_\sigma$-product there holds

$$\begin{aligned}
\|\Psi \star_\sigma \Phi\|_{L^2}^2 &= \iint |\Psi \star_\sigma \Phi|^2 \mathrm{d}x\,\mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^{2N}} \iint \left| \iint \mathcal{F}\Psi(\xi,\eta)\Phi(x-\bar{\sigma}\eta, p-\sigma\xi) e^{\frac{i}{\hbar}(\xi_i x^i - \eta^i p_i)} \mathrm{d}\xi\,\mathrm{d}\eta \right|^2 \mathrm{d}x\,\mathrm{d}p \\
&\leq \frac{1}{(2\pi\hbar)^{2N}} \iint \|\mathcal{F}\Psi\|_{L^1} \iint |\Phi(x-\bar{\sigma}\eta, p-\sigma\xi)|^2 |\mathcal{F}\Psi(\xi,\eta)| \mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}x\,\mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^{2N}} \|\mathcal{F}\Psi\|_{L^1}^2 \|\Phi\|_{L^2}^2.
\end{aligned}$$

Hence

$$\|\Psi \star_\sigma \Phi\|_{L^2} \leq \frac{1}{(2\pi\hbar)^N} \|\mathcal{F}\Psi\|_{L^1} \|\Phi\|_{L^2}.$$

Analogically, one proves that

$$\|\Psi \star_\sigma \Phi\|_{L^2} \leq \frac{1}{(2\pi\hbar)^N} \|\mathcal{F}\Phi\|_{L^1} \|\Psi\|_{L^2}.$$

The above equations show that the $\star_\sigma$-product on $\mathcal{S}(M)$ is separately continuous in the L^2-norm.

Now, assume that $\Psi_{ij} \in \mathcal{S}(M)$ is an orthonormal basis in $L^2(M)$ satisfying

$$\Psi_{ij} \star_\sigma \Psi_{kl} = \frac{1}{(2\pi\hbar)^{N/2}} \delta_{il} \Psi_{kj}. \tag{3.18}$$

Such basis always exists (see Section 4). Every $\Psi \in L^2(M)$ can be expanded in this basis

$$\Psi = \sum_{i,j=0}^{\infty} c_{ij} \Psi_{ij},$$

where the convergence is in the L^2-norm. First note that for $\Psi = \sum_{i,j} c_{ij}\Psi_{ij} \in \mathcal{S}(M)$ and $\Phi = \sum_{k,l} b_{kl}\Psi_{kl} \in \mathcal{S}(M)$, using (3.18) and the continuity of the $\star_\sigma$-product in the L^2-norm, the $\star_\sigma$-product of two Schwartz functions can be written in a form

$$\Psi \star_\sigma \Phi = \left(\sum_{i,j=0}^{\infty} c_{ij}\Psi_{ij} \right) \star_\sigma \left(\sum_{k,l=0}^{\infty} b_{kl}\Psi_{kl} \right) = \frac{1}{(2\pi\hbar)^{N/2}} \sum_{i,j,k=0}^{\infty} c_{ij} b_{ki} \Psi_{kj}. \tag{3.19}$$

Now, for $\Psi = \sum_{i,j} c_{ij}\Psi_{ij} \in L^2(M)$ and $\Phi = \sum_{k,l} b_{kl}\Psi_{kl} \in L^2(M)$ lets define the $\star_\sigma$-product of functions Ψ and Φ by the formula

$$\Psi \star_\sigma \Phi = \frac{1}{(2\pi\hbar)^{N/2}} \sum_{k,j=0}^{\infty} \left(\sum_{i=0}^{\infty} c_{ij} b_{ki} \right) \Psi_{kj}.$$

From (3.19) above definition of the $\star_\sigma$-product, for Schwartz functions, is consistent with the previous one.

From Schwartz inequality it follows that

$$
\begin{aligned}
\|\Psi \star_\sigma \Phi\|_{L^2}^2 &\leq \frac{1}{(2\pi\hbar)^N} \sum_{k,j=0}^{\infty} \left(\sum_{i=0}^{\infty} |c_{ij}||b_{ki}| \right)^2 \leq \frac{1}{(2\pi\hbar)^N} \sum_{i,j=0}^{\infty} |c_{ij}|^2 \sum_{k,l=0}^{\infty} |b_{kl}|^2 \\
&= \frac{1}{(2\pi\hbar)^N} \|\Psi\|_{L^2}^2 \|\Phi\|_{L^2}^2 .
\end{aligned}
$$

Hence, the $\star_\sigma$-product in $L^2(M)$ satisfies the relation (3.17). The uniqueness of the presented extension of the $\star_\sigma$-product is evident from the fact that $\mathcal{S}(M)$ is dense in $L^2(M)$. □

Now, lets introduce the space containing all admissible states for the general (σ, α, β)-ordering. First, lets derive the integral representation of the isomorphism $S_{\alpha,\beta} \colon (\mathcal{A}_Q, \star_\sigma) \to (\mathcal{A}_Q, \star_{\sigma,\alpha,\beta})$ from equation (3.7). There holds

Theorem 3.9. *Let α, β be such matrices, which induced quadratic forms are positive define and let $f \in \mathcal{A}_Q$. Then the integral representation of $S_{\alpha,\beta}$ reads*

$$
\begin{aligned}
S_{\alpha,\beta} f(x,p) = \frac{1}{(2\pi\hbar)^N \sqrt{\det(\alpha\beta)}} & \iint f(x',p') e^{-\frac{1}{2\hbar}\sum_{ij}(\alpha^{-1})_{ij}(x^i - x'^i)(x^j - x'^j)} \\
& \cdot e^{-\frac{1}{2\hbar}\sum_{ij}(\beta^{-1})^{ij}(p_i - p'_i)(p_j - p'_j)} \mathrm{d}x' \, \mathrm{d}p' \, .
\end{aligned}
$$

Proof. From the formula for the Fourier transform of a Gaussian function and the convolution theorem (see Appendix A.1) it follows that

$$
\begin{aligned}
S_{\alpha,\beta} f(x,p) &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} f(x,p) \\
&= \mathcal{F}^{-1}\mathcal{F}\left(e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} f \right)(x,p) \\
&= \mathcal{F}^{-1}\left(e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j} e^{-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j} \mathcal{F} f(\xi,\eta) \right)(x,p) \\
&= \frac{1}{(2\pi\hbar)^N \sqrt{\det(\alpha\beta)}} e^{-\frac{1}{2\hbar}(\alpha^{-1})_{ij}x^i x^j} e^{-\frac{1}{2\hbar}(\beta^{-1})^{ij}p_i p_j} * f(x,p) \\
&= \frac{1}{(2\pi\hbar)^N \sqrt{\det(\alpha\beta)}} \iint f(x',p') e^{-\frac{1}{2\hbar}\sum_{ij}(\alpha^{-1})_{ij}(x^i - x'^i)(x^j - x'^j)} \\
&\qquad \cdot e^{-\frac{1}{2\hbar}\sum_{ij}(\beta^{-1})^{ij}(p_i - p'_i)(p_j - p'_j)} \mathrm{d}x' \, \mathrm{d}p' \, .
\end{aligned}
$$

□

From the integral representation of the isomorphism $S_{\alpha,\beta}\colon (\mathcal{A}_Q, \star_\sigma) \to (\mathcal{A}_Q, \star_{\sigma,\alpha,\beta})$ it can be immediately seen that the isomorphism $S_{\alpha,\beta}$ can also be defined on the Schwartz space $\mathcal{S}(M)$. $\mathcal{S}(M)$ with an L^2-scalar product is an unitary space. The isomorphism $S_{\alpha,\beta}$ induces a scalar product on the space $S_{\alpha,\beta}(\mathcal{S}(M))$ by the formula

$$\langle\Psi|\Phi\rangle = \langle S_{\alpha,\beta}^{-1}\Psi|S_{\alpha,\beta}^{-1}\Phi\rangle_{L^2}, \quad \Psi, \Phi \in S_{\alpha,\beta}(\mathcal{S}(M)),$$

making from $S_{\alpha,\beta}(\mathcal{S}(M))$ an unitary space, which can be completed to a Hilbert space. The completion of $S_{\alpha,\beta}(\mathcal{S}(M))$ will be denoted by $\mathcal{H}$. Thus, $S_{\alpha,\beta}$ is an isometry, hence in particular a continuous map, from $\mathcal{S}(M)$ into $\mathcal{H}$. Now, since $\mathcal{S}(M)$ is dense in $L^2(M)$ and $S_{\alpha,\beta}$ is continuous, $S_{\alpha,\beta}$ can be uniquely extended to a Hilbert space isomorphism defined on the whole space $L^2(M)$. Note that $\mathcal{H} = S_{\alpha,\beta}(L^2(M))$. Thus, as the space, containing all admissible states, for the general (σ, α, β)-ordering the space $\mathcal{H}$ can be chosen. Note that the scalar product on $\mathcal{H}$ satisfies the relation

$$\langle\Psi|\Phi\rangle_{\mathcal{H}} = \langle S_{\alpha,\beta}^{-1}\Psi|S_{\alpha,\beta}^{-1}\Phi\rangle_{L^2}, \quad \Psi, \Phi \in \mathcal{H},$$

Note also that the isomorphism $S_{\alpha,\beta}$ induces the $\star_{\sigma,\alpha,\beta}$-product on $\mathcal{H}$ from the $\star_\sigma$-product on $L^2(M)$ by the formula

$$\Psi \star_{\sigma,\alpha,\beta} \Phi = S_{\alpha,\beta}^{-1}\Psi \star_\sigma S_{\alpha,\beta}^{-1}\Phi, \quad \Psi, \Phi \in \mathcal{H},$$

which satisfies the analogue of relation (3.17)

$$\|\Psi \star_{\sigma,\alpha,\beta} \Phi\|_{\mathcal{H}} \leq \frac{1}{(2\pi\hbar)^{N/2}}\|\Psi\|_{\mathcal{H}}\|\Phi\|_{\mathcal{H}}, \quad \Psi, \Phi \in \mathcal{H}. \tag{3.20}$$

From relation (3.20) it is evident that the $\star_{\sigma,\alpha,\beta}$-product is a continuous mapping $\mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and that $\mathcal{H}$ has a structure of a Hilbert algebra.

Lets see how the scalar product on the Hilbert space $\mathcal{H}$ looks like. Let $\Psi^{\sigma,\alpha,\beta}, \Phi^{\sigma,\alpha,\beta} \in \mathcal{H}$, then since $S_{\alpha,\beta}$ is an isomorphism of $L^2(\mathbb{R}^{2N})$ onto $\mathcal{H}$, there exist $\Psi^\sigma, \Phi^\sigma \in L^2(\mathbb{R}^{2N})$ such that $\Psi^{\sigma,\alpha,\beta} = S_{\alpha,\beta}\Psi^\sigma$ and $\Phi^{\sigma,\alpha,\beta} = S_{\alpha,\beta}\Phi^\sigma$. Now, from definition $\langle\Psi^{\sigma,\alpha,\beta}|\Phi^{\sigma,\alpha,\beta}\rangle_{\mathcal{H}} = \langle\Psi^\sigma|\Phi^\sigma\rangle_{L^2}$. Lets define such measure on the σ-algebra of Borel subsets of $M = \mathbb{R}^{2N}$

$$d\mu(\xi, \eta) = \exp\left(\frac{1}{\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(\frac{1}{\hbar}\beta_{ij}\eta^i\eta^j\right) d\xi\, d\eta\,.$$

The scalar product of functions $\Psi^{\sigma,\alpha,\beta}, \Phi^{\sigma,\alpha,\beta}$ can be written in a form

$$\langle\Psi^{\sigma,\alpha,\beta}|\Phi^{\sigma,\alpha,\beta}\rangle_{\mathcal{H}} = \iint \left(\mathcal{F}\Psi^{\sigma,\alpha,\beta}(\xi, \eta)\right)^* \mathcal{F}\Phi^{\sigma,\alpha,\beta}(\xi, \eta) d\mu(\xi, \eta)\,. \tag{3.21}$$

Indeed, the Fourier transforms of functions $\Psi^{\sigma,\alpha,\beta}, \Phi^{\sigma,\alpha,\beta}$ are equal

$$\mathcal{F}\Psi^{\sigma,\alpha,\beta}(\xi,\eta) = \exp\left(-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j\right)\mathcal{F}\Psi^{\sigma}(\xi,\eta),$$
$$\mathcal{F}\Phi^{\sigma,\alpha,\beta}(\xi,\eta) = \exp\left(-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j\right)\mathcal{F}\Phi^{\sigma}(\xi,\eta).$$

Using above calculations equation (3.21) can be written in a form

$$\begin{aligned}\langle\Psi^{\sigma,\alpha,\beta}|\Phi^{\sigma,\alpha,\beta}\rangle_{\mathcal{H}} &= \iint \exp\left(-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j\right)(\mathcal{F}\Psi^{\sigma}(\xi,\eta))^* \\ &\quad\cdot\exp\left(-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j\right)\mathcal{F}\Phi^{\sigma}(\xi,\eta) \\ &\quad\cdot\exp\left(\frac{1}{\hbar}\alpha^{ij}\xi_i\xi_j\right)\exp\left(\frac{1}{\hbar}\beta_{ij}\eta^i\eta^j\right)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \iint (\mathcal{F}\Psi^{\sigma}(\xi,\eta))^*\,\mathcal{F}\Phi^{\sigma}(\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta \\ &= \langle\mathcal{F}\Psi^{\sigma}|\mathcal{F}\Phi^{\sigma}\rangle_{L^2} = \langle\Psi^{\sigma}|\Phi^{\sigma}\rangle_{L^2}.\end{aligned}$$

From this it can be seen that the Fourier transform of the Hilbert space $\mathcal{H}$ is a space $L^2(\mathbb{R}^{2N},\mu)$ of square integrable functions on the phase space with respect to the measure μ.

Using the integral representation (3.16) of the $\star_{\sigma,\alpha,\beta}$-product it is possible to define a left and right $\star_{\sigma,\alpha,\beta}$-product of a function $A \in \mathcal{A}_Q$ with functions from some subspace of $\mathcal{H}$ receiving again a function from $\mathcal{H}$. Note that in general the function $A \in \mathcal{A}_Q$ cannot be multiplied by every function from $\mathcal{H}$ in such a way, as to receive again a function from $\mathcal{H}$. Above arguments state that operators from the algebra $\hat{\mathcal{A}}_Q$, hence in particular observables, can be treated as operators defined on the Hilbert space $\mathcal{H}$.

It can be shown that a left action of some function $A \in \mathcal{A}_Q$ through the $\star_{\sigma,\alpha,\beta}$-product on some function $\Psi \in \mathcal{H}$ can be treated as an action of an operator function $A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})$ on function Ψ where $\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta}$ are some operators defined on the Hilbert space $\mathcal{H}$. Similarly, a right action of $A \in \mathcal{A}_Q$ through the $\star_{\sigma,\alpha,\beta}$-product on $\Psi \in \mathcal{H}$ can be treated as an action of an operator function $A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha},\hat{p}^*_{\bar{\sigma},\beta})$ on function Ψ. First, note that by using equation (3.15a) and the identity

$$e^{a\partial_x}f(x) = f(x+a), \quad a \in \mathbb{R} \tag{3.22}$$

valid for any smooth function $f\colon \mathbb{R}\to\mathbb{C}$, the $\star_{\sigma,\alpha,\beta}$-product of functions $A \in \mathcal{A}_Q$ and

$\Psi \in \mathcal{H}$ can be formally written in a form

$$A_L \star_{\sigma,\alpha,\beta} \Psi := A \star_{\sigma,\alpha,\beta} \Psi = A(x + i\hbar\sigma \overrightarrow{\partial}_p + \hbar\alpha \overrightarrow{\partial}_x, p - i\hbar\bar{\sigma} \overrightarrow{\partial}_x + \hbar\beta \overrightarrow{\partial}_p)\Psi,$$
$$A_R \star_{\sigma,\alpha,\beta} \Psi := \Psi \star_{\sigma,\alpha,\beta} A = A(x - i\hbar\bar{\sigma} \overrightarrow{\partial}_p + \hbar\alpha \overrightarrow{\partial}_x, p + i\hbar\sigma \overrightarrow{\partial}_x + \hbar\beta \overrightarrow{\partial}_p)\Psi.$$

Lets define the operators $\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}$ by the equations

$$(\hat{q}_{\sigma,\alpha})^i := x^i + i\hbar\sigma^i_j \partial_{p_j} + \hbar\alpha^{ij}\partial_{x^j} = x^i \star_{\sigma,\alpha,\beta} ,$$
$$(\hat{p}_{\sigma,\beta})_i := p_i - i\hbar\bar{\sigma}^j_i \partial_{x^j} + \hbar\beta_{ij}\partial_{p_j} = p_i \star_{\sigma,\alpha,\beta} .$$

It can be easily checked that the operators $\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}$ satisfy the following commutation relations

$$[(\hat{q}_{\sigma,\alpha})^i, (\hat{p}_{\sigma,\beta})_j] = i\hbar\delta^i_j,$$
$$[(\hat{q}_{\sigma,\alpha})^i, (\hat{q}_{\sigma,\alpha})^j] = 0, \qquad [(\hat{p}_{\sigma,\beta})_i, (\hat{p}_{\sigma,\beta})_j] = 0.$$

The operators $\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}$ take then the form

$$(\hat{q}^*_{\bar{\sigma},\alpha})^i := x^i - i\hbar\bar{\sigma}^i_j \partial_{p_j} + \hbar\alpha^{ij}\partial_{x^j},$$
$$(\hat{p}^*_{\bar{\sigma},\beta})_i := p_i + i\hbar\sigma^j_i \partial_{x^j} + \hbar\beta_{ij}\partial_{p_j},$$

and they satisfy the following commutation relations

$$[(\hat{q}^*_{\bar{\sigma},\alpha})^i, (\hat{p}^*_{\bar{\sigma},\beta})_j] = -i\hbar\delta^i_j,$$
$$[(\hat{q}^*_{\bar{\sigma},\alpha})^i, (\hat{q}^*_{\bar{\sigma},\alpha})^j] = 0, \qquad [(\hat{p}^*_{\bar{\sigma},\beta})_i, (\hat{p}^*_{\bar{\sigma},\beta})_j] = 0.$$

There holds

Theorem 3.10. *For any function $A \in \mathcal{A}_Q$ there holds*

$$A_L \star_{\sigma,\alpha,\beta} = A(\overrightarrow{\hat{q}}_{\sigma,\alpha}, \overrightarrow{\hat{p}}_{\sigma,\beta}) = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}),$$
$$A_R \star_{\sigma,\alpha,\beta} = A(\overrightarrow{\hat{q}}^*_{\bar{\sigma},\alpha}, \overrightarrow{\hat{p}}^*_{\bar{\sigma},\beta}) = A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}).$$

Proof. Above theorem can be proved using the integral form (3.16) of the $\star_{\sigma,\alpha,\beta}$-product. First, lets check how the operator $e^{\frac{i}{\hbar}\xi_i(\hat{q}_{\sigma,\alpha})^i} e^{-\frac{i}{\hbar}\eta^i(\hat{p}_{\sigma,\beta})_i}$ acts on some function $\Psi \in \mathcal{H}$. Using the Baker-Campbell-Hausdorff formulae (see Appendix A.4) and the identity (3.22) one receives

$$
\begin{aligned}
&e^{\frac{i}{\hbar}\xi_i(\hat{q}_{\sigma,\alpha})^i}e^{-\frac{i}{\hbar}\eta^i(\hat{p}_{\sigma,\beta})_i}\Psi(x,p) = e^{\frac{i}{\hbar}\sum_i \xi_i(x^i+i\hbar\sigma^i_j\partial_{p_j}+\hbar\alpha^{ij}\partial_{x^j})}e^{-\frac{i}{\hbar}\sum_i \eta^i(p_i-i\hbar\bar{\sigma}^j_i\partial_{x^j}+\hbar\beta_{ij}\partial_{p_j})}\Psi(x,p)\\
&\quad = e^{\frac{i}{\hbar}\xi_i x^i}e^{-\sigma^i_j\xi_i\partial_{p_j}}e^{i\alpha^{ij}\xi_i\partial_{x^j}}e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j}e^{-\frac{i}{\hbar}\eta^i p_i}e^{-\bar{\sigma}^j_i\eta^i\partial_{x^j}}e^{-i\beta_{ij}\eta^i\partial_{p_j}}e^{-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j}\Psi(x,p)\\
&\quad = e^{\frac{i}{\hbar}\xi_i x^i}e^{-\frac{i}{\hbar}\eta^i p_i}e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j}e^{-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j}e^{\frac{i}{\hbar}\sigma^i_j\xi_i\eta^j}e^{-\sigma^i_j\xi_i\partial_{p_j}}e^{i\alpha^{ij}\xi_i\partial_{x^j}}e^{-\bar{\sigma}^j_i\eta^i\partial_{x^j}}e^{-i\beta_{ij}\eta^i\partial_{p_j}}\Psi(x,p)\\
&\quad = e^{\frac{i}{\hbar}\xi_i x^i}e^{-\frac{i}{\hbar}\eta^i p_i}e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j}e^{-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j}e^{\frac{i}{\hbar}\sigma^i_j\xi_i\eta^j}\Psi(x-\bar{\sigma}\eta+i\alpha\xi,p-\sigma\xi-i\beta\eta).
\end{aligned}
$$

Using above equation and equation (3.11) it follows immediately that

$$
\begin{aligned}
A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})\Psi(x,p) = \frac{1}{(2\pi\hbar)^N}\iint &\mathcal{F}A(\xi,\eta)\Psi(x-\bar{\sigma}\eta+i\alpha\xi,p-\sigma\xi-i\beta\eta)\\
&\cdot e^{\frac{i}{\hbar}\xi_i x^i}e^{-\frac{i}{\hbar}\eta^i p_i}\mathrm{d}\xi\,\mathrm{d}\eta\,,
\end{aligned}
$$

which is just the integral form (3.16) of the product $A \star_{\sigma,\alpha,\beta} \Psi$. □

From Theorem 3.10 it follows that operators from the algebra $\hat{\mathcal{A}}_Q$, hence in particular observables, can be written as operator functions of the operators $\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}$.

It is possible to introduce adjoint of left and right $\star_{\sigma,\alpha,\beta}$-multiplication in a standard way

$$
\begin{aligned}
\langle (A_L \star_{\sigma,\alpha,\beta})^\dagger \Psi_1 | \Psi_2 \rangle_{\mathcal{H}} &= \langle \Psi_1 | A_L \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}},\\
\langle (A_R \star_{\sigma,\alpha,\beta})^\dagger \Psi_1 | \Psi_2 \rangle_{\mathcal{H}} &= \langle \Psi_1 | A_R \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}}.
\end{aligned}
$$

From this it then follows that

$$
(A_L \star_{\sigma,\alpha,\beta})^\dagger = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta}) = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta}), \tag{3.23a}
$$

$$
(A_R \star_{\sigma,\alpha,\beta})^\dagger = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha},\hat{p}^*_{\bar{\sigma},\beta}) = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha},\hat{p}^*_{\bar{\sigma},\beta}). \tag{3.23b}
$$

3.4 Pure states, mixed states and expectation values of observables

As was presented earlier all admissible states of the quantum Hamiltonian system are contained in the Hilbert space $\mathcal{H}$. It is necessary to determine which functions from $\mathcal{H}$ can be considered as *pure states* and *mixed states*. *Pure states* will be defined as functions $\Psi_{\text{pure}} \in \mathcal{H}$ which satisfy the following conditions

1. $\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta} = (\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta})^{\dagger}$ (hermiticity),

2. $\Psi_{\text{pure}} \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} = \frac{1}{(2\pi\hbar)^{N/2}} \Psi_{\text{pure}}$ (idempotence),

3. $\|\Psi_{\text{pure}}\|_{\mathcal{H}} = 1$ (normalization).

Mixed states $\Psi_{\text{mix}} \in \mathcal{H}$ will be defined in a standard way, as linear combinations, possibly infinite, of some families of pure states $\Psi_{\text{pure}}^{(\lambda)}$

$$\Psi_{\text{mix}} := \sum_{\lambda} p_{\lambda} \Psi_{\text{pure}}^{(\lambda)},$$

where $0 \leq p_{\lambda} \leq 1$ and $\sum_{\lambda} p_{\lambda} = 1$. Such definition of mixed states reflects the lack of knowledge about the state of the system, where p_{λ} is the probability of finding the system in a state $\Psi_{\text{pure}}^{(\lambda)}$.

In what follows a characterization of mixed states will be given but first, lets introduce a (σ, α, β)-*twisted square root* of functions $\Psi \in \mathcal{H}$. Lets assume that $\Psi \in \mathcal{H}$ is hermitian and positive define, i.e.

1. $\Psi \star_{\sigma,\alpha,\beta} = (\Psi \star_{\sigma,\alpha,\beta})^{\dagger}$ (hermiticity),

2. $\langle \Psi_{\text{pure}} | \Psi \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} \rangle_{\mathcal{H}} \geq 0$ for every pure state Ψ_{pure} (positive definite).

Then, it can be proved (see Theorem 4.3) that there exists exactly one hermitian and positive define function $\Phi \in \mathcal{H}$ such that

$$\Phi \star_{\sigma,\alpha,\beta} \Phi = \frac{1}{(2\pi\hbar)^{N/2}} \Psi.$$

This function Φ will be called a (σ, α, β)-*twisted square root* of function $\Psi \in \mathcal{H}$ and it will be denoted by the same symbol as the ordinary square root, i.e. by $\sqrt{\Psi}$.

Now, it is easy to check that every admissible (pure or mixed) state $\Psi \in \mathcal{H}$ satisfies the following conditions

1. $\Psi \star_{\sigma,\alpha,\beta} = (\Psi \star_{\sigma,\alpha,\beta})^{\dagger}$ (hermiticity),

2. $\langle \Psi_{\text{pure}} | \Psi \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} \rangle_{\mathcal{H}} \geq 0$ for every pure state Ψ_{pure} (positive definite),

3. $\|\sqrt{\Psi}\|_{\mathcal{H}} = 1$ (normalization).

Conversely, every function $\Psi \in \mathcal{H}$ satisfying the above conditions is an admissible (pure or mixed) state (see Theorem 4.4).

For an admissible quantum state $\Psi \in \mathcal{H}$ lets define a *quantum distribution function* ρ on the phase space by the equation

$$\rho := \frac{1}{(2\pi\hbar)^{N/2}}\Psi.$$

Note that from Theorem 4.5 it follows that the function ρ is a quasi-probabilistic distribution function, i.e.

$$\iint \rho(x,p)\mathrm{d}x\,\mathrm{d}p = 1.$$

The quantum distribution functions ρ are the analogue of the classical distribution functions representing states of the classical Hamiltonian system. The difference between classical and quantum distribution functions is that the latter do not have to have values from the range $[0,1]$. Thus, $\rho(x,p)$ cannot be interpreted as a probability density of finding a particle in a point (x,p) of the phase space. This is a reflection of the fact that x and p coordinates do not commute with respect to the $\star_{\sigma,\alpha,\beta}$-multiplication, which yield, from the Heisenberg uncertainty principle, that it is impossible to measure simultaneously the position and momentum of a particle. Hence, the point position of a particle in the phase space does not make sense anymore. On the other hand, from Theorem 4.6 it follows that *marginal distributions*

$$P(x) := \int \rho(x,p)\mathrm{d}p\,, \qquad P(p) := \int \rho(x,p)\mathrm{d}x\,,$$

are probabilistic distribution functions and can be interpreted as probability densities that a particle in the phase space have position x or momentum p. The result is not surprising as each marginal distribution depends on commuting coordinates only.

The expectation value of an observable $\hat{A} \in \hat{\mathcal{A}}_Q$ in an admissible state $\Psi \in \mathcal{H}$ can be defined like in its classical analogue (2.5), i.e. as a mean value with respect to a quantum distribution function $\rho = \frac{1}{(2\pi\hbar)^{N/2}}\Psi$

$$\langle \hat{A} \rangle_\Psi = \iint (\hat{A}\rho)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint (A \star_{\sigma,\alpha,\beta} \rho)(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

3.5 Time evolution of quantum Hamiltonian systems

In this section the time evolution of a quantum Hamiltonian system will be presented. Analogically as in classical mechanics, the time evolution of a quantum Hamiltonian system is governed by a Hamiltonian $\hat{H}$. It will be assumed that $\hat{H} \in \hat{\mathcal{O}}_Q$ and that $\hat{H}$ is self-adjoint in $\mathcal{H}$, i.e. $H = H^*$ and $H_{L,R} \star_{\sigma,\alpha,\beta} = (H_{L,R} \star_{\sigma,\alpha,\beta})^\dagger$. The time evolution of a quantum distribution function ρ is defined like in its classical counterpart (2.12)

$$\begin{gathered} L(H,\rho) := \frac{\partial \rho}{\partial t} - [|H,\rho|] = 0 \\ \Updownarrow \\ i\hbar \frac{\partial \rho}{\partial t} - [H,\rho] = 0. \end{gathered} \tag{3.24}$$

States $\Psi \in \mathcal{H}$ which do not change during the time development, i.e. such that $\frac{\partial \Psi}{\partial t} = 0$ are called *stationary states*. From the time evolution equation (3.24) it follows that stationary states Ψ satisfy

$$[H,\Psi] = 0.$$

If a stationary state Ψ is a pure state then, from Theorem 4.10, it follows that the above equation is equivalent to a pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$H \star_{\sigma,\alpha,\beta} \Psi = E\Psi, \quad \Psi \star_{\sigma,\alpha,\beta} H = E\Psi,$$

for some $E \in \mathbb{R}$. Note that E in the above equations is the expectation value of the Hamiltonian $\hat{H}$ in a stationary state Ψ, hence it is an energy of the system in the state Ψ.

The formal solution of (3.24) takes the form

$$\rho(t) = U(t) \star_{\sigma,\alpha,\beta} \rho(0) \star_{\sigma,\alpha,\beta} U(-t),$$

where

$$U(t) = e_{\star_{\sigma,\alpha,\beta}}^{-\frac{i}{\hbar}tH} := \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{i}{\hbar}t\right)^k \underbrace{H \star_{\sigma,\alpha,\beta} \ldots \star_{\sigma,\alpha,\beta} H}_{k} \tag{3.25}$$

is an unitary function in $\mathcal{H}$ as $\hat{H}$ is self-adjoint. Hence, the time evolution of states can be alternatively expressed in terms of the one parameter group of unitary functions $U(t)$.

From (3.24) it follows that a time dependent expectation value of an observable $\hat{A} \in \hat{\mathcal{A}}_Q$ in a state $\rho(t)$, i.e. $\langle \hat{A} \rangle_{\rho(t)}$, fulfills the following equation of motion

$$\langle \hat{A} \rangle_{L(H,\rho)} = 0 \iff \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{A} \rangle_{\rho(t)} - \langle [|\hat{A}, \hat{H}|] \rangle_{\rho(t)} = 0. \tag{3.26}$$

Indeed

$$\iint \mathrm{d}x\, \mathrm{d}p\, A \star_{\sigma,\alpha,\beta} \frac{\partial \rho}{\partial t}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \iint \mathrm{d}x\, \mathrm{d}p\, A \star_{\sigma,\alpha,\beta} \rho(t) = \frac{\mathrm{d}}{\mathrm{d}t} \langle \hat{A} \rangle_{\rho(t)},$$

$$\begin{aligned} &\iint \mathrm{d}x\, \mathrm{d}p\, A \star_{\sigma,\alpha,\beta} \frac{1}{i\hbar}(H \star_{\sigma,\alpha,\beta} \rho(t) - \rho(t) \star_{\sigma,\alpha,\beta} H) = \\ &= \iint \mathrm{d}x\, \mathrm{d}p\, \frac{1}{i\hbar}(A \star_{\sigma,\alpha,\beta} H - H \star_{\sigma,\alpha,\beta} A) \star_{\sigma,\alpha,\beta} \rho(t) \\ &= \langle [|\hat{A}, \hat{H}|] \rangle_{\rho(t)}. \end{aligned}$$

Equation (3.26) is the quantum analogue of the classical equation (2.13).

Until now the time evolution in the *Schrödinger picture* were considered, i.e. only states undergo a time development. It is also possible to consider a dual approach to the time evolution, namely the *Heisenberg picture*. In this picture states remain still whereas the observables undergo a time development. The time development of an observable $\hat{A} \in \hat{\mathcal{A}}_Q$ is given by the action of the unitary function $U(t)$ from (3.25) on $\hat{A}$

$$\hat{A}(t) = U(-t) \star_{\sigma,\alpha,\beta} A(0) \star_{\sigma,\alpha,\beta} U(t) \star_{\sigma,\alpha,\beta} = \hat{U}(-t)\hat{A}(0)\hat{U}(t). \tag{3.27}$$

Differentiating equation (3.27) with respect to t results in such evolution equation for $\hat{A}$

$$\frac{\mathrm{d}\hat{A}}{\mathrm{d}t}(t) - [|\hat{A}(t), \hat{H}|] = 0. \tag{3.28}$$

Equation (3.28) is the quantum analogue of the classical equation (2.15).

Both presented approaches to the time development yield equal predictions concerning the results of measurements, since

$$\begin{aligned} \langle \hat{A}(0) \rangle_{\rho(t)} &= \iint \mathrm{d}x\, \mathrm{d}p\, A(0) \star_{\sigma,\alpha,\beta} \rho(t) \\ &= \iint \mathrm{d}x\, \mathrm{d}p\, A(0) \star_{\sigma,\alpha,\beta} U(t) \star_{\sigma,\alpha,\beta} \rho(0) \star_{\sigma,\alpha,\beta} U(-t) \\ &= \iint \mathrm{d}x\, \mathrm{d}p\, (U(-t) \star_{\sigma,\alpha,\beta} A(0) \star_{\sigma,\alpha,\beta} U(t)) \star_{\sigma,\alpha,\beta} \rho(0) \\ &= \iint \mathrm{d}x\, \mathrm{d}p\, A(t) \star_{\sigma,\alpha,\beta} \rho(0) = \langle \hat{A}(t) \rangle_{\rho(0)}. \end{aligned}$$

4

Ordinary description of quantum mechanics

In this section it will be shown that from the phase space quantum mechanics immediately follows the ordinary description of quantum mechanics developed by Schrödinger, Dirac and Heisenberg, in which observables and states are defined as operators on the Hilbert space $L^2(\mathbb{R}^N)$. In contrary to a waste of papers devoted to quantum mechanics on a phase space it is shown that the ordinary quantum mechanics naturally follows from the construction of the presented formalism and it is not needed to introduce a morphism between the spaces of observables (usually referred to in the literature as the Wigner map) to show the equivalence between both descriptions of quantum mechanics.

First, it will be shown that $\mathcal{H}$ can be considered as a tensor product of Hilbert spaces $L^2(\mathbb{R}^N)$ and a space dual to it $\big(L^2(\mathbb{R}^N)\big)^*$. It is well known that the Hilbert space $\big(L^2(\mathbb{R}^N)\big)^*$ dual to $L^2(\mathbb{R}^N)$ can be identified with $L^2(\mathbb{R}^N)$ where the anti-linear duality map $*\colon L^2(\mathbb{R}^N) \to \big(L^2(\mathbb{R}^N)\big)^*$ is the complex conjugation of functions. The tensor product of $\big(L^2(\mathbb{R}^N)\big)^*$ and $L^2(\mathbb{R}^N)$ is defined up to an isomorphism. The most natural choice for the tensor product of $\big(L^2(\mathbb{R}^N)\big)^*$ and $L^2(\mathbb{R}^N)$ is the Hilbert space $L^2(\mathbb{R}^{2N})$ where the tensor product of $\varphi \in \big(L^2(\mathbb{R}^N)\big)^*$ and $\psi \in L^2(\mathbb{R}^N)$ is defined as

$$(\varphi \otimes \psi)(x, y) := \varphi^*(x)\psi(y).$$

and the scalar product in $L^2(\mathbb{R}^{2N})$ satisfies the equation

$$\langle \varphi_1 \otimes \psi_1 | \varphi_2 \otimes \psi_2 \rangle_{L^2} = \langle \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2},$$

for $\varphi_1, \varphi_2 \in \big(L^2(\mathbb{R}^N)\big)^*$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^N)$.

Now, an isomorphism of $L^2(\mathbb{R}^{2N})$ onto $\mathcal{H}$ will be defined, which will make from $\mathcal{H}$ a tensor product of $\big(L^2(\mathbb{R}^N)\big)^*$ and $L^2(\mathbb{R}^N)$. First, note that the Fourier transform $\mathcal{F}_y$ is an isomorphism of $L^2(\mathbb{R}^{2N})$. For $\Psi(x,y) \in L^2(\mathbb{R}^{2N})$, the function

$$\Psi(x,p) = \mathcal{F}_y(\Psi(x,y)) = \frac{1}{(2\pi\hbar)^{N/2}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} \Psi(x,y)$$

will be called an (x,p)-representation of $\Psi(x,y)$ and it will be considered as a function on the phase space $M = \mathbb{R}^{2N}$ in the canonical coordinates of position x and momentum p. Lets introduce another isomorphism of $L^2(\mathbb{R}^{2N})$ by the equation

$$T_\sigma \Psi(x,y) := \Psi(x - \bar{\sigma}y, x + \sigma y), \quad \Psi \in L^2(\mathbb{R}^{2N}).$$

As the searched isomorphism of $L^2(\mathbb{R}^{2N})$ onto $\mathcal{H}$ the map $S_{\alpha,\beta}\mathcal{F}_y T_\sigma$ will be taken. A tensor product of $\big(L^2(\mathbb{R}^N)\big)^*$ and $L^2(\mathbb{R}^N)$ induced by this isomorphism will be denoted by $\big(L^2(\mathbb{R}^N)\big)^* \otimes_{\sigma,\alpha,\beta} L^2(\mathbb{R}^N)$ and called a (σ,α,β)-*twisted tensor product* of $\big(L^2(\mathbb{R}^N)\big)^*$ and $L^2(\mathbb{R}^N)$. Hence

$$\mathcal{H} = \big(L^2(\mathbb{R}^N)\big)^* \otimes_{\sigma,\alpha,\beta} L^2(\mathbb{R}^N) = S_{\alpha,\beta}\mathcal{F}_y T_\sigma \left(\big(L^2(\mathbb{R}^N)\big)^* \otimes L^2(\mathbb{R}^N)\right) \tag{4.1}$$

and the scalar product in $\mathcal{H}$ satisfies

$$\langle \varphi_1 \otimes_{\sigma,\alpha,\beta} \psi_1 | \varphi_2 \otimes_{\sigma,\alpha,\beta} \psi_2 \rangle_{\mathcal{H}} = \langle \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2},$$

for $\varphi_1, \varphi_2 \in \big(L^2(\mathbb{R}^N)\big)^* \cong L^2(\mathbb{R}^N)$ and $\psi_1, \psi_2 \in L^2(\mathbb{R}^N)$. The relevance of the representation (4.1) will be revealed in the key theorem 4.8.

The generators of $\mathcal{H}$ are of the form

$$\begin{aligned}
\Psi^{\sigma,\alpha,\beta}(x,p) &= (\varphi \otimes_{\sigma,\alpha,\beta} \psi)(x,p) \\
&= \frac{1}{(2\pi\hbar)^{N/2}} e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} \varphi^*(x - \bar{\sigma}y)\psi(x + \sigma y) \\
&= \frac{1}{(2\pi\hbar)^{3N/2}\sqrt{\det(\alpha\beta)}} \iiint \mathrm{d}x'\, \mathrm{d}p'\, \mathrm{d}y\, \varphi^*(x' - \bar{\sigma}y)\psi(x' + \sigma y) \\
&\quad \cdot e^{-\frac{1}{2\hbar}\sum_{i,j}(\alpha^{-1})_{ij}(x^i - x'^i)(x^j - x'^j)} e^{-\frac{1}{2\hbar}\sum_{i,j}(\beta^{-1})^{ij}(p_i - p'_i)(p_j - p'_j)} e^{-\frac{i}{\hbar}p'_i y^i},
\end{aligned} \tag{4.2}$$

where $\varphi, \psi \in L^2(\mathbb{R}^N)$. In a special case of Weyl ordering $\sigma^i_j = \frac{1}{2}\delta^i_j, \alpha = \beta = 0$ generators $\Psi^{\sigma,\alpha,\beta}$ are the well known Wigner functions related to the Moyal $\star$-product. Many particular examples of the quantum phase-space distribution functions (4.2) considered in the past are listed and described in the review paper [39].

Observe, that if $\{\varphi_i\}$ is an orthonormal basis in $L^2(\mathbb{R}^N)$, then $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ is an orthonormal basis in $\mathcal{H}$ and for any $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi \in \mathcal{H}$ where $\varphi, \psi \in L^2(\mathbb{R}^N)$, one have

$$\varphi = \sum_i b_i \varphi_i, \qquad \psi = \sum_j c_j \varphi_j, \qquad \text{for some } b_i, c_i \in \mathbb{C},$$
$$\Psi = \sum_{i,j} a_{ij} \Psi_{ij}, \qquad a_{ij} = b_i^* c_j.$$

The interesting property of the basis functions Ψ_{ij} is they idempotence. Namely, there holds [8]

Theorem 4.1.

$$\Psi_{ij} \star_{\sigma,\alpha,\beta} \Psi_{kl} = \frac{1}{(2\pi\hbar)^{N/2}} \delta_{il} \Psi_{kj} \tag{4.3}$$

Proof. First lets prove the theorem for the case $\alpha = \beta = 0$. Assume $\Psi_{ij}^{\sigma} \in L^2(\mathbb{R}^{2N})$ are of the form

$$\Psi_{ij}^{\sigma}(x,p) = (\varphi_i \otimes_\sigma \varphi_j)(x,p) = \frac{1}{(2\pi\hbar)^{N/2}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar} p_i y^i} \varphi_i^*(x - \bar{\sigma} y) \varphi_j(x + \sigma y).$$

Using the identity (3.22) and the defining formula (3.15) of the $\star_{\sigma,\alpha,\beta}$-product, the $\star_\sigma$-product of functions Ψ_{ij}^{σ} and Ψ_{kl}^{σ} can be written in a form

$$\begin{aligned}
(\Psi_{ij}^{\sigma} \star_\sigma \Psi_{kl}^{\sigma})(x,p) &= \Psi_{ij}^{\sigma}(x, p - i\hbar\bar{\sigma} \overrightarrow{\partial}_x) \Psi_{kl}^{\sigma}(x, p + i\hbar\sigma \overleftarrow{\partial}_x) \\
&= \frac{1}{(2\pi\hbar)^N} \int \mathrm{d}y\, \varphi_i^*(x - \bar{\sigma} y) \varphi_j(x + \sigma y) e^{-\frac{i}{\hbar} y^i (p_i - i\hbar\bar{\sigma} \overrightarrow{\partial}_x)} \\
&\quad \cdot \int \mathrm{d}y'\, \varphi_k^*(x - \bar{\sigma} y') \varphi_l(x + \sigma y') e^{-\frac{i}{\hbar} y'^i (p_i + i\hbar\sigma \overleftarrow{\partial}_x)} \\
&= \frac{1}{(2\pi\hbar)^N} \iint \mathrm{d}y\, \mathrm{d}y'\, e^{-\frac{i}{\hbar} p_i (y^i + y'^i)} \varphi_i^*(x + \sigma y' - \bar{\sigma} y) \varphi_j(x + \sigma y' + \sigma y) \\
&\quad \cdot \varphi_k^*(x - \bar{\sigma} y - \bar{\sigma} y') \varphi_l(x - \bar{\sigma} y + \sigma y').
\end{aligned}$$

After introducing new coordinates: $z = y + y'$, $z' = x + \sigma y' - \bar{\sigma} y$ the above formula can be written in a form

$$\begin{aligned}
(\Psi_{ij}^{\sigma} \star_\sigma \Psi_{kl}^{\sigma})(x,p) &= \frac{1}{(2\pi\hbar)^N} \int \mathrm{d}z\, e^{-\frac{i}{\hbar} p_i z^i} \varphi_k^*(x - \bar{\sigma} z) \varphi_j(x + \sigma z) \int \mathrm{d}z'\, \varphi_i^*(z') \varphi_l(z') \\
&= \frac{1}{(2\pi\hbar)^{N/2}} \delta_{il} \Psi_{kj}^{\sigma}(x,p).
\end{aligned} \tag{4.4}$$

Now, applying the isomorphism $S_{\alpha,\beta}$ to both sides of equation (4.4), the general formula (4.3) follows immediately. ◻

Hence, if $\Psi_1 = \varphi_1 \otimes_{\sigma,\alpha,\beta} \psi_1$ and $\Psi_2 = \varphi_2 \otimes_{\sigma,\alpha,\beta} \psi_2$ where $\varphi_1, \psi_1, \varphi_2, \psi_2 \in L^2(\mathbb{R}^N)$, then

$$\Psi_1 \star_{\sigma,\alpha,\beta} \Psi_2 = \frac{1}{(2\pi\hbar)^{N/2}} \langle \varphi_1 | \psi_2 \rangle_{L^2} (\varphi_2 \otimes_{\sigma,\alpha,\beta} \psi_1), \tag{4.5a}$$

$$\Psi_2 \star_{\sigma,\alpha,\beta} \Psi_1 = \frac{1}{(2\pi\hbar)^{N/2}} \langle \varphi_2 | \psi_1 \rangle_{L^2} (\varphi_1 \otimes_{\sigma,\alpha,\beta} \psi_2). \tag{4.5b}$$

Using the basis $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ some interesting properties of the admissible states can be proved. Namely, there holds

Theorem 4.2. *Every pure state* $\Psi_{\text{pure}} \in \mathcal{H}$ *is of the form*

$$\Psi_{\text{pure}} = \varphi \otimes_{\sigma,\alpha,\beta} \varphi, \tag{4.6}$$

for some normalized function $\varphi \in L^2(\mathbb{R}^N)$*. Conversely, every function* $\Psi \in \mathcal{H}$ *of the form (4.6) is a pure state.*

Proof. From formula (4.3) it follows that every function $\Psi \in \mathcal{H}$ of the form (4.6) is a pure state. If now one assumes that $\Psi_{\text{pure}} \in \mathcal{H}$ is a pure state then Ψ_{pure} can be written in a form

$$\Psi_{\text{pure}} = \sum_{i,j} c_{ij} \Psi_{ij},$$

where $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ is an induced basis in $\mathcal{H}$ by the basis $\{\varphi_i\}$ in $L^2(\mathbb{R}^N)$. The assumptions that Ψ_{pure} is hermitian, idempotent and normalized can be restated saying that the matrix $\check{c}$ of the coefficients c_{ij} is hermitian ($\check{c} = \check{c}^\dagger$), idempotent ($\check{c}^2 = \check{c}$) and normalized ($\operatorname{tr} \check{c} = 1$). Since the matrix $\check{c}$ is hermitian it can be diagonalized, i.e. there exist an unitary matrix $\check{T}$ such that $c_{ij} = \sum_{k,l} T^\dagger_{ik} (a_k \delta_{kl}) T_{lj} = \sum_k T^*_{ki} a_k T_{kj}$ for some $a_k \in \mathbb{R}$. Hence, Ψ_{pure} takes the form

$$\begin{aligned}\Psi_{\text{pure}} &= \sum_{i,j,k} T^*_{ki} a_k T_{kj} (\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j) = \sum_k a_k \left(\left(\textstyle\sum_i T_{ki} \varphi_i \right) \otimes_{\sigma,\alpha,\beta} \left(\textstyle\sum_j T_{kj} \varphi_j \right) \right) \\ &= \sum_k a_k (\psi_k \otimes_{\sigma,\alpha,\beta} \psi_k),\end{aligned}$$

where $\psi_k = \sum_i T_{ki} \varphi_i$. The conditions that $\check{c}^2 = \check{c}$ and $\operatorname{tr} \check{c} = 1$ give that $a_k^2 = a_k$ and $\sum_k a_k = 1$. Hence $a_k = \delta_{k_0 k}$ for some k_0, from which follows that $\Psi_{\text{pure}} = \psi_{k_0} \otimes_{\sigma,\alpha,\beta} \psi_{k_0}$. □

Theorem 4.3. *Let* $\Psi \in \mathcal{H}$ *be hermitian and positive define, i.e.*

1. $\Psi \star_{\sigma,\alpha,\beta} = (\Psi \star_{\sigma,\alpha,\beta})^\dagger$ *(hermiticity),*

2. $\langle \Psi_{\text{pure}} | \Psi \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} \rangle_{\mathcal{H}} \geq 0$ *for every pure state* Ψ_{pure} *(positive definite).*

Then, there exists exactly one hermitian and positive define function $\Phi \in \mathcal{H}$ *such that*

$$\Phi \star_{\sigma,\alpha,\beta} \Phi = \frac{1}{(2\pi\hbar)^{N/2}} \Psi.$$

Proof. Function Ψ can be written in a form

$$\Psi = \sum_{i,j} c_{ij} \Psi_{ij},$$

where $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ is an induced basis in $\mathcal{H}$ by the basis $\{\varphi_i\}$ in $L^2(\mathbb{R}^N)$ and c_{ij} are coefficients of a complex matrix $\check{c}$ which is hermitian ($\check{c} = \check{c}^\dagger$) and positive define ($c_{ii} \geq 0$). The theorem can be restated saying that there exists exactly one hermitian and positive define matrix $\check{b}$ such that

$$\check{b}^2 = \check{c},$$

which is a well known fact from the linear algebra. The searched function Φ is then equal

$$\Phi = \sum_{i,j} b_{ij} \Psi_{ij}.$$

□

Theorem 4.4. *Every function* $\Psi \in \mathcal{H}$ *satisfying the below conditions is an admissible (pure or mixed) state of the quantum Hamiltonian system*

1. $\Psi \star_{\sigma,\alpha,\beta} = (\Psi \star_{\sigma,\alpha,\beta})^\dagger$ *(hermiticity),*

2. $\langle \Psi_{\text{pure}} | \Psi \star_{\sigma,\alpha,\beta} \Psi_{\text{pure}} \rangle_{\mathcal{H}} \geq 0$ *for every pure state* Ψ_{pure} *(positive definite),*

3. $\|\sqrt{\Psi}\|_{\mathcal{H}} = 1$ *(normalization).*

Proof. Function Ψ can be written in a form

$$\Psi = \sum_{i,j} c_{ij} \Psi_{ij},$$

where $\{\Psi_{ij}\} = \{\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j\}$ is an induced basis in $\mathcal{H}$ by the basis $\{\varphi_i\}$ in $L^2(\mathbb{R}^N)$. The assumptions that Ψ is hermitian, positive define and normalized can be restated saying that the matrix $\check{c}$ of the coefficients c_{ij} is hermitian ($\check{c} = \check{c}^\dagger$), positive define ($c_{ii} \geq 0$) and normalized ($\operatorname{tr} \check{c} = 1$). Since the matrix $\check{c}$ is hermitian it can be diagonalized, i.e. there exist an unitary matrix $\check{T}$ such that $c_{ij} = \sum_{k,l} T^\dagger_{ik}(p_k \delta_{kl}) T_{lj} = \sum_k T^*_{ki} p_k T_{kj}$ for some $p_k \in \mathbb{R}$. Hence, Ψ takes the form

$$\begin{aligned}\Psi &= \sum_{i,j,k} T^*_{ki} p_k T_{kj} (\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j) = \sum_k p_k \left(\left(\textstyle\sum_i T_{ki}\varphi_i \right) \otimes_{\sigma,\alpha,\beta} \left(\textstyle\sum_j T_{kj} \varphi_j \right) \right) \\ &= \sum_k p_k (\psi_k \otimes_{\sigma,\alpha,\beta} \psi_k),\end{aligned}$$

where $\psi_k = \sum_i T_{ki}\varphi_i$. The conditions that $c_{ii} \geq 0$ and $\operatorname{tr}\check{c} = 1$ give that $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$. Hence Ψ is a mixed state. □

Theorem 4.5. *Every admissible (pure or mixed) state $\Psi \in \mathcal{H}$ satisfies*

$$\frac{1}{(2\pi\hbar)^{N/2}} \iint \Psi(x,p) \mathrm{d}x \, \mathrm{d}p = 1.$$

Proof. It is enough to prove the theorem for the case when Ψ is a pure state. In that case Ψ can be written in a form $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$ for some normalized function $\varphi \in L^2(\mathbb{R}^N)$. Hence, one have that

$$\begin{aligned}\frac{1}{(2\pi\hbar)^{N/2}} \iint \Psi(x,p)\mathrm{d}x\,\mathrm{d}p &= (2\pi\hbar)^{N/2} \mathcal{F}\Psi(0) \\ &= (2\pi\hbar)^{N/2} \mathcal{F}\left(e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \mathcal{F}_y T_\sigma(\varphi \otimes \varphi) \right)(0) \\ &= (2\pi\hbar)^{N/2} e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j} e^{-\frac{1}{2\hbar}\beta_{ij}\eta^i\eta^j} \Big|_{\xi=\eta=0} \mathcal{F}\left(\mathcal{F}_y T_\sigma(\varphi\otimes\varphi)\right)(0) \\ &= \frac{1}{(2\pi\hbar)^N} \iiint \mathrm{d}x\,\mathrm{d}p\,\mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} \varphi^*(x - \bar{\sigma}y)\varphi(x+\sigma y) \\ &= \iint \mathrm{d}x\,\mathrm{d}y\,\delta(y)\varphi^*(x-\bar{\sigma}y)\varphi(x+\sigma y) \\ &= \int \mathrm{d}x\, \varphi^*(x)\varphi(x) = 1.\end{aligned}$$

□

Theorem 4.6. *Every admissible (pure or mixed) state $\Psi = \sum_\lambda p_\lambda(\varphi^{(\lambda)} \otimes_{\sigma,\alpha,\beta} \varphi^{(\lambda)})$ for some normalized $\varphi^{(\lambda)} \in L^2(\mathbb{R}^N)$ satisfies*

$$\frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p)\mathrm{d}p = \sum_\lambda p_\lambda e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} |\varphi^{(\lambda)}(x)|^2, \tag{4.7a}$$

$$\frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p)\mathrm{d}x = \sum_\lambda p_\lambda e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} |\tilde{\varphi}^{(\lambda)}(p)|^2, \tag{4.7b}$$

where $\tilde{\varphi}^{(\lambda)}$ denotes the Fourier transform of $\varphi^{(\lambda)}$.

Proof. It is enough to prove the theorem for a pure state $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$. Equation (4.7a) follows from

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p)\mathrm{d}p &= \frac{1}{(2\pi\hbar)^{N}} \int \mathrm{d}p\, e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} \\
&\quad \cdot \varphi^*(x-\bar{\sigma}y)\varphi(x+\sigma y) \\
&= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \frac{1}{(2\pi\hbar)^{N}} \iint \mathrm{d}p\, \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j} \\
&\quad \cdot \varphi^*(x-\bar{\sigma}y)\varphi(x+\sigma y) \\
&= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \int \mathrm{d}y\, \delta(y) e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j} \varphi^*(x-\bar{\sigma}y)\varphi(x+\sigma y) \\
&= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} |\varphi(x)|^2.
\end{aligned}$$

Equation (4.7b) follows from

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p)\mathrm{d}x &= \left(\mathcal{F}_x \Psi(x,p)\right)(0) \\
&= \mathcal{F}_x \left(e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \mathcal{F}_y T_\sigma(\varphi \otimes \varphi)(x,p) \right)(0) \\
&= e^{-\frac{1}{2\hbar}\alpha^{ij}\xi_i\xi_j}\Big|_{\xi=0} \mathcal{F}_x \left(e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \mathcal{F}_y T_\sigma(\varphi \otimes \varphi)(x,p) \right)(0) \\
&= e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \frac{1}{(2\pi\hbar)^{N}} \iint \mathrm{d}x\, \mathrm{d}p\, e^{-\frac{i}{\hbar}p_i y^i} \varphi^*(x-\bar{\sigma}y)\varphi(x+\sigma y).
\end{aligned}$$

Introducing new coordinates $x_1 = x - \bar{\sigma}y, x_2 = x + \sigma y$ gives

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p)\mathrm{d}x &= e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \frac{1}{(2\pi\hbar)^{N}} \iint \mathrm{d}x_1\, \mathrm{d}x_2\, e^{-\frac{i}{\hbar}p_i x_2^i} e^{\frac{i}{\hbar}p_i x_1^i} \varphi^*(x_1)\varphi(x_2) \\
&= e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} |\tilde{\varphi}(p)|^2.
\end{aligned}$$

□

From Theorem 4.2 follows that there is a one to one correspondence between pure states of the phase space quantum mechanics and the normalized functions from the Hilbert space $L^2(\mathbb{R}^N)$.

Elements of the algebra $\hat{\mathcal{A}}_Q$, hence in particular observables, are operators on $\mathcal{H}$ of the form $A \star_{\sigma,\alpha,\beta}$. In the canonical case discussed so far, from Theorem 3.10, these operators are equal to operator functions $A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})$. Moreover, states $\Psi \in \mathcal{H}$ can also be considered as operators on $\mathcal{H}$ given by the formula

$$\hat{\Psi} = (2\pi\hbar)^{N/2}\Psi \star_{\sigma,\alpha,\beta} . \tag{4.8}$$

The space of all operators $\hat{\Psi}$ given by (4.8) will be denoted by $\hat{\mathcal{H}}$. Note that $\hat{\mathcal{H}}$ inherits from $\mathcal{H}$ a structure of a Hilbert algebra with the scalar product of $\hat{\Psi}_1 = (2\pi\hbar)^{N/2}\Psi_1\star_{\sigma,\alpha,\beta}$ and $\hat{\Psi}_2 = (2\pi\hbar)^{N/2}\Psi_2 \star_{\sigma,\alpha,\beta}$ defined by

$$\langle\hat{\Psi}_1|\hat{\Psi}_2\rangle_{\hat{\mathcal{H}}} := \langle\Psi_1|\Psi_2\rangle_{\mathcal{H}}.$$

Note also, that from (3.20) $\|\cdot\|_{\hat{\mathcal{H}}}$ satisfies the following relation

$$\|\hat{\Psi}_1\hat{\Psi}_2\|_{\hat{\mathcal{H}}} \leq \|\hat{\Psi}_1\|_{\hat{\mathcal{H}}}\|\hat{\Psi}_2\|_{\hat{\mathcal{H}}}.$$

Now, it will be proved that operators from $\hat{\mathcal{H}}$ can be naturally identified with Hilbert-Schmidt operators defined on the Hilbert space $L^2(\mathbb{R}^N)$. The space of Hilbert-Schmidt operators $\mathcal{S}^2(L^2(\mathbb{R}^N))$ is a space of all bounded operators $\hat{A} \in \mathcal{B}(L^2(\mathbb{R}^N))$ for which $\|\hat{A}\|_{\mathcal{S}^2} < \infty$, where $\|\cdot\|_{\mathcal{S}^2}$ is a norm induced by a scalar product

$$\langle\hat{A}|\hat{B}\rangle_{\mathcal{S}^2} := \operatorname{tr}(\hat{A}^\dagger\hat{B}), \quad \hat{A},\hat{B} \in \mathcal{S}^2(L^2(\mathbb{R}^N)). \tag{4.9}$$

The space of Hilbert-Schmidt operators $\mathcal{S}^2(L^2(\mathbb{R}^N))$ with the scalar product (4.9) is a Hilbert algebra. From the well known relation between the $\mathcal{S}^2$-norm and the usual operator norm

$$\|\hat{A}\| \leq \|\hat{A}\|_{\mathcal{S}^2}, \quad \hat{A} \in \mathcal{S}^2(L^2(\mathbb{R}^N))$$

it follows that the inclusion $\mathcal{S}^2(L^2(\mathbb{R}^N)) \subset \mathcal{B}(L^2(\mathbb{R}^N))$ is continuous and hence, every sequence of Hilbert-Schmidt operators convergent in $\mathcal{S}^2(L^2(\mathbb{R}^N))$ is also convergent in $\mathcal{B}(L^2(\mathbb{R}^N))$.

There holds

Theorem 4.7. *For every $\hat{\Psi} \in \hat{\mathcal{H}}$*

$$\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}, \tag{4.10}$$

where $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}^N))$ is some Hilbert-Schmidt operator defined on the Hilbert space $L^2(\mathbb{R}^N)$. Conversely, for every $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}^N))$ the operator $\hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}$ is an element of $\hat{\mathcal{H}}$.

In particular, for $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ *the corresponding operator* $\hat{\Psi}$ *takes the form*

$$\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}, \tag{4.11}$$

where $\hat{\rho} = \langle\varphi|\cdot\rangle_{L^2}\psi$.

Moreover, for $\hat{\Psi}_1 = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_1$ *and* $\hat{\Psi}_2 = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_2$

$$\langle\hat{\Psi}_1|\hat{\Psi}_2\rangle_{\hat{\mathcal{H}}} = \langle\hat{\rho}_1|\hat{\rho}_2\rangle_{\mathcal{S}^2}. \tag{4.12}$$

Proof. From equation (4.5a) for $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ and the basis functions $\Psi_{ij} = \varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j$ it follows that

$$\begin{aligned}\hat{\Psi}\Psi_{ij} &= (2\pi\hbar)^{N/2}(\varphi \otimes_{\sigma,\alpha,\beta} \psi) \star_{\sigma,\alpha,\beta} (\varphi_i \otimes_{\sigma,\alpha,\beta} \varphi_j) = \langle\varphi|\varphi_j\rangle_{L^2}(\varphi_i \otimes_{\sigma,\alpha,\beta} \psi)\\ &= \varphi_i \otimes_{\sigma,\alpha,\beta} (\hat{\rho}\varphi_j) = (\hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho})\Psi_{ij},\end{aligned}$$

where $\hat{\rho} = \langle\varphi|\cdot\rangle_{L^2}\psi$, which proves formula (4.11).

It is sufficient to prove formula (4.12) for basis functions Ψ_{ij}. From (4.11) it follows that operators corresponding to the basis functions Ψ_{ij} can be written in a form

$$\hat{\Psi}_{ij} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_{ij},$$

where $\hat{\rho}_{ij} = \langle\varphi_i|\cdot\rangle_{L^2}\varphi_j$. This implies that

$$\langle\hat{\Psi}_{ij}|\hat{\Psi}_{kl}\rangle_{\hat{\mathcal{H}}} = \delta_{ik}\delta_{jl} = \langle\hat{\rho}_{ij}|\hat{\rho}_{kl}\rangle_{\mathcal{S}^2},$$

which proves formula (4.12). Formula (4.10) is an immediate consequence of formulae (4.11) and (4.12). □

From the above theorem follows that states can be naturally identified with appropriate operators on the Hilbert space $L^2(\mathbb{R}^N)$. For instance, if $\Psi_{\text{pure}} = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$ is a pure state then $\hat{\Psi}_{\text{pure}} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_{\text{pure}}$ where $\hat{\rho}_{\text{pure}} = \langle\varphi|\cdot\rangle_{L^2}\varphi$. Moreover, from the defining relations of pure states follows the following characterisation of the pure state operators $\hat{\rho}_{\text{pure}}$

1. $\hat{\rho}_{\text{pure}} = \hat{\rho}_{\text{pure}}^\dagger$ (hermiticity),
2. $\hat{\rho}_{\text{pure}}^2 = \hat{\rho}_{\text{pure}}$ (idempotence),
3. $\|\hat{\rho}_{\text{pure}}\|_{\mathcal{S}^2}^2 = \operatorname{tr}\hat{\rho}_{\text{pure}} = 1$ (normalization).

If $\Psi_{\text{mix}} = \sum_\lambda p_\lambda \Psi_{\text{pure}}^{(\lambda)} = \sum_\lambda p_\lambda \varphi^{(\lambda)} \otimes_{\sigma,\alpha,\beta} \varphi^{(\lambda)}$ is a mixed state then $\hat{\Psi}_{\text{mix}} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}_{\text{mix}}$ where

$$\hat{\rho}_{\text{mix}} = \sum_\lambda p_\lambda \hat{\rho}_{\text{pure}}^{(\lambda)} = \sum_\lambda p_\lambda \langle \varphi^{(\lambda)} | \cdot \rangle_{L^2} \varphi^{(\lambda)}.$$

Pure and mixed state operators $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}^N))$ are called *density operators*.

From the below theorem follows that observables can be naturally identified with operators defined on the Hilbert space $L^2(\mathbb{R}^N)$. This theorem is also the key theorem from which formulae for the expectation values of observables and the time evolution of the observables and states, represented as operators in $L^2(\mathbb{R}^N)$, follows.

Theorem 4.8. *Let $A \in \mathcal{A}_Q$ and $\Psi \in \mathcal{H}$ be such that $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ for $\varphi, \psi \in L^2(\mathbb{R}^N)$, then*

$$A_L \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta})\Psi = \varphi \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\psi, \qquad \text{if } \psi \in D(A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})),$$
$$A_R \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta})\Psi = A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\varphi \otimes_{\sigma,\alpha,\beta} \psi, \qquad \text{if } \varphi \in D(A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})),$$

where $A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ is a (σ, α, β)-ordered operator function of canonical operators of position $\hat{q} = x$ and momentum $\hat{p} = -i\hbar\partial_x$, acting in the Hilbert space $L^2(\mathbb{R}^N)$, and $D(\hat{A})$ denotes a domain of an operator $\hat{A}$.

Since the proof of the above theorem is quite long and tedious it was moved to Appendix A.8.

Corollary 4.1. *Every solution of the $\star_{\sigma,\alpha,\beta}$-genvalue equation*

$$A \star_{\sigma,\alpha,\beta} \Psi = a\Psi \tag{4.13}$$

for $A \in \mathcal{A}_Q$ and $a \in \mathbb{C}$ is of the form

$$\Psi = \sum_i \varphi_i \otimes_{\sigma,\alpha,\beta} \psi_i, \tag{4.14}$$

where $\varphi_i \in L^2(\mathbb{R}^N)$ are arbitrary and $\psi_i \in L^2(\mathbb{R}^N)$ are the eigenvectors of the operator $A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ corresponding to the eigenvalue a spanning the subspace of all eigenvectors of $A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})$ corresponding to the eigenvalue a, i.e. ψ_i satisfy the eigenvalue equation

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p})\psi_i = a\psi_i.$$

In particular, when a is nondegenerate, every solution of (4.13) is of the form

$$\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi,$$

where $\varphi \in L^2(\mathbb{R}^N)$ is arbitrary and $\psi \in L^2(\mathbb{R}^N)$ satisfy the eigenvalue equation

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\psi = a\psi.$$

Similarly, every solution of the $\star_{\sigma,\alpha,\beta}$-genvalue equation

$$\Psi \star_{\sigma,\alpha,\beta} B = b\Psi \tag{4.15}$$

for $B \in \mathcal{A}_Q$ and $b \in \mathbb{C}$ is of the form

$$\Psi = \sum_i \psi_i \otimes_{\sigma,\alpha,\beta} \varphi_i,$$

where $\varphi_i \in L^2(\mathbb{R}^N)$ are arbitrary and $\psi_i \in L^2(\mathbb{R}^N)$ are the eigenvectors of the operator $B^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ corresponding to the eigenvalue b^* spanning the subspace of all eigenvectors of $B^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ corresponding to the eigenvalue b^*, i.e. ψ_i satisfy the eigenvalue equation

$$B^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\psi_i = b^*\psi_i.$$

In particular, when b^* is nondegenerate, every solution of (4.15) is of the form

$$\Psi = \psi \otimes_{\sigma,\alpha,\beta} \varphi,$$

where $\varphi \in L^2(\mathbb{R}^N)$ is arbitrary and $\psi \in L^2(\mathbb{R}^N)$ satisfy the eigenvalue equation

$$B^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\psi = b^*\psi.$$

Proof. Replacing Φ_L, in the proof of Theorem 4.8, by $a\Psi$ one gets from equation (A.14a) that the $\star_{\sigma,\alpha,\beta}$-genvalue equation (4.13) is equivalent to the following equation

$$A_{\sigma,\alpha,\beta}(\xi, -i\hbar\partial_\xi)\tilde{\Psi}_1(\xi,z) = a\tilde{\Psi}_1(\xi,z),$$

i.e. $\tilde{\Psi}_1(\xi,z)$ is an eigenvector of the operator $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ for every z. If $\{\psi_i \in L^2(\mathbb{R}^N)\}$ is the basis in the subspace of all eigenvectors of the operator $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ corresponding to the eigenvalue a then $\tilde{\Psi}_1(\xi,z)$, for every z, can be written as a linear combination of the basis vectors ψ_i

$$\tilde{\Psi}_1(\xi,z) = \sum_i \kappa_i(z)\psi_i(\xi), \tag{4.16}$$

where the coefficients $\kappa_i(z) \in \mathbb{C}$ depend on z. Since $\Psi \in \mathcal{H}$ the functions $\kappa_i \in L^2(\mathbb{R}^N)$. Now, from equation (4.16), using the analogous arguments as in the proof of Theorem 4.8 it can be proved that Ψ is of the form (4.14) where $\varphi_i^*(x - \bar{\sigma}y) = \kappa_i(y - \bar{\sigma}^{-1}x)$. The second part of the corollary can be proved analogically. □

From Theorem 4.8 it follows that for $\Psi_1 = \varphi_1 \otimes_{\sigma,\alpha,\beta} \psi_1$ and $\Psi_2 = \varphi_2 \otimes_{\sigma,\alpha,\beta} \psi_2$ where $\varphi_1, \psi_1, \varphi_2, \psi_2 \in L^2(\mathbb{R}^N)$

$$\langle \Psi_1 | A_L \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}} = \langle \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \psi_2 \rangle_{L^2}, \tag{4.17a}$$

$$\begin{aligned} \langle \Psi_1 | A_R \star_{\sigma,\alpha,\beta} \Psi_2 \rangle_{\mathcal{H}} &= \langle A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \varphi_2 | \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2}, \\ &= \langle \varphi_2 | A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \varphi_1 \rangle_{L^2} \langle \psi_1 | \psi_2 \rangle_{L^2}. \end{aligned} \tag{4.17b}$$

From equations (3.23) it follows that

$$(A_L \star_{\sigma,\alpha,\beta})^\dagger \Psi = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}) \Psi = \varphi \otimes_{\sigma,\alpha,\beta} A^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \psi, \tag{4.18a}$$

$$(A_R \star_{\sigma,\alpha,\beta})^\dagger \Psi = A^*_{\bar{\sigma},\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta}) \Psi = A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \varphi \otimes_{\sigma,\alpha,\beta} \psi, \tag{4.18b}$$

for $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$ where $\varphi, \psi \in L^2(\mathbb{R}^N)$. Note, that Corollary 4.1 implies that in the nondegenerate case the solution Ψ to the following pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$A \star_{\sigma,\alpha,\beta} \Psi = a\Psi, \qquad \Psi \star_{\sigma,\alpha,\beta} B = b\Psi, \tag{4.19}$$

is unique up to a multiplication constant and is of the form $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \psi$, where $\varphi, \psi \in L^2(\mathbb{R}^N)$ satisfy the following eigenvalue equations

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \psi = a\psi, \qquad B^\dagger_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \varphi = b^* \varphi. \tag{4.20}$$

Hence, the pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations (4.19) is equivalent to the pair of eigenvalue equations (4.20). In particular, from formula (4.18b) it follows that a pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$A_L \star_{\sigma,\alpha,\beta} \Psi = a\Psi, \qquad (A_R \star_{\sigma,\alpha,\beta})^\dagger \Psi = a^* \Psi$$

have a solution Ψ in the form of a pure state $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$, where $\varphi \in L^2(\mathbb{R}^N)$ is a solution to the eigenvalue equation

$$A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}) \varphi = a\varphi.$$

From Theorem 4.8 follows also that operators $\hat{A} = A \star_{\sigma,\alpha,\beta}$, hence in particular observables, can be written as

$$A \star_{\sigma,\alpha,\beta} = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta}) = \hat{1} \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q}, \hat{p}).$$

Hence, operators from $\hat{\mathcal{A}}_Q$ can be naturally identified with operator functions $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ defined on the Hilbert space $L^2(\mathbb{R}^N)$. Moreover, from Theorems 4.7 and 4.8 it follows that the action of observables, treated as operator functions $A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})$, on states, treated as operators $\hat{\Psi} \in \hat{\mathcal{H}}$, is equivalent to the action of observables, treated as operator functions $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$, on states, treated as operators $\hat{\rho} \in \mathcal{S}^2(L^2(\mathbb{R}^N))$. In fact, there holds

$$
\begin{aligned}
A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})\hat{\Psi} &= \hat{1} \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\hat{\rho}, \\
\hat{\Psi} A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta}) &= \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}),
\end{aligned}
$$

where $\hat{\Psi} = \hat{1} \otimes_{\sigma,\alpha,\beta} \hat{\rho}$.

Using Theorem 4.8 a formula for the expectation value of observables represented as operators on the Hilbert space $L^2(\mathbb{R}^N)$ can be derived. Namely, there holds

Theorem 4.9. *Let $\hat{A} \in \hat{\mathcal{A}}_Q$ be some observable and $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ be a corresponding operator in the Hilbert space $L^2(\mathbb{R}^N)$. Moreover, let $\Psi = \sum_\lambda p_\lambda \Psi^{(\lambda)}_{\text{pure}} = \sum_\lambda p_\lambda (\varphi^{(\lambda)} \otimes_{\sigma,\alpha,\beta} \varphi^{(\lambda)}) \in \mathcal{H}$ be some mixed state and $\hat{\rho} = \sum_\lambda p_\lambda \langle \varphi^{(\lambda)} | \cdot \rangle_{L^2} \varphi^{(\lambda)}$ the corresponding density operator. Then there holds*

$$
\langle \hat{A} \rangle_\Psi = \sum_\lambda p_\lambda \langle \varphi^{(\lambda)} | A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) \varphi^{(\lambda)} \rangle_{L^2} = \operatorname{tr}(\hat{\rho} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})). \tag{4.21}
$$

Proof. From Theorem 4.8 it follows

$$
\begin{aligned}
\langle \hat{A} \rangle_\Psi &= \frac{1}{(2\pi\hbar)^{N/2}} \iint \mathrm{d}x\,\mathrm{d}p\,(A \star_{\sigma,\alpha,\beta} \Psi)(x,p) \\
&= \frac{1}{(2\pi\hbar)^{N/2}} \sum_\lambda p_\lambda \iint \mathrm{d}x\,\mathrm{d}p\,(A \star_{\sigma,\alpha,\beta} \Psi^{(\lambda)}_{\text{pure}})(x,p) \\
&= \frac{1}{(2\pi\hbar)^{N/2}} \sum_\lambda p_\lambda \iint \mathrm{d}x\,\mathrm{d}p\,(\varphi^{(\lambda)} \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)})(x,p) \\
&= \frac{1}{(2\pi\hbar)^{N}} \sum_\lambda p_\lambda \iint \mathrm{d}x\,\mathrm{d}p\, e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} \varphi^{(\lambda)*}(x - \bar{\sigma}y) \\
&\quad \cdot A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}(x + \sigma y) \\
&= \frac{1}{(2\pi\hbar)^{N}} \sum_\lambda p_\lambda \iint \mathrm{d}x\,\mathrm{d}p\, e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}p_i y^i} e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j} \varphi^{(\lambda)*}(x - \bar{\sigma}y) \\
&\quad \cdot A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}(x + \sigma y)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_\lambda p_\lambda \iint \mathrm{d}x\,\mathrm{d}y\, e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\delta(y)e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j}\varphi^{(\lambda)*}(x-\bar{\sigma}y)A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}(x+\sigma y) \\
&= \sum_\lambda p_\lambda \int \mathrm{d}x\, e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\varphi^{(\lambda)*}(x)A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}(x) \\
&= \sum_\lambda p_\lambda \int \mathrm{d}x\, \varphi^{(\lambda)*}(x)A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}(x) \\
&= \sum_\lambda p_\lambda \langle \varphi^{(\lambda)}, A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}\rangle_{L^2} = \operatorname{tr}(\hat{\rho}A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})).
\end{aligned}
$$

□

Corollary 4.2.

$$
\langle \hat{A}\rangle_\Psi = \sum_\lambda p_\lambda \langle \Psi^{(\lambda)}_{\text{pure}}|A_L \star_{\sigma,\alpha,\beta} \Psi^{(\lambda)}_{\text{pure}}\rangle_{\mathcal{H}} = \sum_\lambda p_\lambda \langle \Psi^{(\lambda)}_{\text{pure}}|A_R \star_{\sigma,\alpha,\beta} \Psi^{(\lambda)}_{\text{pure}}\rangle_{\mathcal{H}} \tag{4.22}
$$

Proof. Equation (4.22) follows immediately from (4.21) and (4.17) as from one side

$$
\begin{aligned}
\langle \hat{A}\rangle_\Psi &= \sum_\lambda p_\lambda \langle \varphi^{(\lambda)}|A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}\rangle_{L^2} = \sum_\lambda p_\lambda \langle \varphi^{(\lambda)}|\varphi^{(\lambda)}\rangle_{L^2}\langle \varphi^{(\lambda)}|A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}\rangle_{L^2} \\
&= \sum_\lambda p_\lambda \langle \Psi^{(\lambda)}_{\text{pure}}|A_L \star_{\sigma,\alpha,\beta} \Psi^{(\lambda)}_{\text{pure}}\rangle_{\mathcal{H}}
\end{aligned}
$$

and from the other side

$$
\begin{aligned}
\langle \hat{A}\rangle_\Psi &= \sum_\lambda p_\lambda \langle \varphi^{(\lambda)}|A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}\rangle_{L^2} = \sum_\lambda p_\lambda \langle A^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi^{(\lambda)}|\varphi^{(\lambda)}\rangle_{L^2}\langle \varphi^{(\lambda)}|\varphi^{(\lambda)}\rangle_{L^2} \\
&= \sum_\lambda p_\lambda \langle \Psi^{(\lambda)}_{\text{pure}}|A_R \star_{\sigma,\alpha,\beta} \Psi^{(\lambda)}_{\text{pure}}\rangle_{\mathcal{H}}.
\end{aligned}
$$

□

Using the results of this section it is possible to prove a useful property of pure states used in Section 3.5. Namely

Theorem 4.10. *Let $A \in \mathcal{A}_Q$. A pure state function $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi \in \mathcal{H}$ satisfies the equation*

$$
[A, \Psi] = 0 \tag{4.23}
$$

if and only if it satisfies the following pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$
A \star_{\sigma,\alpha,\beta} \Psi = a\Psi, \quad \Psi \star_{\sigma,\alpha,\beta} A = a\Psi, \tag{4.24}
$$

for some $a \in \mathbb{C}$.

Proof. It is obvious that if Ψ satisfies (4.24) then it also satisfies (4.23). Lets assume that Ψ satisfies (4.23). Hence, it also satisfies

$$A \star_{\sigma,\alpha,\beta} \Psi \star_{\sigma,\alpha,\beta} \Psi = \Psi \star_{\sigma,\alpha,\beta} A \star_{\sigma,\alpha,\beta} \Psi.$$

From the idempotent property of pure states the above equation implies

$$\frac{1}{(2\pi\hbar)^{N/2}} A \star_{\sigma,\alpha,\beta} \Psi = \Psi \star_{\sigma,\alpha,\beta} A \star_{\sigma,\alpha,\beta} \Psi. \tag{4.25}$$

From Theorem 4.8 it follows that

$$A \star_{\sigma,\alpha,\beta} \Psi = \varphi \otimes_{\sigma,\alpha,\beta} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi. \tag{4.26}$$

Now, equations (4.25) and (4.26) with the help of (4.5) give

$$A \star_{\sigma,\alpha,\beta} \Psi = (2\pi\hbar)^{N/2} \Psi \star_{\sigma,\alpha,\beta} (A \star_{\sigma,\alpha,\beta} \Psi) = \langle \varphi | A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi \rangle_{L^2} \Psi = a\Psi,$$

where $a = \langle \varphi | A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi \rangle_{L^2}$. The second $\star_{\sigma,\alpha,\beta}$-genvalue equation can be derived analogically. □

Finally, lets derive the time evolution of the observables and states represented as operators on the Hilbert space $L^2(\mathbb{R}^N)$. In this case the time evolution is governed by a Hermitian operator $H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ corresponding to the Hamiltonian $\hat{H}$. From the time evolution equation (3.24) one receives the following evolution equation for density operators $\hat{\rho}$, called the *von Neumann equation*

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} - [H_{\sigma,\alpha,\beta}(\hat{q},\hat{p}), \hat{\rho}] = 0. \tag{4.27}$$

For a pure state density operator $\hat{\rho} = \langle \varphi | \cdot \rangle_{L^2} \varphi$ equation (4.27) takes the form

$$i\hbar \left\langle \frac{\partial \varphi}{\partial t} \middle| \cdot \right\rangle_{L^2} \varphi + i\hbar \langle \varphi | \cdot \rangle_{L^2} \frac{\partial \varphi}{\partial t} - \langle \varphi | \cdot \rangle_{L^2} H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi + \langle H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi | \cdot \rangle_{L^2} \varphi = 0$$

$$\Updownarrow$$

$$\left\langle -i\hbar \frac{\partial \varphi}{\partial t} + H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi \middle| \cdot \right\rangle_{L^2} \varphi + \langle \varphi | \cdot \rangle_{L^2} \left(i\hbar \frac{\partial \varphi}{\partial t} - H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi \right) = 0$$

$$\Updownarrow$$

$$i\hbar \frac{\partial \varphi}{\partial t} = H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi. \tag{4.28}$$

The above equation is called the *time dependent Schrödinger equation.*

The equation for stationary states takes now the form

$$[H_{\sigma,\alpha,\beta}(\hat{q},\hat{p}),\hat{\rho}] = 0,$$

which for pure states $\hat{\rho} = \langle\varphi|\cdot\rangle_{L^2}\varphi$ is equivalent to such eigenvalue equation

$$H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi = E\varphi$$

called the *stationary Schrödinger equation*.

The representation in the Hilbert space $L^2(\mathbb{R}^N)$ of the one parameter group of unitary functions $U(t)$ from equation (3.25) is a one parameter group of unitary operators

$$U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t) = e^{-\frac{i}{\hbar}tH_{\sigma,\alpha,\beta}(\hat{q},\hat{p})}.$$

The time evolution of a density operator $\hat{\rho}$ can be alternatively expressed by the equation

$$\hat{\rho}(t) = U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t)\hat{\rho}(0)U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},-t).$$

It is then easy to check that the above equation is indeed a solution to the von Neumann equation (4.27). Using the unitary operators $U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t)$ also the time evolution of observables $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ can be expressed, similarly as in equation (3.27)

$$A_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t) = U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},-t)A_{\sigma,\alpha,\beta}(\hat{q},\hat{p},0)U_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t).$$

The corresponding time evolution equation for observables $A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})$ from equation (3.28) takes the form

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}A_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t) - [A_{\sigma,\alpha,\beta}(\hat{q},\hat{p},t),H_{\sigma,\alpha,\beta}(\hat{q},\hat{p})] = 0.$$

The above equation is called the *Heisenberg equation*.

5

Examples

In this section some examples of the presented formalism of the phase space quantum mechanics will be given. First a free particle will be considered and then a simple harmonic oscillator.

5.1 Free particle

In this section a free particle will be considered. For simplicity a one dimensional particle (the case of $N = 1$) will be considered. The free particle is a system, which time evolution is governed by a Hamiltonian $\hat{H}$ induced by the function

$$H(x, p) = \frac{1}{2}p^2, \tag{5.1}$$

where the mass of the particle $m = 1$. This Hamiltonian describes only the kinetic energy of the particle. It does not contain any terms describing the potential energy, i.e. there are no forces acting on the particle (the particle is free).

First, lets find a time evolution of a free particle being initially in an arbitrary pure state. To do this it is necessary to solve the time evolution equation (3.24)

$$i\hbar\frac{\partial \Psi}{\partial t} - [H, \Psi] = 0, \tag{5.2}$$

with H given by (5.1) and with the assumption that the solution Ψ is in a form of a pure state, i.e. $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$ for some function $\varphi \in L^2(\mathbb{R})$. From Section 4 it is known that the function $\Psi = \varphi \otimes_{\sigma,\alpha,\beta} \varphi$ is a solution to (5.2) if and only if function φ is a solution to the Schrödinger equation (4.28), which for H given by (5.1) takes the form

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2}\frac{\partial^2\varphi}{\partial x^2}. \tag{5.3}$$

The simplest solution to equation (5.3) is of the form of a plain wave

$$\varphi_p(x,t) = e^{\frac{i}{\hbar}(px - E(p)t)},$$

where $p \in \mathbb{R}$ and $E(p) = \frac{1}{2}p^2$. The general solution to (5.3) is in the form of a linear combination of the plain wave solutions φ_p, i.e. in the form of a wave packet

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int f(p)e^{\frac{i}{\hbar}(px-E(p)t)}\mathrm{d}p = \frac{1}{\sqrt{2\pi\hbar}}\int g(p,t)e^{\frac{i}{\hbar}px}\mathrm{d}p\,, \tag{5.4}$$

where $f \in L^2(\mathbb{R})$ and $g(p,t) = f(p)e^{-\frac{i}{\hbar}E(p)t}$. From (5.4) the solution Ψ of (5.2) reads

$$\begin{aligned}
\Psi(x,p,t) = (\varphi \otimes_{\sigma,\alpha,\beta} \varphi)(x,p,t) &= \frac{1}{\sqrt{2\pi\hbar}}S_{\alpha,\beta}\int \mathrm{d}y\, e^{-\frac{i}{\hbar}py}\varphi^*(x-\bar{\sigma}y,t)\varphi(x+\sigma y,t) \\
&= \frac{1}{(2\pi\hbar)^{3/2}}S_{\alpha,\beta}\int \mathrm{d}y\, e^{-\frac{i}{\hbar}py}\int \mathrm{d}p'\, g^*(p',t)e^{-\frac{i}{\hbar}p'(x-\bar{\sigma}y)}\int \mathrm{d}p''\, g(p'',t)e^{\frac{i}{\hbar}p''(x+\sigma y)} \\
&= \frac{1}{(2\pi\hbar)^{3/2}}S_{\alpha,\beta}\int \mathrm{d}p'\int \mathrm{d}p''\, g^*(p',t)g(p'',t)e^{\frac{i}{\hbar}(p''-p')x}\int \mathrm{d}y\, e^{-\frac{i}{\hbar}py}e^{\frac{i}{\hbar}(\sigma p''+\bar{\sigma}p')y} \\
&= \frac{1}{(2\pi\hbar)^{3/2}}S_{\alpha,\beta}\int \mathrm{d}p_1\int \mathrm{d}p_2\, g^*(p_2-\sigma p_1,t)g(p_2+\bar{\sigma}p_1,t)e^{\frac{i}{\hbar}p_1x}\int \mathrm{d}y\, e^{-\frac{i}{\hbar}(p-p_2)y} \\
&= \frac{1}{\sqrt{2\pi\hbar}}S_{\alpha,\beta}\int \mathrm{d}p_1\int \mathrm{d}p_2\, g^*(p_2-\sigma p_1,t)g(p_2+\bar{\sigma}p_1,t)e^{\frac{i}{\hbar}p_1x}\delta(p-p_2) \\
&= \frac{1}{\sqrt{2\pi\hbar}}S_{\alpha,\beta}\int \mathrm{d}p_1\, g^*(p-\sigma p_1,t)g(p+\bar{\sigma}p_1,t)e^{\frac{i}{\hbar}p_1x},
\end{aligned} \tag{5.5}$$

where new coordinates $p_1 = p'' - p', p_2 = \sigma p'' + \bar{\sigma}p'$ were used.

Lets consider some particular cases of the solution (5.5). Assume that

$$f(p) = \frac{1}{(2\pi)^{1/4}(\Delta p)^{1/2}}e^{-\frac{(p-p_0)^2}{4(\Delta p)^2}}$$

is a Gaussian function. By (5.4) the solution φ of the Schrödinger equation (5.3) takes the form

$$\varphi(x,t) = \frac{1}{(2\pi)^{1/4}\sqrt{\Delta x + i\Delta pt}}\exp\left(-\frac{p_0^2}{4(\Delta p)^2}\right)\exp\left(-\frac{(x-i\frac{\Delta x}{\Delta p}p_0)^2}{4(\Delta x)^2+4i\Delta x\Delta pt}\right),$$

where $\Delta x = \frac{\hbar}{2\Delta p}$. This solution describes the time evolution of a wave packet initially in the form of a Gaussian-like function

$$\varphi(x,0) = \frac{1}{(2\pi)^{1/4}(\Delta x)^{1/2}}e^{-\frac{x^2}{4(\Delta x)^2}}e^{\frac{i}{\hbar}p_0x}.$$

It is now possible to calculate the solution Ψ of the time evolution equation (5.2). For simplicity only the case $\alpha = \beta = 0$ will be considered. The function Ψ takes the form

$$\Psi(x,p,t) = \frac{1}{\sqrt{2\pi\left((\bar{\sigma}^2+\sigma^2)\Delta x\Delta p + i(1-2\sigma)(\Delta p)^2 t\right)}} \exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right) \cdot \exp\left(-\frac{(x - pt + i(1-2\sigma)\frac{\Delta x}{\Delta p}(p-p_0))^2}{4(\bar{\sigma}^2+\sigma^2)(\Delta x)^2 + 4i(1-2\sigma)\Delta x\Delta p t}\right). \tag{5.6}$$

This solution describes the time evolution of a free particle initially in the state

$$\Psi(x,p,0) = \frac{1}{\sqrt{2\pi(\bar{\sigma}^2+\sigma^2)\Delta x\Delta p}} \exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right) \cdot \exp\left(-\frac{(x + i(1-2\sigma)\frac{\Delta x}{\Delta p}(p-p_0))^2}{4(\bar{\sigma}^2+\sigma^2)(\Delta x)^2}\right).$$

The state Ψ from (5.6) greatly simplifies in the case $\sigma = \frac{1}{2}$. For this special case the state Ψ reads

$$\Psi(x,p,t) = \frac{1}{\sqrt{\pi\Delta x\Delta p}} \exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right) \exp\left(-\frac{(x-pt)^2}{2(\Delta x)^2}\right).$$

Lets calculate the expectation values and uncertainties of the position x and momentum p of a free particle described by the state (5.6). One easily calculates that

$$\begin{aligned}
\langle x\rangle_{\Psi(t)} &= \iint x \star_\sigma \Psi(t)\mathrm{d}x\,\mathrm{d}p \ = p_0 t, \\
\langle x^2\rangle_{\Psi(t)} &= \iint x^2 \star_\sigma \Psi(t)\mathrm{d}x\,\mathrm{d}p \ = (\Delta x)^2 + (\Delta p)^2 t^2 + p_0^2 t^2, \\
\Delta x(t) &= \sqrt{\langle x^2\rangle_{\Psi(t)} - \langle x\rangle_{\Psi(t)}^2} = \sqrt{(\Delta x)^2 + (\Delta p)^2 t^2}, \\
\langle p\rangle_{\Psi(t)} &= \iint p \star_\sigma \Psi(t)\mathrm{d}x\,\mathrm{d}p \ = p_0, \\
\langle p^2\rangle_{\Psi(t)} &= \iint p^2 \star_\sigma \Psi(t)\mathrm{d}x\,\mathrm{d}p \ = (\Delta p)^2 + p_0^2, \\
\Delta p(t) &= \sqrt{\langle p^2\rangle_{\Psi(t)} - \langle p\rangle_{\Psi(t)}^2} = \Delta p.
\end{aligned}$$

Note, that during the time evolution the uncertainty of the momentum $\Delta p(t)$ of the free particle described by the state (5.6) do not change in time and is equal to its initial value Δp, whereas the uncertainty of the position $\Delta x(t)$ initially equal Δx increases in time. Note also, that the uncertainties of the position and momentum satisfy the Heisenberg uncertainty principle, i.e. $\Delta x(t)\Delta p(t) \geq \frac{\hbar}{2}$. Moreover, initially the free particle is in a state which minimizes the Heisenberg uncertainty principle since $\Delta x(0)\Delta p(0) = \Delta x \Delta p = \frac{\hbar}{2}$. Worth noting is also the fact that the expectation value of the momentum $\langle p\rangle_{\Psi(t)}$ is constant and equal p_0, whereas the expectation value of the position $\langle x\rangle_{\Psi(t)}$ is equal $p_0 t$. Hence, the time evolution of the free particle described by the state (5.6) can be interpreted as the movement of the particle along a straight line with the constant momentum equal p_0, similarly as in the classical case. The difference between the classical and quantum case is that in the quantum case there is some uncertainty of the position and momentum, in contrast to the classical case where the position and momentum is known precisely.

It is interesting to calculate to which classical state converges the state (5.6) in the limit $\hbar \to 0^+$. Assume that $\Delta x \propto \sqrt{\hbar}$ and $\Delta p \propto \sqrt{\hbar}$, then $\frac{\Delta p}{\Delta x} = c = \text{const}$. To calculate to which classical state converges the state Ψ from (5.6) in the limit $\hbar \to 0^+$ it is necessary to calculate the limit $\lim_{\hbar\to 0^+} \rho$, where $\rho = \frac{1}{\sqrt{2\pi\hbar}}\Psi$ is the quantum distribution function induced by Ψ. One easily calculates that

$$\begin{aligned}\rho(x,p,t) &= \frac{1}{\sqrt{2\pi\hbar}}\Psi(x,p,t)\\ &= \frac{1}{2\pi\Delta x\Delta p\sqrt{2(\bar{\sigma}^2+\sigma^2)+2i(1-2\sigma)\frac{\Delta p}{\Delta x}t}}\exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right)\\ &\quad\cdot\exp\left(-\frac{(x-pt+i(1-2\sigma)\frac{\Delta x}{\Delta p}(p-p_0))^2}{4(\bar{\sigma}^2+\sigma^2)(\Delta x)^2+4i(1-2\sigma)\Delta x\Delta p t}\right).\end{aligned}$$

The limit $\lim_{\hbar\to 0^+} \rho$ has to be calculated in the distributional sense, i.e. one have to calculate the limit $\lim_{\hbar\to 0^+}\langle \rho, \phi\rangle$ for every test function ϕ. One easily calculates that

$$\begin{aligned}\lim_{\hbar\to 0^+}\langle\rho,\phi\rangle &= \lim_{\hbar\to 0^+}\iint \frac{1}{2\pi\Delta x\Delta p\sqrt{2(\bar{\sigma}^2+\sigma^2)+2i(1-2\sigma)\frac{\Delta p}{\Delta x}t}}\exp\left(-\frac{(p-p_0)^2}{2(\Delta p)^2}\right)\\ &\quad\cdot\exp\left(-\frac{(x-pt+i(1-2\sigma)\frac{\Delta x}{\Delta p}(p-p_0))^2}{4(\bar{\sigma}^2+\sigma^2)(\Delta x)^2+4i(1-2\sigma)\Delta x\Delta p t}\right)\phi(x,p)\mathrm{d}x\,\mathrm{d}p\\ &= \lim_{\hbar\to 0^+}\frac{1}{2\pi\sqrt{2(\bar{\sigma}^2+\sigma^2)+2i(1-2\sigma)ct}}\iint\exp\left(-\frac{p'^2}{2}\right)\\ &\quad\cdot\exp\left(-\frac{(x'-p'ct+i(1-2\sigma)p')^2}{4(\bar{\sigma}^2+\sigma^2)+4i(1-2\sigma)ct}\right)\phi(x'\Delta x+p_0t,p'\Delta p+p_0)\mathrm{d}x'\,\mathrm{d}p'\end{aligned}$$

$$= \phi(p_0 t, p_0) \frac{1}{2\pi\sqrt{2(\bar{\sigma}^2+\sigma^2)+2i(1-2\sigma)ct}} \iint \exp\left(-\frac{p'^2}{2}\right)$$
$$\cdot \exp\left(-\frac{(x'-p'ct+i(1-2\sigma)p')^2}{4(\bar{\sigma}^2+\sigma^2)+4i(1-2\sigma)ct}\right) \mathrm{d}x'\,\mathrm{d}p'$$
$$= \phi(p_0 t, p_0),$$

where new coordinates $x' = (x - p_0 t)/\Delta x$ and $p' = (p - p_0)/\Delta p$ were used. Hence

$$\lim_{\hbar \to 0^+} \rho(x,p,t) = \delta(x - p_0 t)\delta(p - p_0).$$

Above equation implies that the state Ψ from (5.6) describing the free particle converges in the limit $\hbar \to 0^+$ to the classical pure state describing the free particle moving along a straight line with the constant momentum equal p_0.

Lets consider now another particular case of the solution (5.5). Assume that

$$f(p) = \delta(p - p_0).$$

By (5.4) the solution φ of the Schrödinger equation (5.3) takes the form of a plain wave

$$\varphi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}(p_0 x - E(p_0)t)}. \tag{5.7}$$

From (5.7) the solution Ψ of (5.2), for the case $\beta > 0$, reads

$$\begin{aligned}
\Psi(x,p,t) &= (\varphi \otimes_{\sigma,\alpha,\beta} \varphi)(x,p,t) \\
&= \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{1}{2}\hbar\alpha\partial_x^2} e^{\frac{1}{2}\hbar\beta\partial_p^2} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}py} \varphi^*(x - \bar{\sigma}y, t)\varphi(x + \sigma y, t) \\
&= \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{1}{2}\hbar\alpha\partial_x^2} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}py} e^{-\frac{1}{2\hbar}\beta y^2} e^{-\frac{i}{\hbar}(p_0(x-\bar{\sigma}y)-E(p_0)t)} e^{\frac{i}{\hbar}(p_0(x+\sigma y)-E(p_0)t)} \\
&= \frac{1}{(2\pi\hbar)^{3/2}} e^{\frac{1}{2}\hbar\alpha\partial_x^2} \int \mathrm{d}y\, e^{-\frac{i}{\hbar}(p-p_0)y} e^{-\frac{1}{2\hbar}\beta y^2} \\
&= \frac{1}{2\pi\hbar\sqrt{\beta}} e^{-\frac{1}{2\hbar\beta}(p-p_0)^2}.
\end{aligned} \tag{5.8}$$

In the limit $\beta \to 0^+$ equation (5.8) takes the form

$$\Psi(x,p,t) = \frac{1}{\sqrt{2\pi\hbar}} \delta(p - p_0).$$

Note, that Ψ is not a proper state since it does not belong to the space of states $\mathcal{H}$. Hence, Ψ does not describe a physical system. It, however, describes an idealized situation of a particle with the momentum known precisely and the position not known at all. Note also, that Ψ does not depend on time t, i.e. Ψ can be thought of as a stationary state of the system. In fact, Ψ is a formal $\star_{\sigma,\alpha,\beta}$-genfunction of p and H, i.e. Ψ formally satisfies the following $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$p \star_{\sigma,\alpha,\beta} \Psi = p_0 \Psi, \qquad \Psi \star_{\sigma,\alpha,\beta} p = p_0 \Psi,$$
$$H \star_{\sigma,\alpha,\beta} \Psi = E(p_0) \Psi, \qquad \Psi \star_{\sigma,\alpha,\beta} H = E(p_0) \Psi.$$

Hence, p_0 and $E(p_0)$ can be interpreted as the momentum and energy of the particle.

5.2 Simple harmonic oscillator

5.2.1 Stationary states of the harmonic oscillator

Lets consider a Hamiltonian system describing a one dimensional ($N = 1$) simple harmonic oscillator. Its Hamiltonian $\hat{H}$ is induced by the function

$$H(x,p) = \frac{1}{2}\left(p^2 + \omega^2 x^2\right),$$

where ω is the frequency of oscillations. Note that H is a Hermitian function for every (σ, α, β)-ordering, i.e. $H \star_{\sigma,\alpha,\beta} = (H \star_{\sigma,\alpha,\beta})^\dagger$. Lets try to find stationary pure states of the harmonic oscillator. From Section 3.5 it is known that the stationary pure states are precisely the solutions of the following pair of $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$H \star_{\sigma,\alpha,\beta} \Psi = E\Psi, \tag{5.9a}$$
$$\Psi \star_{\sigma,\alpha,\beta} H = E\Psi, \tag{5.9b}$$

for $E \in \mathbb{R}$. To solve the above equations it is convenient to introduce new coordinates called *holomorphic coordinates* [38]

$$a(x,p) = \frac{\omega x + ip}{\sqrt{2\hbar\omega}}, \qquad \bar{a}(x,p) = \frac{\omega x - ip}{\sqrt{2\hbar\omega}},$$

from which follows that

$$x = \sqrt{\frac{\hbar}{2\omega}}\left(a(x,p) + \bar{a}(x,p)\right), \qquad p = -i\sqrt{\frac{\hbar\omega}{2}}\left(a(x,p) - \bar{a}(x,p)\right),$$
$$\frac{\partial}{\partial a} = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{1}{\omega}\frac{\partial}{\partial x} - i\frac{\partial}{\partial p}\right), \qquad \frac{\partial}{\partial \bar{a}} = \sqrt{\frac{\hbar\omega}{2}}\left(\frac{1}{\omega}\frac{\partial}{\partial x} + i\frac{\partial}{\partial p}\right),$$
$$\frac{\partial}{\partial x} = \sqrt{\frac{\omega}{2\hbar}}\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial \bar{a}}\right), \qquad \frac{\partial}{\partial p} = \frac{i}{\sqrt{2\hbar\omega}}\left(\frac{\partial}{\partial a} - \frac{\partial}{\partial \bar{a}}\right).$$

Note, that $a \star_{\sigma,\alpha,\beta} = (\bar{a} \star_{\sigma,\alpha,\beta})^{\dagger}$ and $\bar{a} \star_{\sigma,\alpha,\beta} = (a \star_{\sigma,\alpha,\beta})^{\dagger}$. In this new coordinates the function H inducing the Hamiltonian $\hat{H}$ takes the form

$$H(a,\bar{a}) = \hbar\omega a\bar{a}. \tag{5.10}$$

Note, that for appropriate functions $\Psi \in \mathcal{H}$ there holds

$$\begin{aligned} a \star_{\sigma,\alpha,\beta} \Psi &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega x \star_{\sigma,\alpha,\beta} \Psi + ip \star_{\sigma,\alpha,\beta} \Psi\right) = \frac{1}{\sqrt{2\hbar\omega}} \left(\omega \hat{q}_{\sigma,\alpha}\Psi + i\hat{p}_{\sigma,\beta}\Psi\right) \\ &= \frac{1}{\sqrt{2\hbar\omega}} \left(\omega x\Psi + i\hbar\omega\sigma\frac{\partial\Psi}{\partial p} + \hbar\omega\alpha\frac{\partial\Psi}{\partial x} + ip\Psi + \hbar\bar{\sigma}\frac{\partial\Psi}{\partial x} + i\hbar\beta\frac{\partial\Psi}{\partial p}\right) \\ &= a\Psi + \frac{1}{2}\left(1 - 2\sigma + \omega\alpha - \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial a} + \frac{1}{2}\left(1 + \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial\bar{a}} \end{aligned} \tag{5.11a}$$

and similarly

$$\Psi \star_{\sigma,\alpha,\beta} a = a\Psi + \frac{1}{2}\left(1 - 2\sigma + \omega\alpha - \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial a} - \frac{1}{2}\left(1 - \omega\alpha - \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial\bar{a}}, \tag{5.11b}$$

$$\bar{a} \star_{\sigma,\alpha,\beta} \Psi = \bar{a}\Psi - \frac{1}{2}\left(1 - 2\sigma - \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial\bar{a}} - \frac{1}{2}\left(1 - \omega\alpha - \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial a}, \tag{5.11c}$$

$$\Psi \star_{\sigma,\alpha,\beta} \bar{a} = \bar{a}\Psi - \frac{1}{2}\left(1 - 2\sigma - \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial\bar{a}} + \frac{1}{2}\left(1 + \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi}{\partial a}. \tag{5.11d}$$

By replacing Ψ by $\bar{a}$ in equations (5.11a) and (5.11b) one gets

$$\bar{a} \star_{\sigma,\alpha,\beta} a = \bar{a}a - \bar{\lambda}, \tag{5.12a}$$

$$a \star_{\sigma,\alpha,\beta} \bar{a} = a\bar{a} + \lambda, \tag{5.12b}$$

where $\lambda = \frac{1}{2}(1+\omega\alpha+\omega^{-1}\beta)$ and $\bar{\lambda} := 1-\lambda = \frac{1}{2}(1-\omega\alpha-\omega^{-1}\beta)$. From above equations it follows, by using (5.10), that the function H takes the form

$$H(a,\bar{a}) = \hbar\omega\left(\bar{a} \star_{\sigma,\alpha,\beta} a + \bar{\lambda}\right) = \hbar\omega\left(a \star_{\sigma,\alpha,\beta} \bar{a} - \lambda\right). \tag{5.13}$$

From equations (5.12) follows also the commutation relation between functions a and $\bar{a}$, namely

$$[a,\bar{a}] = a \star_{\sigma,\alpha,\beta} \bar{a} - \bar{a} \star_{\sigma,\alpha,\beta} a = 1. \tag{5.14}$$

First, lets prove that the $\star_{\sigma,\alpha,\beta}$-genvalues E of H are greater than or equal to $\bar{\lambda}\hbar\omega$

$$E \geq \bar{\lambda}\hbar\omega. \tag{5.15}$$

Indeed, let Ψ be a normalized solution to (5.9a). Then

$$\langle\Psi|H\star_{\sigma,\alpha,\beta}\Psi\rangle_{\mathcal{H}} = E\langle\Psi|\Psi\rangle_{\mathcal{H}}$$
$$\Updownarrow$$
$$\langle\Psi|\hbar\omega(\bar{a}\star_{\sigma,\alpha,\beta}a+\bar{\lambda})\star_{\sigma,\alpha,\beta}\Psi\rangle_{\mathcal{H}} = E$$
$$\Updownarrow$$
$$\hbar\omega\langle\Psi|\bar{a}\star_{\sigma,\alpha,\beta}a\star_{\sigma,\alpha,\beta}\Psi\rangle_{\mathcal{H}} + \bar{\lambda}\hbar\omega\langle\Psi|\Psi\rangle_{\mathcal{H}} = E$$
$$\Updownarrow$$
$$\hbar\omega\langle(\bar{a}\star_{\sigma,\alpha,\beta})^{\dagger}\Psi|a\star_{\sigma,\alpha,\beta}\Psi\rangle_{\mathcal{H}} = E-\bar{\lambda}\hbar\omega$$
$$\Updownarrow$$
$$\hbar\omega\langle a\star_{\sigma,\alpha,\beta}\Psi|a\star_{\sigma,\alpha,\beta}\Psi\rangle_{\mathcal{H}} = E-\bar{\lambda}\hbar\omega$$
$$\Downarrow$$
$$E-\bar{\lambda}\hbar\omega \geq 0.$$

Now, lets assume that $\Psi_{mn}\in\mathcal{H}$ are normalized solutions to

$$H\star_{\sigma,\alpha,\beta}\Psi_{mn} = E_m\Psi_{mn}, \tag{5.16a}$$
$$\Psi_{mn}\star_{\sigma,\alpha,\beta}H = E_n\Psi_{mn}, \tag{5.16b}$$

where m,n are numbering the $\star_{\sigma,\alpha,\beta}$-genvalues of H. Multiplying (5.16a) from the left by a and using (5.13) and the commutation relation (5.14) results in

$$\begin{aligned}
E_m a\star_{\sigma,\alpha,\beta}\Psi_{mn} &= a\star_{\sigma,\alpha,\beta}H\star_{\sigma,\alpha,\beta}\Psi_{mn}\\
&= \hbar\omega a\star_{\sigma,\alpha,\beta}\left(\bar{a}\star_{\sigma,\alpha,\beta}a+\frac{1}{2}\left(1-\omega\alpha-\omega^{-1}\beta\right)\right)\star_{\sigma,\alpha,\beta}\Psi_{mn}\\
&= \hbar\omega\left(\bar{a}\star_{\sigma,\alpha,\beta}a+1\right)\star_{\sigma,\alpha,\beta}a\star_{\sigma,\alpha,\beta}\Psi_{mn}\\
&\quad+\frac{1}{2}\hbar\omega\left(1-\omega\alpha-\omega^{-1}\beta\right)a\star_{\sigma,\alpha,\beta}\Psi_{mn}\\
&= H\star_{\sigma,\alpha,\beta}a\star_{\sigma,\alpha,\beta}\Psi_{mn}+\hbar\omega a\star_{\sigma,\alpha,\beta}\Psi_{mn},
\end{aligned}$$

from which follows that

$$H\star_{\sigma,\alpha,\beta}(a\star_{\sigma,\alpha,\beta}\Psi_{mn}) = (E_m-\hbar\omega)(a\star_{\sigma,\alpha,\beta}\Psi_{mn}). \tag{5.17a}$$

Similarly one gets that

$$(\Psi_{mn} \star_{\sigma,\alpha,\beta} a) \star_{\sigma,\alpha,\beta} H = (E_n + \hbar\omega)(\Psi_{mn} \star_{\sigma,\alpha,\beta} a), \tag{5.17b}$$

$$H \star_{\sigma,\alpha,\beta} (\bar{a} \star_{\sigma,\alpha,\beta} \Psi_{mn}) = (E_m + \hbar\omega)(\bar{a} \star_{\sigma,\alpha,\beta} \Psi_{mn}), \tag{5.17c}$$

$$(\Psi_{mn} \star_{\sigma,\alpha,\beta} \bar{a}) \star_{\sigma,\alpha,\beta} H = (E_n - \hbar\omega)(\Psi_{mn} \star_{\sigma,\alpha,\beta} \bar{a}). \tag{5.17d}$$

From equations (5.17a) and (5.17c) it follows that $a \star_{\sigma,\alpha,\beta} \Psi_{mn}$ and $\bar{a} \star_{\sigma,\alpha,\beta} \Psi_{mn}$ are the solutions to (5.16a) with energy respectively decreased and increased by $\hbar\omega$. Similarly, from equations (5.17b) and (5.17d) it follows that $\Psi_{mn} \star_{\sigma,\alpha,\beta} a$ and $\Psi_{mn} \star_{\sigma,\alpha,\beta} \bar{a}$ are the solutions to (5.16b) with energy respectively increased and decreased by $\hbar\omega$. First of all, this shows that the spectrum of energies is discrete with the spacing between energies equal $\hbar\omega$. Hence, m, n numbering the $\star_{\sigma,\alpha,\beta}$-genvalues of H are some integer numbers. Secondly, one gets an action of a and $\bar{a}$ on Ψ_{mn} given by the formulae

$$\Psi_{mn} \star_{\sigma,\alpha,\beta} a = A_n \Psi_{m,n+1}, \qquad \bar{a} \star_{\sigma,\alpha,\beta} \Psi_{mn} = B_m \Psi_{m+1,n}, \tag{5.18a}$$

$$a \star_{\sigma,\alpha,\beta} \Psi_{mn} = C_m \Psi_{m-1,n}, \qquad \Psi_{mn} \star_{\sigma,\alpha,\beta} \bar{a} = D_n \Psi_{m,n-1}, \tag{5.18b}$$

where A_m, B_m, C_n, D_n are some normalization constants. The functions a and $\bar{a}$ are called the *annihilation* and *creation* functions since they decrease and increase the number of excitations of the vibrational mode with frequency ω (annihilate and create the quanta of vibrations).

From equations (5.17a) and (5.17d) follows that the left action of a and the right action of $\bar{a}$ on Ψ_{mn} creates a state with energy decreased by $\hbar\omega$, so one could thought that it is possible to create a state with arbitrarily small energy. But energy spectrum is bounded from below according to equation (5.15), hence it is necessary that for some state $\Psi_{m_0 n_0}$ there holds

$$a \star_{\sigma,\alpha,\beta} \Psi_{m_0 n_0} = 0 \quad \text{and} \quad \Psi_{m_0 n_0} \star_{\sigma,\alpha,\beta} \bar{a} = 0. \tag{5.19}$$

It is natural to enumerate the states Ψ_{mn} in such a way that $m_0 = 0$ and $n_0 = 0$. Equations (5.19) take then the form

$$a \star_{\sigma,\alpha,\beta} \Psi_{00} = 0 \quad \text{and} \quad \Psi_{00} \star_{\sigma,\alpha,\beta} \bar{a} = 0. \tag{5.20}$$

The state Ψ_{00} satisfying (5.20) has the lowest energy E_0. It is called a *ground state* and the energy E_0 the *ground energy*. Lets calculate the value of E_0. There holds

$$H \star_{\sigma,\alpha,\beta} \Psi_{00} = \hbar\omega(\bar{a} \star_{\sigma,\alpha,\beta} a + \bar{\lambda}) \star_{\sigma,\alpha,\beta} \Psi_{00} = \bar{\lambda}\hbar\omega\Psi_{00},$$

hence $E_0 = \bar{\lambda}\hbar\omega$. From this it can be immediately seen that $E_n = (n+\bar{\lambda})\hbar\omega$. Since $H = \hbar\omega(\bar{a} \star_{\sigma,\alpha,\beta} a + \bar{\lambda})$, $\star_{\sigma,\alpha,\beta}$-genvalues of the function $N := \bar{a} \star_{\sigma,\alpha,\beta} a$ are the natural numbers $n = 0, 1, 2, \ldots$ and $\star_{\sigma,\alpha,\beta}$-genfunctions are the $\star_{\sigma,\alpha,\beta}$-genfunctions Ψ_{mn} of H, i.e.

$$N \star_{\sigma,\alpha,\beta} \Psi_{mn} = m\Psi_{mn}, \quad \Psi_{mn} \star_{\sigma,\alpha,\beta} N = n\Psi_{mn}.$$

Hence, the function $N = \bar{a} \star_{\sigma,\alpha,\beta} a$ can be interpreted as an observable of the number of excitations of the vibrational mode with frequency ω.

It is now possible to calculate the normalization constants A_m, B_m, C_n, D_n from equations (5.18). For example one gets that

$$\langle a \star_{\sigma,\alpha,\beta} \Psi_{mn} | a \star_{\sigma,\alpha,\beta} \Psi_{mn} \rangle_{\mathcal{H}} = |C_m|^2 \langle \Psi_{m-1,n} | \Psi_{m-1,n} \rangle_{\mathcal{H}},$$

from which follows that

$$\begin{aligned} |C_m|^2 &= \langle \Psi_{mn} | (a \star_{\sigma,\alpha,\beta})^\dagger a \star_{\sigma,\alpha,\beta} \Psi_{mn} \rangle_{\mathcal{H}} = \langle \Psi_{mn} | \bar{a} \star_{\sigma,\alpha,\beta} a \star_{\sigma,\alpha,\beta} \Psi_{mn} \rangle_{\mathcal{H}} \\ &= m \langle \Psi_{mn} | \Psi_{mn} \rangle_{\mathcal{H}} = m. \end{aligned}$$

Hence, C_m can be chosen to be equal $C_m = \sqrt{m}$. Analogically one finds that $A_n = \sqrt{n+1}$, $B_m = \sqrt{m+1}$ and $D_n = \sqrt{n}$. Equations (5.18) take then the form

$$\Psi_{mn} \star_{\sigma,\alpha,\beta} a = \sqrt{n+1}\Psi_{m,n+1}, \qquad \bar{a} \star_{\sigma,\alpha,\beta} \Psi_{mn} = \sqrt{m+1}\Psi_{m+1,n}, \tag{5.21a}$$

$$a \star_{\sigma,\alpha,\beta} \Psi_{mn} = \sqrt{m}\Psi_{m-1,n}, \qquad \Psi_{mn} \star_{\sigma,\alpha,\beta} \bar{a} = \sqrt{n}\Psi_{m,n-1}. \tag{5.21b}$$

From equations (5.21a) it can be seen that from the ground state Ψ_{00} all other states Ψ_{mn} can be reconstructed. In fact

$$\Psi_{mn} = \frac{1}{\sqrt{m!n!}} \underbrace{\bar{a} \star_{\sigma,\alpha,\beta} \ldots \star_{\sigma,\alpha,\beta} \bar{a}}_{m} \star_{\sigma,\alpha,\beta} \Psi_{00} \star_{\sigma,\alpha,\beta} \underbrace{a \star_{\sigma,\alpha,\beta} \ldots \star_{\sigma,\alpha,\beta} a}_{n}, \tag{5.22}$$

The ground state Ψ_{00} can be calculated from equations (5.20). These equations, by using formulae (5.11a) and (5.11d), can be rewritten in the form

$$\begin{aligned} a\Psi_{00} + \frac{1}{2}\left(1 - 2\sigma + \omega\alpha - \omega^{-1}\beta\right)\frac{\partial\Psi_{00}}{\partial a} + \frac{1}{2}\left(1 + \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi_{00}}{\partial \bar{a}} &= 0, \\ \bar{a}\Psi_{00} - \frac{1}{2}\left(1 - 2\sigma - \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi_{00}}{\partial \bar{a}} + \frac{1}{2}\left(1 + \omega\alpha + \omega^{-1}\beta\right)\frac{\partial\Psi_{00}}{\partial a} &= 0. \end{aligned}$$

The normalized solution to this system of partial differential equations reads

$$\Psi_{00}(a,\bar{a}) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{(1-2\sigma)^2 + (1+2\omega\alpha)(1+2\omega^{-1}\beta)}}{\bar{\sigma}^2 + \sigma^2 + 2\alpha\beta + \omega\alpha + \omega^{-1}\beta}$$
$$\cdot \exp\left(\frac{-(1+\omega\alpha+\omega^{-1}\beta)a\bar{a}}{\bar{\sigma}^2 + \sigma^2 + 2\alpha\beta + \omega\alpha + \omega^{-1}\beta}\right)$$
$$\cdot \exp\left(\frac{-\frac{1}{2}(1-2\sigma-\omega\alpha+\omega^{-1}\beta)a^2 + \frac{1}{2}(1-2\sigma+\omega\alpha-\omega^{-1}\beta)\bar{a}^2}{\bar{\sigma}^2 + \sigma^2 + 2\alpha\beta + \omega\alpha + \omega^{-1}\beta}\right) \tag{5.23a}$$

or after the change of coordinates

$$\Psi_{00}(x,p) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\sqrt{(1-2\sigma)^2 + (1+2\omega\alpha)(1+2\omega^{-1}\beta)}}{\bar{\sigma}^2 + \sigma^2 + 2\alpha\beta + \omega\alpha + \omega^{-1}\beta}$$
$$\cdot \exp\left(\frac{-(1+2\omega^{-1}\beta)\omega^2 x^2 - (1+2\omega\alpha)p^2 - i2(1-2\sigma)\omega xp}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2 + 2\alpha\beta + \omega\alpha + \omega^{-1}\beta)}\right). \tag{5.23b}$$

In what follows the $\star_{\sigma,\alpha,\beta}$-genfunctions Ψ_{mn} will be calculated using equation (5.22). To simplify calculations the special case of the (σ,α,β)-ordering will be considered, namely the case with $\sigma = \frac{1}{2}$ and $\beta = \omega^2\alpha$. To simplify the notation the $\star_{\frac{1}{2},\alpha,\omega^2\alpha}$-product will be denoted by $\star$. For this special ordering equations (5.23) for the ground state take the form

$$\Psi_{00}(a,\bar{a}) = \frac{1}{\sqrt{2\pi\hbar\lambda}} \exp\left(-\frac{a\bar{a}}{\lambda}\right),$$
$$\Psi_{00}(x,p) = \frac{1}{\sqrt{2\pi\hbar\lambda}} \exp\left(-\frac{p^2+\omega^2x^2}{2\lambda\hbar\omega}\right),$$

where now $\lambda = \frac{1}{2}(1+2\omega\alpha)$. Also, equations (5.11) reduce to the form

$$a \star \Psi = a\Psi + \lambda\frac{\partial\Psi}{\partial\bar{a}}, \qquad \bar{a} \star \Psi = \bar{a}\Psi - \bar{\lambda}\frac{\partial\Psi}{\partial a}, \tag{5.24a}$$
$$\Psi \star a = a\Psi - \bar{\lambda}\frac{\partial\Psi}{\partial\bar{a}}, \qquad \Psi \star \bar{a} = \bar{a}\Psi + \lambda\frac{\partial\Psi}{\partial a}. \tag{5.24b}$$

From equations (5.24a) or (5.24b) it follows that

$$\underbrace{a \star \ldots \star a}_{n} = a^n, \qquad \underbrace{\bar{a} \star \ldots \star \bar{a}}_{m} = \bar{a}^m,$$

thus equation (5.22) reduces to the form

$$\Psi_{mn} = \frac{1}{\sqrt{m!n!}}\bar{a}^m \star \Psi_{00} \star a^n. \tag{5.25}$$

To calculate two $\star$-products from equation (5.25) below formulae, following easily from (5.24), will be used

$$\bar{a} \star (g\Psi_{00}) = \left(\frac{\bar{a}g}{\lambda} - \bar{\lambda}\frac{\partial g}{\partial a}\right)\Psi_{00}, \tag{5.26a}$$

$$(g\Psi_{00}) \star a = \left(\frac{ag}{\lambda} - \bar{\lambda}\frac{\partial g}{\partial \bar{a}}\right)\Psi_{00}, \tag{5.26b}$$

where g is some function on the phase space. By taking $g(a,\bar{a}) = \bar{a}^m$ in (5.26a) one gets

$$\bar{a} \star (\bar{a}^m\Psi_{00}) = \frac{\bar{a}^{m+1}}{\lambda}\Psi_{00},$$

from which follows that

$$\bar{a}^m \star \Psi_{00} = \left(\frac{\bar{a}}{\lambda}\right)^m \Psi_{00}. \tag{5.27}$$

Now, by taking $g(a,\bar{a}) = \bar{a}^m$ in (5.26b) one gets

$$(\bar{a}^m\Psi_{00}) \star a^{n+1} = ((\bar{a}^m\Psi_{00}) \star a) \star a^n = \left(\left(\frac{a\bar{a}^m}{\lambda} - \bar{\lambda}m\bar{a}^{m-1}\right)\Psi_{00}\right) \star a^n,$$

from which follows that

$$(\bar{a}^m\Psi_{00}) \star a^n = \sum_{k=0}^{n}(-1)^k k!\binom{m}{k}\binom{n}{k}\bar{\lambda}^k\left(\frac{1}{\lambda}\right)^{n-k}\bar{a}^{m-k}a^{n-k}\Psi_{00}. \tag{5.28}$$

By using (5.27) and (5.28) equation (5.25) takes the form

$$\Psi_{mn}(a,\bar{a}) = \frac{1}{\sqrt{m!n!}}\sum_{k=0}^{n}(-1)^k k!\binom{m}{k}\binom{n}{k}\bar{\lambda}^k\left(\frac{1}{\lambda}\right)^{m+n-k}\bar{a}^{m-k}a^{n-k}\Psi_{00}(a,\bar{a}). \tag{5.29}$$

Above equation can be written differently when passing to the polar coordinates (r,θ)

$$\omega x + ip = re^{i\theta}.$$

In this new coordinates

$$a(r,\theta) = \frac{1}{\sqrt{2\hbar\omega}}re^{i\theta}, \quad \bar{a}(r,\theta) = \frac{1}{\sqrt{2\hbar\omega}}re^{-i\theta},$$
$$r^2 = p^2 + \omega^2x^2,$$

and equation (5.29) takes the form

$$\Psi_{mn}(r,\theta) = \frac{1}{\sqrt{2\pi\hbar\lambda}}(-1)^n\sqrt{\frac{n!}{m!}}\frac{\bar{\lambda}^n}{\lambda^m}\left(\frac{r}{\sqrt{2\hbar\omega}}\right)^{m-n}L_n^{m-n}\left(\frac{r^2}{2\hbar\omega\lambda\bar{\lambda}}\right)$$
$$\cdot\, e^{-i(m-n)\theta}\exp\left(-\frac{r^2}{2\hbar\omega\lambda}\right), \tag{5.30}$$

where

$$L_n^s(x) = \frac{x^{-s}e^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}x^{n+s}\right) = \sum_{k=0}^{n}(-1)^k\frac{(n+s)!}{(n-k)!(s+k)!k!}x^k$$

are the Laguerre's polynomials. This result for $\lambda = \frac{1}{2}$ was derived in [3], [40] and [5]. The stationary pure states of the harmonic oscillator are the functions

$$\Psi_{nn}(r,\theta) = \frac{1}{\sqrt{2\pi\hbar\lambda}}(-1)^n\left(\frac{\bar{\lambda}}{\lambda}\right)^n L_n\left(\frac{r^2}{2\hbar\omega\lambda\bar{\lambda}}\right)\exp\left(-\frac{r^2}{2\hbar\omega\lambda}\right), \tag{5.31}$$

where $L_n(x) = L_n^0(x)$. Equation (5.31) can be also written in the following form, independent on a coordinate system

$$\Psi_{nn} = \frac{1}{\sqrt{2\pi\hbar\lambda}}(-1)^n\left(\frac{\bar{\lambda}}{\lambda}\right)^n L_n\left(\frac{H}{\hbar\omega\lambda\bar{\lambda}}\right)\exp\left(-\frac{H}{\hbar\omega\lambda}\right).$$

Equations (5.30) and (5.31) are valid for $\lambda \neq 0, 1$ but it can be easily calculated how this equations look like in the limits $\lambda \to 0$ and $\lambda \to 1$. For the case $\lambda \to 1$ one gets

$$\begin{aligned}\Psi_{mn}(r,\theta) &= \frac{1}{\sqrt{2\pi\hbar m!n!}}\left(\frac{r}{\sqrt{2\hbar\omega}}\right)^{m+n} e^{-i(m-n)\theta}\exp\left(-\frac{r^2}{2\hbar\omega}\right),\\ \Psi_{nn}(r,\theta) &= \frac{1}{\sqrt{2\pi\hbar n!}}\left(\frac{r^2}{2\hbar\omega}\right)^n\exp\left(-\frac{r^2}{2\hbar\omega}\right).\end{aligned}$$

Moreover, for the case $\lambda \to 0$ one gets

$$\begin{aligned}\Psi_{mn}(a,\bar{a}) &= \frac{1}{2\hbar\sqrt{m!n!}}(-1)^{m+n}\sum_{k=0}^{n}k!\binom{m}{k}\binom{n}{k}\frac{\partial^{m-k}}{\partial a^{m-k}}\frac{\partial^{n-k}}{\partial\bar{a}^{n-k}}\delta^{(2)}(a),\\ \Psi_{nn}(a,\bar{a}) &= \frac{1}{2\hbar}\sum_{k=0}^{n}\frac{1}{k!}\binom{n}{k}\frac{\partial^{2k}}{\partial a^k\partial\bar{a}^k}\delta^{(2)}(a).\end{aligned}$$

It is worth noting that the stationary states Ψ_{nn}, for the case $\lambda = 0$, are some distributions which cannot be identified with actual functions. This shows that the space $\mathcal{H}$ is in general, for certain orderings (for (σ,α,β)-orderings for which α,β induce quadratic forms which are not positive define), the space of distributions.

It is interesting to check to which classical states converge quantum states Ψ_{nn} in the limit $\hbar \to 0^+$. A quantum distribution function $\rho_n = \frac{1}{\sqrt{2\pi\hbar}}\Psi_{nn}$ reads

$$\rho_n(x,p) = \frac{1}{2\pi\hbar\lambda}(-1)^n\left(\frac{\bar{\lambda}}{\lambda}\right)^n L_n\left(\frac{p^2+\omega^2x^2}{2\hbar\omega\lambda\bar{\lambda}}\right)\exp\left(-\frac{p^2+\omega^2x^2}{2\hbar\omega\lambda}\right).$$

The limit $\hbar \to 0^+$ has to be calculated in a distributional sense, hence the limit $\lim_{\hbar\to 0^+}\langle \rho_n, \phi\rangle$ has to be calculated for every test function ϕ. One have that

$$\begin{aligned}\lim_{\hbar\to 0^+}\langle \rho_n, \phi\rangle &= \lim_{\hbar\to 0^+}\iint \rho_n(x,p)\phi(x,p)\mathrm{d}x\,\mathrm{d}p\\ &= \lim_{\hbar\to 0^+}\iint \frac{1}{2\pi\hbar\lambda}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n L_n\left(\frac{p^2+\omega^2x^2}{2\hbar\omega\lambda\bar\lambda}\right)\exp\left(-\frac{p^2+\omega^2x^2}{2\hbar\omega\lambda}\right)\\ &\quad\cdot\phi(x,p)\mathrm{d}x\,\mathrm{d}p\,.\end{aligned}$$

After the change of the coordinates from (x,p) to $(\omega x/\sqrt{2\hbar\omega\lambda}, p/\sqrt{2\hbar\omega\lambda})$

$$\begin{aligned}\lim_{\hbar\to 0^+}\langle \rho_n, \phi\rangle &= \lim_{\hbar\to 0^+}\iint \frac{1}{2\pi\hbar\lambda}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n 2\hbar\lambda L_n\left(\frac{x^2+p^2}{\bar\lambda}\right)e^{-(x^2+p^2)}\\ &\quad\cdot\phi\left(\sqrt{\frac{2\hbar\lambda}{\omega}}x, \sqrt{2\hbar\omega\lambda}p\right)\mathrm{d}x\,\mathrm{d}p\\ &= \iint \frac{1}{\pi}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n L_n\left(\frac{x^2+p^2}{\bar\lambda}\right)e^{-(x^2+p^2)}\\ &\quad\cdot\lim_{\hbar\to 0^+}\phi\left(\sqrt{\frac{2\hbar\lambda}{\omega}}x, \sqrt{2\hbar\omega\lambda}p\right)\mathrm{d}x\,\mathrm{d}p\\ &= \phi(0,0)\iint \frac{1}{\pi}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n L_n\left(\frac{x^2+p^2}{\bar\lambda}\right)e^{-(x^2+p^2)}\mathrm{d}x\,\mathrm{d}p\\ &= \phi(0,0) = \langle\delta_{(0,0)},\phi\rangle \end{aligned}\tag{5.32}$$

since

$$\begin{aligned}&\iint \frac{1}{\pi}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n L_n\left(\frac{x^2+p^2}{\bar\lambda}\right)e^{-(x^2+p^2)}\mathrm{d}x\,\mathrm{d}p =\\ &= \frac{1}{\pi}(-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n\int_0^\infty\int_0^{2\pi} L_n\left(\frac{r^2}{\bar\lambda}\right)e^{-r^2}r\mathrm{d}r\,\mathrm{d}\theta\\ &= (-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n\int_0^\infty L_n\left(\frac{z}{\bar\lambda}\right)e^{-z}\mathrm{d}z\\ &= (-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n\int_0^\infty \frac{1}{n!}e^{\frac{z}{\bar\lambda}}\frac{\mathrm{d}^n}{\mathrm{d}(z/\bar\lambda)^n}\left(e^{-\frac{z}{\bar\lambda}}\left(\frac{z}{\bar\lambda}\right)^n\right)e^{-z}\mathrm{d}z\\ &= (-1)^n\left(\frac{\bar\lambda}{\lambda}\right)^n\frac{1}{n!}\int_0^\infty e^{\frac{\lambda}{\bar\lambda}z}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(e^{-\frac{z}{\bar\lambda}}z^n\right)\mathrm{d}z\\ &= \left(\frac{\bar\lambda}{\lambda}\right)^n\frac{1}{n!}\int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(e^{\frac{\lambda}{\bar\lambda}z}\right)e^{-\frac{z}{\bar\lambda}}z^n\mathrm{d}z\\ &= \frac{1}{n!}\int_0^\infty e^{\frac{\lambda}{\bar\lambda}z}e^{-\frac{z}{\bar\lambda}}z^n\mathrm{d}z = \frac{1}{n!}\int_0^\infty e^{-z}z^n\mathrm{d}z = 1,\end{aligned}$$

where first, the change to polar coordinates (r,θ) were applied, and then to $z = r^2$. Equation (5.32) implies that

$$\lim_{\hbar\to 0^+} \rho_n = \delta_{(0,0)},$$

i.e. the quantum stationary pure states ρ_n of the harmonic oscillator converge, in the limit $\hbar \to 0^+$, to the classical state $(x = 0, p = 0)$ describing a particle with the position and momentum equal 0. This result is not surprising as the state $(x = 0, p = 0)$ is the only classical stationary pure state of the harmonic oscillator.

5.2.2 Coherent states of the harmonic oscillator

Coherent states of the harmonic oscillator are functions $\Psi_{z_1,z_2} \in \mathcal{H}$ which satisfy the following $\star_{\sigma,\alpha,\beta}$-genvalue equations

$$a_L \star_{\sigma,\alpha,\beta} \Psi_{z_1,z_2} = z_1 \Psi_{z_1,z_2}, \tag{5.33a}$$

$$\bar{a}_R \star_{\sigma,\alpha,\beta} \Psi_{z_1,z_2} = z_2^* \Psi_{z_1,z_2}, \tag{5.33b}$$

where $z_1, z_2 \in \mathbb{C}$. Functions $\Psi_z := \Psi_{z,z}$ are then the admissible pure states. It will be shown later that coherent states are states which resemble the classical pure states the most. In fact, their time evolution is close to the time evolution of the classical pure states. Moreover, it can be shown that coherent states minimize the Heisenberg uncertainty principle, i.e. $\Delta x \Delta p = \hbar/2$. This once again shows that coherent states are the best realization of the classical pure states. Indeed, the classical pure states are those states for which the position and momentum is known precisely. However, in quantum mechanics we cannot know the precise position and momentum of a particle due to the Heisenberg uncertainty principle $\Delta x \Delta p \geq \hbar/2$, hence the states which minimize the uncertainty principle are the best realizations of the classical pure states.

Since

$$\begin{aligned}
a_L \star_{\sigma,\alpha,\beta} \Psi_{z_1,z_2} &= a_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha}, \hat{p}_{\sigma,\beta})\Psi_{z_1,z_2} = \frac{\omega \hat{q}_{\sigma,\alpha} + i\hat{p}_{\sigma,\beta}}{\sqrt{2\hbar\omega}} \Psi_{z_1,z_2} \\
&= \frac{\omega x + i\hbar\omega\sigma\partial_p + \hbar\omega\alpha\partial_x + ip + \hbar\bar{\sigma}\partial_x + i\hbar\beta\partial_p}{\sqrt{2\hbar\omega}} \Psi_{z_1,z_2} \\
&= \frac{(\omega x + ip)\Psi_{z_1,z_2} + \hbar((\bar{\sigma} + \omega\alpha)\partial_x + i(\sigma\omega + \beta)\partial_p)\Psi_{z_1,z_2}}{\sqrt{2\hbar\omega}},
\end{aligned}$$

$$\bar{a}_R \star_{\sigma,\alpha,\beta} \Psi_{z_1,z_2} = \bar{a}_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha}, \hat{p}^*_{\bar{\sigma},\beta})\Psi_{z_1,z_2} = \frac{\omega\hat{q}^*_{\bar{\sigma},\alpha} - i\hat{p}^*_{\bar{\sigma},\beta}}{\sqrt{2\hbar\omega}}\Psi_{z_1,z_2}$$
$$= \frac{\omega x - i\hbar\omega\bar{\sigma}\partial_p + \hbar\omega\alpha\partial_x - ip + \hbar\sigma\partial_x - i\hbar\beta\partial_p}{\sqrt{2\hbar\omega}}\Psi_{z_1,z_2}$$
$$= \frac{(\omega x - ip)\Psi_{z_1,z_2} + \hbar((\sigma + \omega\alpha)\partial_x - i(\bar{\sigma}\omega + \beta)\partial_p)\Psi_{z_1,z_2}}{\sqrt{2\hbar\omega}},$$

equations (5.33), for $z_1 = (\omega x_1 + ip_1)/\sqrt{2\hbar\omega}$ and $z_2 = (\omega x_2 + ip_2)/\sqrt{2\hbar\omega}$, are equivalent to a system of the following two differential equations:

$$(\omega(x - x_1) + i(p - p_1))\Psi_{z_1,z_2} + \hbar((\bar{\sigma} + \omega\alpha)\partial_x + i(\sigma\omega + \beta)\partial_p)\Psi_{z_1,z_2} = 0, \tag{5.34a}$$
$$(\omega(x - x_2) - i(p - p_2))\Psi_{z_1,z_2} + \hbar((\sigma + \omega\alpha)\partial_x - i(\bar{\sigma}\omega + \beta)\partial_p)\Psi_{z_1,z_2} = 0. \tag{5.34b}$$

To not receive to complicated equations only the case of $\alpha = \beta = 0$ will be considered. So, the solution to the system of differential equations (5.34) for $\alpha = \beta = 0$ reads

$$\Psi_{z_1,z_2}(x,p) = \frac{1}{\sqrt{\pi\hbar\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(\frac{i(p_1 - p_2)x}{\hbar}\right) \exp\left(-\omega\frac{(x - x_1)^2 + (x - x_2)^2}{2\hbar}\right)$$
$$\cdot \exp\left(\frac{(\sigma(\omega(x - x_1) + i(p - p_1)) - \bar{\sigma}(\omega(x - x_2) - i(p - p_2)))^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right). \tag{5.35}$$

Functions Ψ_{z_1,z_2} which correspond to actual states are those for which $z_1 = z_2 = \bar{z} = (\omega\bar{x} + i\bar{p})/\sqrt{2\hbar\omega}$. The equation (5.35) can be written then in the form

$$\Psi_{\bar{z}}(x,p) = \frac{1}{\sqrt{\pi\hbar\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(-\frac{\omega^2(x - \bar{x})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right) \exp\left(-\frac{(p - \bar{p})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right)$$
$$\cdot \exp\left(i\frac{2(2\sigma - 1)\omega(x - \bar{x})(p - \bar{p})}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right).$$

A quantum distribution function induced by $\Psi_{\bar{z}}$ is then given by

$$\rho(x,p) = \frac{1}{\sqrt{2\pi\hbar}}\Psi_{\bar{z}}(x,p)$$
$$= \frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(-\frac{\omega^2(x - \bar{x})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right) \exp\left(-\frac{(p - \bar{p})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right)$$
$$\cdot \exp\left(i\frac{2(2\sigma - 1)\omega(x - \bar{x})(p - \bar{p})}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right). \tag{5.36}$$

Note that the expectation value of the position and momentum in a coherent state $\Psi_{\bar{z}}$ is equal respectively $\bar{x}$ and $\bar{p}$.

Lets consider now the time evolution of the quantum distribution functions ρ from equation (5.36). To find out how the functions ρ develop in time it is necessary to solve the time evolution equation (3.24)

$$i\hbar\frac{\partial\rho}{\partial t} - [H,\rho] = 0, \tag{5.37}$$

where $H(x,p) = \frac{1}{2}(\omega^2x^2 + p^2)$. From the definition of the $\star_\sigma$-product it easily follows that

$$\begin{aligned}
H \star_\sigma \rho &= H\rho - i\hbar\bar{\sigma}p\frac{\partial\rho}{\partial x} - \frac{1}{2}\hbar^2\bar{\sigma}^2\frac{\partial^2\rho}{\partial x^2} + i\hbar\sigma\omega^2x\frac{\partial\rho}{\partial p} - \frac{1}{2}\hbar^2\sigma^2\omega^2\frac{\partial^2\rho}{\partial p^2},\\
\rho \star_\sigma H &= H\rho - i\hbar\bar{\sigma}\omega^2x\frac{\partial\rho}{\partial p} - \frac{1}{2}\hbar^2\bar{\sigma}^2\omega^2\frac{\partial^2\rho}{\partial p^2} + i\hbar\sigma p\frac{\partial\rho}{\partial x} - \frac{1}{2}\hbar^2\sigma^2\frac{\partial^2\rho}{\partial x^2}.
\end{aligned}$$

From this follows that

$$[H,\rho] = i\hbar\omega^2x\frac{\partial\rho}{\partial p} - i\hbar p\frac{\partial\rho}{\partial x} - \hbar\omega^2\frac{1}{2}(2\sigma-1)\frac{\partial^2\rho}{\partial p^2} + \hbar\frac{1}{2}(2\sigma-1)\frac{\partial^2\rho}{\partial x^2},$$

and the time evolution equation (5.37) takes the form

$$\frac{\partial\rho}{\partial t} - \omega^2x\frac{\partial\rho}{\partial p} + p\frac{\partial\rho}{\partial x} - i\hbar\omega^2\frac{1}{2}(2\sigma-1)\frac{\partial^2\rho}{\partial p^2} + i\hbar\frac{1}{2}(2\sigma-1)\frac{\partial^2\rho}{\partial x^2} = 0.$$

The solution of the above equation with the initial condition equal

$$\begin{aligned}
\rho(x,p,0) = &\frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2+\sigma^2)}}\exp\left(-\frac{\omega^2(x-x_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)\exp\left(-\frac{(p-p_0)^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)\\
&\cdot\exp\left(i\frac{2(2\sigma-1)\omega(x-x_0)(p-p_0)}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right),
\end{aligned}$$

reads

$$\begin{aligned}
\rho(x,p,t) = &\frac{1}{\pi\hbar\sqrt{2\omega(\bar{\sigma}^2+\sigma^2)}}\exp\left(-\frac{\omega^2(x-\bar{x}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)\exp\left(-\frac{(p-\bar{p}(t))^2}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right)\\
&\cdot\exp\left(i\frac{2(2\sigma-1)\omega(x-\bar{x}(t))(p-\bar{p}(t))}{2\hbar\omega(\bar{\sigma}^2+\sigma^2)}\right),
\end{aligned}$$

where

$$\begin{aligned}
\bar{x}(t) &= x_0\cos\omega t + \frac{p_0}{\omega}\sin\omega t,\\
\bar{p}(t) &= -\omega x_0\sin\omega t + p_0\cos\omega t.
\end{aligned}$$

Hence, it can be seen that the expectation values $\bar{x}(t)$ and $\bar{p}(t)$ of the position and momentum evolve in time like classical pure states of the harmonic oscillator. But, it has to be remembered that the coherent states have some uncertainty Δx and Δp of the position and momentum in contrary to the classical pure states. Hence, even though the coherent states resemble the classical pure states they differ from them.

In what follows it will be shown that the coherent states ρ from equation (5.36) converge, in the limit $\hbar \to 0^+$, to the classical pure states $(\bar{x}, \bar{p})$ describing a particle with the position and momentum equal $\bar{x}$ and $\bar{p}$. To prove this it is necessary to prove that

$$\lim_{\hbar \to 0^+} \langle \rho, \phi \rangle = \langle \delta_{(\bar{x},\bar{p})}, \phi \rangle = \phi(\bar{x}, \bar{p})$$

for every test function ϕ. One have that

$$\begin{aligned}\lim_{\hbar \to 0^+} \langle \rho, \phi \rangle = \lim_{\hbar \to 0^+} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi \hbar \sqrt{2\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(-\frac{\omega^2 (x - \bar{x})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right) \\ &\cdot \exp\left(-\frac{(p - \bar{p})^2}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right) \exp\left(i \frac{2(2\sigma - 1)\omega(x - \bar{x})(p - \bar{p})}{2\hbar\omega(\bar{\sigma}^2 + \sigma^2)}\right) \phi(x, p) \mathrm{d}x \, \mathrm{d}p \,.\end{aligned}$$

Changing the variables x and p to $x' = (x - \bar{x})/\sqrt{\hbar}$ and $p' = (p - \bar{p})/\sqrt{\hbar}$ one receives

$$\begin{aligned}\lim_{\hbar \to 0^+} \langle \rho, \phi \rangle = \lim_{\hbar \to 0^+} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\pi \sqrt{2\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(-\frac{\omega^2 x'^2}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \\ &\cdot \exp\left(-\frac{p'^2}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \exp\left(i \frac{2(2\sigma - 1)\omega x' p'}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \\ &\cdot \phi(\sqrt{\hbar} x' + \bar{x}, \sqrt{\hbar} p' + \bar{p}) \mathrm{d}x' \, \mathrm{d}p' \,.\end{aligned}$$

The only term under the integral which depends on $\hbar$ is $\phi(\sqrt{\hbar}x' + \bar{x}, \sqrt{\hbar}p' + \bar{p})$ so $\lim_{\hbar \to 0^+}$ works only on this term giving from continuity of ϕ

$$\lim_{\hbar \to 0^+} \phi(\sqrt{\hbar} x' + \bar{x}, \sqrt{\hbar} p' + \bar{p}) = \phi(\bar{x}, \bar{p}).$$

Hence, this term can be put in front of the integral sign. The only thing left to calculate is the integral

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} &\frac{1}{\pi \sqrt{2\omega(\bar{\sigma}^2 + \sigma^2)}} \exp\left(-\frac{\omega^2 x'^2}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \exp\left(-\frac{p'^2}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \\ &\cdot \exp\left(i \frac{2(2\sigma - 1)\omega x' p'}{2\omega(\bar{\sigma}^2 + \sigma^2)}\right) \mathrm{d}x' \, \mathrm{d}p' \,,\end{aligned}$$

which happens to be equal 1 as shows a simple calculation. From this one receives that

$$\lim_{\hbar \to 0^+} \langle \rho, \phi \rangle = \phi(\bar{x}, \bar{p}),$$

which ends the proof.

6

Conclusions

In the paper the quantization procedure of a classical Hamiltonian system was presented. In full details the quantization was performed in the canonical coordinates for the case of the Hamiltonian system without any constrains, i.e. the case when a phase space was a Poisson manifold equal $\mathbb{R}^{2N}$. In addition to this, it was shown that in the canonical regime, from the presented quantization scheme, immediately follows the ordinary description of quantum mechanics developed by Schrödinger, Dirac and Heisenberg. Finally, there were given some examples of the presented formalism, namely the free particle and the simple harmonic oscillator.

In the presented quantization procedure (σ, α, β)-parameter family of $\star$-products were considered. It was shown that all these $\star$-products, and thus all quantizations induced by them, are mathematically equivalent (gauge equivalent). Arises now the question whether or not all received quantizations give the same physical results. It is clear that different $\star_{\sigma,\alpha,\beta}$-products give different spectra and hence different expectation values of observables. In fact, for the case $\sigma \neq \frac{1}{2}$ the received spectra and expectation values will be in general complex. This distinguish the family of $\star$-products for which $\sigma = \frac{1}{2}$. In this case, according to (3.13), all observables $A \star_{\frac{1}{2},\alpha,\beta} = A_{\frac{1}{2},\alpha,\beta}(\hat{q}_{\frac{1}{2},\alpha}, \hat{p}_{\frac{1}{2},\beta}) \in \hat{\mathcal{O}}_Q$ are Hermitian and hence have real spectra. But, it still does not prove that only $\star$-products with $\sigma = \frac{1}{2}$ are admissible, since one could always assume that the physical meaning have only real parts of the spectra of observables. In this case two $\star$-products could induce physically equivalent quantizations if the real parts of spectra of a given Hamiltonian $\hat{H}$ were shifted by a constant value.

Introducing the time evolution of a quantum Hamiltonian system the assumption that a Hamiltonian $\hat{H}$ is self-adjoint, i.e. $(H_{L,R} \star_{\sigma,\alpha,\beta})^\dagger = H_{L,R} \star_{\sigma,\alpha,\beta}$, was used. This assumption guaranties that, during the time evolution, functions $\Psi \in \mathcal{H}$ which were initially admissible (pure or mixed) states remain admissible states. In other words, the one parameter group of functions $U(t) = e_{\star_{\sigma,\alpha,\beta}}^{-\frac{i}{\hbar}tH}$, which induce the time evolution, is unitary. It seems that it is not possible to describe the time evolution using non-hermitian Hamiltonians. This distinguishes the family of $\star$-products for which $\sigma = \frac{1}{2}$.

From the above considerations it follows that the admissible $\star$-products are those for which $\sigma = \frac{1}{2}$. The question is whether or not this family of $\star$-products is physically equivalent. It seems that different $\star$-products from this family give different physical results since spectra, and hence expectation values, of a given observable are not, in general, shifted by a constant value. For example, it can be easily calculated that all polynomials, except the ones linear and quadratic in x and p variables, give spectra which are not shifted by a constant value. This ambiguity of physically nonequivalent $\star$-products rises a problem of selecting one of them, which would yield equal predictions concerning the results of measurements. It would need to be checked if the family of the $\star$-products, for which $\sigma = \frac{1}{2}$, is indeed physically nonequivalent and, if yes, if the $\star$-product reproducing the results of measurements is distinguished in some way from other products in this family.

As an example consider a Hamiltonian system with a natural Hamiltonian

$$H(x,p) = \frac{1}{2}p^2 + V(x).$$

This Hamiltonian is Hermitian for every σ, α, β, hence for the class of natural Hamiltonians all $\star_{\sigma,\alpha,\beta}$-products are admissible. Note also that the spectrum of this Hamiltonian is σ-independent, hence for a fixed α, β all $\star_{\sigma,\alpha,\beta}$-products are physically equivalent.

In conclusion, the paper presents the natural quantization scheme of classical Hamiltonian systems without any constrains and in canonical regime. It is natural to develop the presented formalism to general Hamiltonian system with constrains. It is also natural to check how this formalism would look like after the change of coordinates. Especially interesting would be non-canonical formulation of quantum mechanics. The further development of the presented formalism would be an incorporation of the spin degree of freedom. Some attempts to generalize the presented quantization scheme were already made [41, 12, 42], but they still need to be systematize and refined.

A

Appendix

A.1 Notation and used conventions

In the paper by the symbols $M_N(\mathbb{N})$ and $M_N(\mathbb{R})$ the sets of all $N \times N$ matrices with natural and real coefficients are denoted. Moreover, for $\sigma \in M_N(\mathbb{R})$ the symbol $\bar{\sigma}$ denotes the matrix $\bar{\sigma}_{ij} = \delta_{ij} - \sigma_{ij}$ and for $n \in M_N(\mathbb{N})$ the symbol n^T denotes the transposition of the matrix n.

In the paper the Einstein summation convention is used, i.e. the summation symbol is skipped in terms where the summation index is written in the subscript and superscript, e.g. $a_i b^i \equiv \sum_i a_i b^i$. Moreover, for $(x,p) = (x^1, \dots, x^N, p_1, \dots, p_N) \in \mathbb{R}^{2N}$ and some smooth vector fields $X_1, \dots, X_N$ on $\mathbb{R}^{2N}$, in particular $\partial_{x^1}, \dots, \partial_{x^N}$ and $\partial_{p_1}, \dots, \partial_{p_N}$, the following notation is used

$$X = (X_1, \dots, X_N),$$
$$\partial_x = (\partial_{x^1}, \dots, \partial_{x^N}), \qquad \partial_p = (\partial_{p_1}, \dots, \partial_{p_N}).$$

Furthermore, for multi-indices $n, m \in \mathbb{N}^N$ and $a = (a_1, \dots, a_N) \in \mathbb{R}^N$ the following multi-index notation is used

$$|n| = \sum_i n_i, \qquad n! = \prod_i n_i!, \qquad \binom{n}{m} = \prod_i \binom{n_i}{m_i},$$
$$x^n = \prod_i (x^i)^{n_i}, \qquad p^n = \prod_i (p_i)^{n_i}, \qquad a^n = \prod_i a_i^{n_i},$$
$$X^n = \prod_i X_i^{n_i}, \qquad \partial_x^n = \prod_i \partial_{x^i}^{n_i}, \qquad \partial_p^n = \prod_i \partial_{p_i}^{n_i}.$$

The analogous notation is used in the case when multi-indices are $N\times N$ matrices with natural coefficients, i.e. when $n,m\in M_N(\mathbb{N})$

$$
\begin{aligned}
|n| &= \sum_{i,j} n_{ij}, & n! &= \prod_{i,j} n_{ij}!, & \binom{n}{m} &= \prod_{i,j}\binom{n_{ij}}{m_{ij}},\\
x^n &= \prod_{i,j}(x^i)^{n_{ij}}, & p^n &= \prod_{i,j}(p_i)^{n_{ij}}, & a^n &= \prod_{i,j} a_{ij}^{n_{ij}},\\
X^n &= \prod_{i,j} X_i^{n_{ij}}, & \partial_x^n &= \prod_{i,j}\partial_{x^i}^{n_{ij}}, & \partial_p^n &= \prod_{i,j}\partial_{p_i}^{n_{ij}},
\end{aligned}
$$

where now $a\in M_N(\mathbb{R})$.

The following notations and conventions for the Fourier transform and convolution are used in the paper. For $f\in L^2(\mathbb{R}^{2N})$ the Fourier transform $\mathcal{F}f=g$ and the inverse Fourier transform $\mathcal{F}^{-1}g=f$ are defined by the equations

$$
\begin{aligned}
\mathcal{F}f(\xi,\eta) &:= \frac{1}{(2\pi\hbar)^N}\iint f(x,p)e^{-\frac{i}{\hbar}(\xi_i x^i-\eta^i p_i)}\mathrm{d}x\,\mathrm{d}p\,,\\
\mathcal{F}^{-1}g(x,p) &:= \frac{1}{(2\pi\hbar)^N}\iint g(\xi,\eta)e^{\frac{i}{\hbar}(\xi_i x^i-\eta^i p_i)}\mathrm{d}\xi\,\mathrm{d}\eta\,.
\end{aligned}
$$

Also the partial Fourier transforms $\mathcal{F}_1 f=g$, $\mathcal{F}_2 f=h$ and they inverses $\mathcal{F}_1^{-1}g=f$, $\mathcal{F}_2^{-1}h=f$ are defined by the equations

$$
\begin{aligned}
\mathcal{F}_1 f(p,y) &:= \frac{1}{(2\pi\hbar)^{N/2}}\int f(x,y)e^{-\frac{i}{\hbar}x^i p_i}\mathrm{d}x\,, & \mathcal{F}_1^{-1}g(x,y) &:= \frac{1}{(2\pi\hbar)^{N/2}}\int g(p,y)e^{\frac{i}{\hbar}x^i p_i}\mathrm{d}p\,,\\
\mathcal{F}_2 f(x,p) &:= \frac{1}{(2\pi\hbar)^{N/2}}\int f(x,y)e^{-\frac{i}{\hbar}y^i p_i}\mathrm{d}y\,, & \mathcal{F}_2^{-1}h(x,y) &:= \frac{1}{(2\pi\hbar)^{N/2}}\int h(x,p)e^{\frac{i}{\hbar}y^i p_i}\mathrm{d}p\,.
\end{aligned}
$$

Note that $\mathcal{F}f=\mathcal{F}_1\mathcal{F}_2^{-1}f$ and $\mathcal{F}^{-1}f=\mathcal{F}_1^{-1}\mathcal{F}_2 f$. The partial Fourier transforms $\mathcal{F}_1$, $\mathcal{F}_1^{-1}$, $\mathcal{F}_2$, $\mathcal{F}_2^{-1}$ are also denoted by $\mathcal{F}_x$, $\mathcal{F}_p^{-1}$, $\mathcal{F}_y$, $\mathcal{F}_p^{-1}$. The Fourier transform have the following properties

$$
\begin{aligned}
\mathcal{F}(\partial_x^n\partial_p^m f)(\xi,\eta) &= \left(\frac{i}{\hbar}\xi\right)^n\left(-\frac{i}{\hbar}\eta\right)^m\mathcal{F}f(\xi,\eta),\\
\mathcal{F}^{-1}(\partial_\xi^n\partial_\eta^m g)(x,p) &= \left(-\frac{i}{\hbar}x\right)^n\left(\frac{i}{\hbar}p\right)^m\mathcal{F}^{-1}g(x,p),\\
\mathcal{F}(x^n p^m f)(\xi,\eta) &= (i\hbar\partial_\xi)^n\,(-i\hbar\partial_\eta)^m\,\mathcal{F}f(\xi,\eta),\\
\mathcal{F}^{-1}(\xi^n\eta^m g)(x,p) &= (-i\hbar\partial_x)^n\,(i\hbar\partial_p)^m\,\mathcal{F}^{-1}g(x,p).
\end{aligned}
$$

Moreover, the following convention for Dirac delta distribution is used

$$\begin{aligned}
\delta(x-x_0) &= \frac{1}{(2\pi\hbar)^N}\int e^{\frac{i}{\hbar}\sum_i \xi_i(x^i-x_0^i)}\mathrm{d}\xi\,,\\
\delta(p-p_0) &= \frac{1}{(2\pi\hbar)^N}\int e^{-\frac{i}{\hbar}\sum_i \eta^i(p_i-p_{0i})}\mathrm{d}\eta\,,\\
\delta(\xi-\xi_0) &= \frac{1}{(2\pi\hbar)^N}\int e^{-\frac{i}{\hbar}\sum_i x^i(\xi_i-\xi_{0i})}\mathrm{d}x\,,\\
\delta(\eta-\eta_0) &= \frac{1}{(2\pi\hbar)^N}\int e^{\frac{i}{\hbar}\sum_i p_i(\eta^i-\eta_0^i)}\mathrm{d}p\,.
\end{aligned}$$

A convolution of functions $f, g \in L^2(\mathbb{R}^{2N})$ is denoted by $f * g$ and defined by the equation

$$(f*g)(x,p) := \iint f(x',p')g(x-x',p-p')\mathrm{d}x'\,\mathrm{d}p' \equiv \iint f(x-x',p-p')g(x',p')\mathrm{d}x'\,\mathrm{d}p'\,.$$

There holds the convolution theorem

$$\mathcal{F}(f\cdot g) = \frac{1}{(2\pi\hbar)^N}\mathcal{F}f * \mathcal{F}g, \qquad \mathcal{F}(f*g) = (2\pi\hbar)^N\mathcal{F}f\cdot\mathcal{F}g.$$

The scalar product in some Hilbert space $\mathcal{H}$ is denoted by $\langle\,\cdot\,|\,\cdot\,\rangle_{\mathcal{H}}$. Moreover, the symbol $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the bilinear map defined on the Schwartz space $\mathcal{S}(\mathbb{R}^{2N})$ by the formula

$$\langle f, g\rangle := \iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\,,$$

for $f, g \in \mathcal{S}(\mathbb{R}^{2N})$.

A.2 Proof of Theorem 3.2

Equation (3.9a) follows from the following relation [30]

$$\exp(aXY)(fg) = f\exp(a(XY\otimes 1 + X\otimes Y + Y\otimes X + 1\otimes XY))g$$

valid for commuting vector fields X, Y and $a \in \mathbb{C}$. Using the above relation and the defining equation (3.8) one gets

$$
\begin{aligned}
f \star_{\sigma,\alpha,\beta} g = f \exp &\left(i\hbar\sigma^{ij}(X_iY_j \otimes 1 + X_i \otimes Y_j + Y_j \otimes X_i + 1 \otimes X_iY_j)\right)\\
&\cdot \exp\left(\tfrac{1}{2}\hbar\alpha^{ij}(X_iX_j \otimes 1 + X_i \otimes X_j + X_j \otimes X_i + 1 \otimes X_iX_j)\right)\\
&\cdot \exp\left(\tfrac{1}{2}\hbar\beta^{ij}(Y_iY_j \otimes 1 + Y_i \otimes Y_j + Y_j \otimes Y_i + 1 \otimes Y_iY_j)\right)\\
&\cdot \exp\left(-i\hbar\sigma^{ij}X_iY_j \otimes 1 - \tfrac{1}{2}\hbar\alpha^{ij}X_iX_j \otimes 1 - \tfrac{1}{2}\hbar\beta^{ij}Y_iY_j \otimes 1\right)\\
&\cdot \exp\left(-i\hbar\delta^{ij}Y_i \otimes X_j\right)\\
&\cdot \exp\left(-i\hbar\sigma^{ij}1 \otimes X_iY_j - \tfrac{1}{2}\hbar\alpha^{ij}1 \otimes X_iX_j - \tfrac{1}{2}\hbar\beta^{ij}1 \otimes Y_iY_j\right) g\\
= f \exp &\left(i\hbar\sigma^{ij}X_i \otimes Y_j - i\hbar\bar{\sigma}^{ij}Y_j \otimes X_i + \hbar\alpha^{ij}X_i \otimes X_j + \hbar\beta^{ij}Y_i \otimes Y_j\right) g.
\end{aligned}
$$

Equation (3.9b) follows from (3.9a) as

$$
\begin{aligned}
f \star_{\sigma,\alpha,\beta} g &= f \prod_{i,j} \exp\left(i\hbar\sigma^{ij}\overleftarrow{X}_i\overrightarrow{Y}_j\right)\exp\left(-i\hbar\bar{\sigma}^{ij}\overleftarrow{Y}_i\overrightarrow{X}_j\right)\exp\left(\hbar\alpha^{ij}\overleftarrow{X}_i\overrightarrow{X}_j\right)\\
&\quad\cdot \exp\left(\hbar\beta^{ij}\overleftarrow{Y}_i\overrightarrow{Y}_j\right) g\\
&= f \prod_{i,j} \sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!}(i\hbar\sigma^{ij})^{n_{ij}}\overleftarrow{X}_i^{n_{ij}}\overrightarrow{Y}_j^{n_{ij}} \sum_{m_{ij}=0}^{\infty} \frac{1}{m_{ij}!}(-i\hbar\bar{\sigma}^{ij})^{m_{ij}}\overleftarrow{Y}_i^{m_{ij}}\overrightarrow{X}_j^{m_{ij}}\\
&\quad\cdot \sum_{r_{ij}=0}^{\infty} \frac{1}{r_{ij}!}(\hbar\alpha^{ij})^{r_{ij}}\overleftarrow{X}_i^{r_{ij}}\overrightarrow{X}_j^{r_{ij}} \sum_{s_{ij}=0}^{\infty} \frac{1}{s_{ij}!}(\hbar\beta^{ij})^{s_{ij}}\overleftarrow{Y}_i^{s_{ij}}\overrightarrow{Y}_j^{s_{ij}}\\
&= f \sum_{\substack{n,m,r,s\\ \in M_N(\mathbb{N})}} \prod_{i,j} (-1)^{m_{ij}}(i\hbar)^{n_{ij}+m_{ij}}\hbar^{r_{ij}+s_{ij}} \frac{(\sigma^{ij})^{n_{ij}}(\bar{\sigma}^{ij})^{m_{ij}}(\alpha^{ij})^{r_{ij}}(\beta^{ij})^{s_{ij}}}{n_{ij}!m_{ij}!r_{ij}!s_{ij}!}\\
&\quad\cdot \overleftarrow{X}_i^{n_{ij}+r_{ij}}\overleftarrow{Y}_i^{m_{ij}+s_{ij}}\overrightarrow{X}_j^{m_{ij}+r_{ij}}\overrightarrow{Y}_j^{n_{ij}+s_{ij}} g\\
&= \sum_{\substack{n,m,r,s\\ \in M_N(\mathbb{N})}} (-1)^{|m|}(i\hbar)^{|n|+|m|}\hbar^{|r|+|s|}\frac{\sigma^n\bar{\sigma}^m\alpha^r\beta^s}{n!m!r!s!}(X^{n+r}Y^{m+s}f)(X^{m^T+r^T}Y^{n^T+s^T}g).
\end{aligned}
$$

Equation (3.9c) follows from the fact that a summation over $n, m \in M_N(\mathbb{N})$ can be replaced by a summation over $k \in M_N(\mathbb{N})$ and $m \in M_N(k)$ where $k = m + n$. From this $n = k - m$ and equation (3.9b) can be rewritten in a form

$$
\begin{aligned}
f \star_{\sigma,\alpha,\beta} g = \sum_{k\in M_N(\mathbb{N})} \sum_{m\in M_N(k)} \sum_{r,s\in M_N(\mathbb{N})} &(-1)^{|m|}(i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\sigma^{k-m}\bar{\sigma}^m\alpha^r\beta^s}{(k-m)!m!r!s!}\\
&\cdot (X^{k-m+r}Y^{m+s}f)(X^{m^T+r^T}Y^{k^T-m^T+s^T}g).
\end{aligned}
$$

From the identity

$$\frac{1}{(k-m)!m!} = \frac{1}{k!}\binom{k}{m}$$

previous equation takes a form

$$f \star_{\sigma,\alpha,\beta} g = \sum_{\substack{k,r,s\\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\alpha^r\beta^s}{k!r!s!}\sum_{m\in M_N(k)}\binom{k}{m}\sigma^{k-m}(-\bar{\sigma})^m(X^{k-m+r}Y^{m+s}f)$$
$$\cdot (X^{m^T+r^T}Y^{k^T-m^T+s^T}g).$$

A.3 Proof of Theorem 3.4

From (3.12) it follows that

$$(\hat{q}^n\hat{p}^m)_{\sigma,\alpha,\beta} = (-1)^{|n|}(i\hbar)^{|n|+|m|}\partial_{\xi_1}^{n_1}\ldots\partial_{\xi_N}^{n_N}\partial_{\eta_1}^{m_1}\ldots\partial_{\eta_N}^{m_N}e^{\frac{i}{\hbar}\xi_i\hat{q}^i}e^{-\frac{i}{\hbar}\eta^i\hat{p}_i}$$
$$\cdot e^{-\frac{i}{\hbar}\sum_i\sigma_i\xi_i\eta^i+\frac{1}{2\hbar}\alpha^i\xi_i^2+\frac{1}{2\hbar}\beta_i(\eta^i)^2}\bigg|_{\xi=\eta=0}.$$

Using Leibniz's formula the above equation can be written in a form

$$(\hat{q}^n\hat{p}^m)_{\sigma,\alpha,\beta} = (-1)^{|n|}(i\hbar)^{|n|+|m|}\sum_{k_1=0}^{n_1}\cdots\sum_{k_N=0}^{n_N}\sum_{l_1=0}^{m_1}\cdots\sum_{l_N=0}^{m_N}\binom{n_1}{k_1}\cdots\binom{n_N}{k_N}\binom{m_1}{l_1}\cdots\binom{m_N}{l_N}$$
$$\cdot\left(\partial_{\xi_1}^{k_1}\ldots\partial_{\xi_N}^{k_N}\partial_{\eta_1}^{l_1}\ldots\partial_{\eta_N}^{l_N}e^{-\frac{i}{\hbar}\sum_i\sigma_i\xi_i\eta^i}\right)\left(\partial_{\xi_1}^{n_1-k_1}\ldots\partial_{\xi_N}^{n_N-k_N}e^{\frac{i}{\hbar}\xi_i\hat{q}^i}e^{\frac{1}{2\hbar}\alpha^i\xi_i^2}\right)$$
$$\cdot\left(\partial_{\eta_1}^{m_1-l_1}\ldots\partial_{\eta_N}^{m_N-l_N}e^{-\frac{i}{\hbar}\eta^i\hat{p}_i}e^{\frac{1}{2\hbar}\beta_i(\eta^i)^2}\right)\bigg|_{\xi=\eta=0}. \tag{A.1}$$

The first derivative can be easily calculated giving

$$\partial_{\xi_1}^{k_1}\ldots\partial_{\xi_N}^{k_N}\partial_{\eta_1}^{l_1}\ldots\partial_{\eta_N}^{l_N}e^{-\frac{i}{\hbar}\sum_i\sigma_i\xi_i\eta^i}\bigg|_{\xi=\eta=0} = \prod_{j=1}^{N}\left(-\frac{i}{\hbar}\right)^{k_j}\sigma_j^{k_j}k_j!\delta_{k_jl_j}. \tag{A.2}$$

The second derivative can be rewritten using Leibniz's formula as

$$\partial_{\xi_1}^{n_1-k_1}\ldots\partial_{\xi_N}^{n_N-k_N}e^{\frac{i}{\hbar}\xi_i\hat{q}^i}e^{\frac{1}{2\hbar}\alpha^i\xi_i^2} = \prod_{j=1}^{N}\sum_{r_j=0}^{n_j-k_j}\binom{n_j-k_j}{r_j}\partial_{\xi_j}^{n_j-k_j-r_j}e^{\frac{i}{\hbar}\xi_j\hat{q}^j}\partial_{\xi_j}^{r_j}e^{\frac{1}{2\hbar}\alpha^j\xi_j^2}. \tag{A.3}$$

Calculation of the first derivative from (A.3) gives

$$\partial_{\xi_j}^{n_j-k_j-r_j}e^{\frac{i}{\hbar}\xi_j\hat{q}^j}\bigg|_{\xi=0} = \left(\frac{i}{\hbar}\hat{q}^j\right)^{n_j-k_j-r_j}. \tag{A.4}$$

The second derivative from (A.3) gives

$$\partial^{r_j}_{\xi_j} e^{\frac{1}{2\hbar}\alpha^j \xi_j^2}\Big|_{\xi=0} = \begin{cases} (r_j-1)!!\left(\frac{1}{\hbar}\alpha^j\right)^{r_j/2} & \text{for } r_j \text{ even} \\ 0 & \text{for } r_j \text{ odd} \end{cases} . \tag{A.5}$$

Putting equations (A.4) and (A.5) into (A.3) one receives

$$\begin{aligned} \partial^{n_1-k_1}_{\xi_1} \dots \partial^{n_N-k_N}_{\xi_N} e^{\frac{i}{\hbar}\xi_i \hat{q}^i} e^{\frac{1}{2\hbar}\alpha^i \xi_i^2} = \prod_{j=1}^{N} \sum_{r_j=0}^{[n_j-k_j/2]} & \binom{n_j-k_j}{2r_j}(2r_j-1)!! \\ & \cdot \left(\frac{1}{\hbar}\alpha^j\right)^{r_j} \left(\frac{i}{\hbar}\hat{q}^j\right)^{n_j-k_j-2r_j} . \end{aligned} \tag{A.6}$$

Analogically, calculation of the third derivative from (A.1) gives

$$\begin{aligned} \partial^{m_1-l_1}_{\eta_1} \dots \partial^{m_N-l_N}_{\eta_N} e^{-\frac{i}{\hbar}\eta^i \hat{p}_i} e^{\frac{1}{2\hbar}\beta_i (\eta^i)^2} = \prod_{j=1}^{N} \sum_{s_j=0}^{[m_j-l_j/2]} & \binom{m_j-l_j}{2s_j}(2s_j-1)!! \\ & \cdot \left(\frac{1}{\hbar}\beta_j\right)^{s_j} \left(-\frac{i}{\hbar}\hat{p}_j\right)^{m_j-l_j-2s_j} . \end{aligned} \tag{A.7}$$

Putting equations (A.2), (A.6) and (A.7) into (A.1) gives the searched equation.

A.4 Baker-Campbell-Hausdorff formulae

For two operators $\hat{A}$ and $\hat{B}$ defined on some Hilbert space, such that $[\hat{A},[\hat{A},\hat{B}]] = \text{const}$ and $[\hat{B},[\hat{A},\hat{B}]] = \text{const}$, there holds

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} e^{\frac{1}{6}[\hat{A},[\hat{A},\hat{B}]]+\frac{1}{3}[\hat{B},[\hat{A},\hat{B}]]}, \\ e^{\hat{A}} e^{\hat{B}} &= e^{\hat{B}} e^{\hat{A}} e^{[\hat{A},\hat{B}]} e^{-\frac{1}{2}[\hat{A},[\hat{A},\hat{B}]]-\frac{1}{2}[\hat{B},[\hat{A},\hat{B}]]}. \end{aligned}$$

Above equations are called the Baker-Campbell-Hausdorff formulae.

A.5 Proof of Theorem 3.5

Calculating the Fourier transform of (3.15b) gives

$$
\begin{aligned}
\mathcal{F}(f \star_{\sigma,\alpha,\beta} g)(\xi,\eta) &= \frac{1}{(2\pi\hbar)^N} \sum_{\substack{n,m,r,s \\ \in M_N(\mathbb{N})}} (-1)^{|m|} (i\hbar)^{|n|+|m|} \hbar^{|r|+|s|} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} \\
&\quad \cdot \left(\left(\left(\tfrac{i}{\hbar}\xi\right)^{n+r} \left(-\tfrac{i}{\hbar}\eta\right)^{m+s} \mathcal{F}f \right) * \left(\left(\tfrac{i}{\hbar}\xi\right)^{m^T+r^T} \left(-\tfrac{i}{\hbar}\eta\right)^{n^T+s^T} \mathcal{F}g \right) \right)(\xi,\eta) \\
&= \frac{1}{(2\pi\hbar)^N} \sum_{\substack{n,m,r,s \\ \in M_N(\mathbb{N})}} (-1)^{|m|} (i\hbar)^{|n|+|m|} \hbar^{|r|+|s|} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} \iint \left(\tfrac{i}{\hbar}\xi'\right)^{n+r} \left(-\tfrac{i}{\hbar}\eta'\right)^{m+s} \\
&\quad \cdot \mathcal{F}f(\xi',\eta') \left(\tfrac{i}{\hbar}(\xi-\xi')\right)^{m^T+r^T} \left(-\tfrac{i}{\hbar}(\eta-\eta')\right)^{n^T+s^T} \mathcal{F}g(\xi-\xi',\eta-\eta') \mathrm{d}\xi' \, \mathrm{d}\eta' \\
&= \frac{1}{(2\pi\hbar)^N} \iint \mathcal{F}f(\xi',\eta') \mathcal{F}g(\xi-\xi',\eta-\eta') e^{\frac{i}{\hbar}\sum_{i,j}(\sigma^i_j \xi'_i(\eta^j-\eta'^j) - \bar{\sigma}^j_i \eta'^i(\xi_j-\xi'_j))} \\
&\quad \cdot e^{-\frac{1}{\hbar}\sum_{i,j}(\alpha^{ij}\xi'_i(\xi_j-\xi'_j) + \beta_{ij}\eta'^i(\eta^j-\eta'^j))} \mathrm{d}\xi' \, \mathrm{d}\eta' \,.
\end{aligned}
$$

Now, calculating the inverse Fourier transform of the above equation gives

$$
\begin{aligned}
(f \star_{\sigma,\alpha,\beta} g)(x,p) &= \frac{1}{(2\pi\hbar)^{2N}} \iiiint \mathcal{F}f(\xi',\eta') \mathcal{F}g(\xi-\xi',\eta-\eta') \\
&\quad \cdot e^{\frac{i}{\hbar}\sum_{i,j}(\sigma^i_j \xi'_i(\eta^j-\eta'^j) - \bar{\sigma}^j_i \eta'^i(\xi_j-\xi'_j))} e^{-\frac{1}{\hbar}\sum_{i,j}(\alpha^{ij}\xi'_i(\xi_j-\xi'_j) + \beta_{ij}\eta'^i(\eta^j-\eta'^j))} \\
&\quad \cdot e^{\frac{i}{\hbar}(\xi_i x^i - \eta^i p_i)} \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\xi' \, \mathrm{d}\eta' \,.
\end{aligned}
$$

After the change of coordinates: $\xi' = \xi', \eta' = \eta'$ and $\xi = \xi' + \xi'', \eta = \eta' + \eta''$ the above equation can be written in a form

$$
\begin{aligned}
(f \star_{\sigma,\alpha,\beta} g)(x,p) &= \frac{1}{(2\pi\hbar)^{2N}} \iiiint \mathcal{F}f(\xi',\eta') \mathcal{F}g(\xi'',\eta'') e^{\frac{i}{\hbar}\sum_i \xi''_i(x^i - \bar{\sigma}^i_j \eta'^j + i\alpha^{ij}\xi'_j)} \\
&\quad \cdot e^{-\frac{i}{\hbar}\sum_i \eta''^i(p_i - \sigma^j_i \xi'_j - i\beta_{ij}\eta'^j)} e^{\frac{i}{\hbar}(\xi'_i x^i - \eta'^i p_i)} \mathrm{d}\xi' \, \mathrm{d}\eta' \, \mathrm{d}\xi'' \, \mathrm{d}\eta'' \\
&= \frac{1}{(2\pi\hbar)^N} \iint \mathcal{F}f(\xi',\eta') g(x - \bar{\sigma}\eta' + i\alpha\xi', p - \sigma\xi' - i\beta\eta') \\
&\quad \cdot e^{\frac{i}{\hbar}(\xi'_i x^i - \eta'^i p_i)} \mathrm{d}\xi' \, \mathrm{d}\eta' \\
&= \frac{1}{(2\pi\hbar)^N} \iint f(x + \sigma\eta'' + i\alpha\xi'', p + \bar{\sigma}\xi'' - i\beta\eta'') \mathcal{F}g(\xi'',\eta'') \\
&\quad \cdot e^{\frac{i}{\hbar}(\xi''_i x^i - \eta''^i p_i)} \mathrm{d}\xi'' \, \mathrm{d}\eta'' \,.
\end{aligned}
$$

Now, assume that $\alpha = \beta = 0$ and $\det\sigma \neq 0, \det\bar{\sigma} \neq 0$. With these assumptions equation (3.16) can be written in a form

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^{2N}} \iiiint f(x',p') g(x - \bar{\sigma}\eta', p - \sigma\xi') \cdot e^{-\frac{i}{\hbar}\sum_i (\xi'_i(x'^i - x^i) - \eta'^i(p'_i - p_i))} \mathrm{d}x' \, \mathrm{d}p' \, \mathrm{d}\xi' \, \mathrm{d}\eta' \,. \tag{A.8}$$

Lets change the coordinates from ξ'_i, η'^i to $x''^i = x^i - \bar{\sigma}^i_j \eta'^j$, $p''_i = p_i - \sigma^j_i \xi'_j$. From this follows that $\xi'_i = (\sigma^{-1})^j_i(p_j - p''_j)$, $\eta'^i = (\bar{\sigma}^{-1})^i_j(x^j - x''^j)$. The Jacobian of this transformation is equal

$$J = \begin{pmatrix} 0 & -\sigma^{-1} \\ -\bar{\sigma}^{-1} & 0 \end{pmatrix},$$

hence $|\det J| = |\det(\sigma^{-1}\bar{\sigma}^{-1})| = |\det(\sigma\bar{\sigma})|^{-1}$. After such change of the coordinates (A.8) takes the form

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^{2N}|\det(\sigma\bar{\sigma})|} \iiiint f(x',p') g(x'',p'') e^{\frac{i}{\hbar}\sum_{i,j}(\sigma^{-1})^j_i(x^i - x'^i)(p_j - p''_j)} \cdot e^{-\frac{i}{\hbar}\sum_{i,j}(\bar{\sigma}^{-1})^i_j(p_i - p'_i)(x^j - x''^j)} \mathrm{d}x' \, \mathrm{d}p' \, \mathrm{d}x'' \, \mathrm{d}p'' \,.$$

Now, lets assume that $N = 1$ and $\alpha\beta \neq \sigma\bar{\sigma}$. With these assumptions (3.16) can be written in a form

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^2} \iiiint f(x',p') g(x - \bar{\sigma}\eta' + i\alpha\xi', p - \sigma\xi' - i\beta\eta') \cdot e^{-\frac{i}{\hbar}(\xi'(x'-x) - \eta'(p'-p))} \mathrm{d}x' \, \mathrm{d}p' \, \mathrm{d}\xi' \, \mathrm{d}\eta' \,. \tag{A.9}$$

Lets change the coordinates from ξ', η' to $x'' = x - \bar{\sigma}\eta' + i\alpha\xi'$, $p'' = p - \sigma\xi' - i\beta\eta'$. From this follows that $\xi' = \frac{1}{\alpha\beta - \sigma\bar{\sigma}}(\bar{\sigma}(p'' - p) - i\beta(x'' - x))$, $\eta' = \frac{1}{\alpha\beta - \sigma\bar{\sigma}}(\sigma(x'' - x) + i\alpha(p'' - p))$. The Jacobian of this transformation is equal

$$J = \frac{1}{\alpha\beta - \sigma\bar{\sigma}} \begin{pmatrix} -i\beta & \bar{\sigma} \\ \sigma & i\alpha \end{pmatrix},$$

hence $|\det J| = \frac{1}{|\alpha\beta - \sigma\bar{\sigma}|}$. After such change of the coordinates (A.9) takes the form

$$(f \star_{\sigma,\alpha,\beta} g)(x,p) = \frac{1}{(2\pi\hbar)^2|\alpha\beta - \sigma\bar{\sigma}|} \iiiint f(x',p') g(x'',p'') \exp\biggl(\frac{-i}{\hbar(\alpha\beta - \sigma\bar{\sigma})} \cdot \Bigl((\bar{\sigma}(p'' - p) - i\beta(x'' - x))(x' - x) - (\sigma(x'' - x) + i\alpha(p'' - p))(p' - p)\Bigr)\biggr) \mathrm{d}x' \, \mathrm{d}p' \, \mathrm{d}x'' \, \mathrm{d}p'' \,.$$

A.6 Proof of Theorem 3.6

Integrating (3.15b) and using the fact that derivatives under the integral sign can be rearranged gives

$$\begin{aligned}
\iint (f \star_{\sigma,\alpha,\beta} g)(x,p) \mathrm{d}x \, \mathrm{d}p &= \sum_{\substack{n,m,r,s \\ \in M_N(\mathbb{N})}} (-1)^{|m|} (i\hbar)^{|n|+|m|} \hbar^{|r|+|s|} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} \\
&\quad \cdot \iint (\partial_x^{n+r} \partial_p^{m+s} f)(x,p) (\partial_x^{m^T+r^T} \partial_p^{n^T+s^T} g)(x,p) \mathrm{d}x \, \mathrm{d}p \\
&= \sum_{\substack{n,m,r,s \\ \in M_N(\mathbb{N})}} (-1)^{|m|} (i\hbar)^{|n|+|m|} \hbar^{|r|+|s|} \frac{\sigma^n \bar{\sigma}^m \alpha^r \beta^s}{n!m!r!s!} \\
&\quad \cdot \iint (\partial_x^{m^T+r^T} \partial_p^{n^T+s^T} f)(x,p) (\partial_x^{n+r} \partial_p^{m+s} g)(x,p) \mathrm{d}x \, \mathrm{d}p \\
&= \iint (g \star_{\sigma,\alpha,\beta} f)(x,p) \mathrm{d}x \, \mathrm{d}p \, .
\end{aligned}$$

Now, integrating (3.15c) gives

$$\begin{aligned}
\iint (f \star_{\sigma,\alpha,\beta} g)(x,p) \mathrm{d}x \, \mathrm{d}p &= \sum_{\substack{k,r,s \\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|} \hbar^{|r|+|s|} \frac{\alpha^r \beta^s}{k!r!s!} \sum_{m \in M_N(k)} \binom{k}{m} \sigma^{k-m} (-\bar{\sigma})^m \\
&\quad \cdot \iint (\partial_x^{k-m+r} \partial_p^{m+s} f)(x,p) (\partial_x^{m^T+r^T} \partial_p^{k^T-m^T+s^T} g)(x,p) \mathrm{d}x \, \mathrm{d}p \, .
\end{aligned}$$

Lets assume that σ is a diagonal matrix. In this case all terms of the sum in which k is not a diagonal matrix will vanish. For k diagonal also all $m \in M_N(k)$ are diagonal hence in the previous equation k^T and m^T can be replaced by k and m giving

$$\begin{aligned}
\iint (f \star_{\sigma,\alpha,\beta} g)(x,p)\mathrm{d}x\,\mathrm{d}p &= \sum_{\substack{k,r,s\\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\alpha^r\beta^s}{k!r!s!}\sum_{m\in M_N(k)}\binom{k}{m}\sigma^{k-m}(-\bar{\sigma})^m \\
&\quad \cdot \iint (\partial_x^{k-m+r}\partial_p^{m+s}f)(x,p)(\partial_x^{m+r^T}\partial_p^{k-m+s^T}g)(x,p)\mathrm{d}x\,\mathrm{d}p \\
&= \sum_{\substack{k,r,s\\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\alpha^r\beta^s}{k!r!s!}\sum_{m\in M_N(k)}\binom{k}{m}\sigma^{k-m}(-\bar{\sigma})^m \\
&\quad \cdot \iint (\partial_x^{k+r+r^T}f)(x,p)(\partial_p^{k+s+s^T}g)(x,p)\mathrm{d}x\,\mathrm{d}p \\
&= \sum_{\substack{k,r,s\\ \in M_N(\mathbb{N})}} (i\hbar)^{|k|}\hbar^{|r|+|s|}\frac{\alpha^r\beta^s}{k!r!s!}(\sigma-\bar{\sigma})^k \\
&\quad \cdot \iint (\partial_x^{k+r+r^T}f)(x,p)(\partial_p^{k+s+s^T}g)(x,p)\mathrm{d}x\,\mathrm{d}p\,.
\end{aligned}$$

Now it can be easily seen that for the case when $\sigma^i_j = \frac{1}{2}\delta^i_j$ and $\alpha = \beta = 0$ all terms of the sum in the above equation will vanish except the first term for which $k = r = s = 0$. Hence in this case one receives

$$\iint (f \star_{\sigma,\alpha,\beta} g)(x,p)\mathrm{d}x\,\mathrm{d}p = \iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\,.$$

A.7 Jensen's inequality

Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\Omega,\mu)$ and if $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex, then [43]

$$\varphi\left(\int_\Omega f\mathrm{d}\mu\right) \leq \int_\Omega (\varphi \circ f)\mathrm{d}\mu\,.$$

Above inequality is called the *Jensen's inequality.*

From Jensen's inequality another useful inequality can be derived. Namely, for $f, g \in \mathcal{S}(\mathbb{R}^{2N})$ there holds

$$\left|\iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\right|^2 \leq \iint |g(x,p)|\mathrm{d}x\,\mathrm{d}p \iint |f(x,p)|^2|g(x,p)|\mathrm{d}x\,\mathrm{d}p\,.$$

Indeed, by taking $\Omega = \mathbb{R}^{2N}$, $\mathfrak{M} = \mathfrak{B}(\mathbb{R}^{2N})$, $\varphi(x) = x^2$ and

$$\mathrm{d}\mu(x,p) = \frac{|g(x,p)|\mathrm{d}x\,\mathrm{d}p}{\iint |g(x,p)|\mathrm{d}x\,\mathrm{d}p}$$

from Jensen's inequality there holds

$$\begin{aligned}\left|\iint f(x,p)g(x,p)\mathrm{d}x\,\mathrm{d}p\right|^2 &\leq \left(\iint |f(x,p)||g(x,p)|\mathrm{d}x\,\mathrm{d}p\right)^2\\ &= \left(\iint |g(x,p)|\mathrm{d}x\,\mathrm{d}p\right)^2\left(\iint |f(x,p)|\mathrm{d}\mu(x,p)\right)^2\\ &\leq \left(\iint |g(x,p)|\mathrm{d}x\,\mathrm{d}p\right)^2\iint |f(x,p)|^2\mathrm{d}\mu(x,p)\\ &= \iint |g(x,p)|\mathrm{d}x\,\mathrm{d}p\iint |f(x,p)|^2|g(x,p)|\mathrm{d}x\,\mathrm{d}p\,.\end{aligned}$$

A.8 Proof of Theorem 4.8

For some $\Psi \in \mathcal{H}$ let

$$A_L \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})\Psi = \Phi_L, \tag{A.10a}$$

$$A_R \star_{\sigma,\alpha,\beta} \Psi = A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha},\hat{p}^*_{\bar{\sigma},\beta})\Psi = \Phi_R. \tag{A.10b}$$

For equation (A.10a) let

$$\begin{aligned}\Psi(x,p) &= e^{\frac{1}{2}(a_1)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_1)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_1)_i^j x^i p_j}\Psi_1(x,p),\\ \Phi_L(x,p) &= e^{\frac{1}{2}(a_1)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_1)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_1)_i^j x^i p_j}\Phi_1(x,p)\end{aligned}$$

and for equation (A.10b) respectively

$$\begin{aligned}\Psi(x,p) &= e^{\frac{1}{2}(a_2)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_2)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_2)_i^j x^i p_j}\Psi_2(x,p),\\ \Phi_R(x,p) &= e^{\frac{1}{2}(a_2)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_2)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_2)_i^j x^i p_j}\Phi_2(x,p),\end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are some complex symmetric matrices. Notice that by using the Baker-Campbell-Hausdorff formulae (see Appendix A.4) one finds

$$\begin{aligned}&e^{\frac{i}{\hbar}\xi_i(\hat{q}_{\sigma,\alpha})^i}e^{-\frac{i}{\hbar}\eta^i(\hat{p}_{\sigma,\beta})_i}e^{\frac{1}{2}(a_1)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_1)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_1)_i^j x^i p_j} =\\ &= e^{\frac{1}{2}(a_1)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_1)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_1)_i^j x^i p_j}e^{\frac{i}{\hbar}\xi_i(\hat{Q}_{\sigma,\alpha})^i}e^{-\frac{i}{\hbar}\eta^i(\hat{P}_{\sigma,\beta})_i},\end{aligned}$$

where $(\hat{Q}_{\sigma,\alpha})^i = (\hat{q}_{\sigma,\alpha})^i - (a_1)^{ij}\partial_{x^j} + i\hbar\sigma^i_j(c_1)^j_k x^k + (\hbar\alpha^{ij} - (a_1)^{ij})(c_1)^k_j p_k$, $(\hat{P}_{\sigma,\beta})_i = (\hat{p}_{\sigma,\beta})_i - (b_1)_{ij}\partial_{p_j} - i\hbar\bar{\sigma}^j_i(c_1)^k_j p_k + (\hbar\beta_{ij} - (b_1)_{ij})(c_1)^j_k x^k$, $[(\hat{Q}_{\sigma,\alpha})^i, (\hat{P}_{\sigma,\beta})_j] = i\hbar\delta^i_j$ and similarly

$$\begin{aligned}&e^{\frac{i}{\hbar}\xi_i(\hat{q}^*_{\bar{\sigma},\alpha})^i}e^{-\frac{i}{\hbar}\eta^i(\hat{p}^*_{\bar{\sigma},\beta})_i}e^{\frac{1}{2}(a_2)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_2)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_2)_i^j x^i p_j} =\\ &= e^{\frac{1}{2}(a_2)^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}(b_2)_{ij}\partial_{p_i}\partial_{p_j}}e^{(c_2)_i^j x^i p_j}e^{\frac{i}{\hbar}\xi_i(\hat{Q}^*_{\bar{\sigma},\alpha})^i}e^{-\frac{i}{\hbar}\eta^i(\hat{P}^*_{\bar{\sigma},\beta})_i},\end{aligned}$$

where $(\hat{Q}^*_{\bar{\sigma},\alpha})^i = (\hat{q}^*_{\bar{\sigma},\alpha})^i - (a_2)^{ij}\partial_{x^j} - i\hbar\bar{\sigma}^i_j(c_2)^j_k x^k + (\hbar\alpha^{ij} - (a_2)^{ij})(c_2)^k_j p_k$, $(\hat{P}^*_{\bar{\sigma},\beta})_i = (\hat{p}^*_{\bar{\sigma},\beta})_i - (b_2)_{ij}\partial_{p_j} + i\hbar\sigma^j_i(c_2)^k_j p_k + (\hbar\beta_{ij} - (b_2)_{ij})(c_2)^j_k x^k$, $[(\hat{Q}^*_{\bar{\sigma},\alpha})^i, (\hat{P}^*_{\bar{\sigma},\beta})_j] = -i\hbar\delta^i_j$. Now, using formulae (3.11) and (3.14) one gets from (A.10)

$$A_{\sigma,\alpha,\beta}(\hat{Q}_{\sigma,\alpha}, \hat{P}_{\sigma,\beta})\Psi_1 = \Phi_1, \tag{A.11a}$$

$$A_{\sigma,\alpha,\beta}(\hat{Q}^*_{\bar{\sigma},\alpha}, \hat{P}^*_{\bar{\sigma},\beta})\Psi_2 = \Phi_2, \tag{A.11b}$$

where

$$\begin{aligned}
\hat{Q}_{\sigma,\alpha} &= x + i\hbar\sigma c_1 x + (\hbar\alpha - a_1)c_1 p + i\hbar\sigma\partial_p + (\hbar\alpha - a_1)\partial_x, \\
\hat{P}_{\sigma,\beta} &= p - i\hbar\bar{\sigma} c_1 p + (\hbar\beta - b_1)c_1 x - i\hbar\bar{\sigma}\partial_x + (\hbar\beta - b_1)\partial_p, \\
\hat{Q}^*_{\bar{\sigma},\alpha} &= x - i\hbar\bar{\sigma} c_2 x + (\hbar\alpha - a_2)c_2 p - i\hbar\bar{\sigma}\partial_p + (\hbar\alpha - a_2)\partial_x, \\
\hat{P}^*_{\bar{\sigma},\beta} &= p + i\hbar\sigma c_2 p + (\hbar\beta - b_2)c_2 x + i\hbar\sigma\partial_x + (\hbar\beta - b_2)\partial_p.
\end{aligned}$$

Under the choice $a_1 = a_2 = \hbar\alpha$, $b_1 = b_2 = \hbar\beta$, $c_1 = -\frac{i}{\hbar}\bar{\sigma}^{-1}$ and $c_2 = \frac{i}{\hbar}\sigma^{-1}$ formulae (A.11) take a form

$$A_{\sigma,\alpha,\beta}(\bar{\sigma}^{-1}x + i\hbar\sigma\partial_p, -i\hbar\bar{\sigma}\partial_x)\Psi_1 = \Phi_1, \tag{A.12a}$$

$$A_{\sigma,\alpha,\beta}(\sigma^{-1}x - i\hbar\bar{\sigma}\partial_p, i\hbar\sigma\partial_x)\Psi_2 = \Phi_2. \tag{A.12b}$$

Now, taking the inverse Fourier transform of both equations (A.12) with respect to p variable one gets

$$A_{\sigma,\alpha,\beta}(\bar{\sigma}^{-1}x + \sigma z, -i\hbar\bar{\sigma}\partial_x)\tilde{\Psi}_1(x,z) = \tilde{\Phi}_1(x,z), \tag{A.13a}$$

$$A_{\sigma,\alpha,\beta}(\sigma^{-1}x - \bar{\sigma} z, i\hbar\sigma\partial_x)\tilde{\Psi}_2(x,z) = \tilde{\Phi}_2(x,z). \tag{A.13b}$$

Lets introduce new coordinates

$$\begin{aligned}
\xi &= \bar{\sigma}^{-1}x + \sigma z, & z &= z, & \partial_x &= \bar{\sigma}^{-1}\partial_\xi, \\
\eta &= \sigma^{-1}x - \bar{\sigma} z, & z &= z, & \partial_x &= \sigma^{-1}\partial_\eta,
\end{aligned}$$

then equations (A.13) can be written as

$$A_{\sigma,\alpha,\beta}(\xi, -i\hbar\partial_\xi)\tilde{\Psi}_1(\xi,z) = \tilde{\Phi}_1(\xi,z), \tag{A.14a}$$

$$A_{\sigma,\alpha,\beta}(\eta, i\hbar\partial_\eta)\tilde{\Psi}_2(\eta,z) = \tilde{\Phi}_2(\eta,z). \tag{A.14b}$$

Now, lets restrict functions Ψ to the class for which $\tilde{\Psi}_1(\xi,z) = \varphi_1(\xi)\kappa_1(z)$ and $\tilde{\Psi}_2(\eta,z) = \varphi_2^*(\eta)\kappa_2(z)$, where $\varphi_1, \varphi_2, \kappa_1, \kappa_2 \in L^2(\mathbb{R}^N)$. Then, obviously $\tilde{\Phi}_1(\xi,z) = \psi_1(\xi)\kappa_1(z)$ and

$\tilde{\Phi}_2(\eta, z) = \psi_2^*(\eta)\kappa_2(z)$ for some $\psi_1, \psi_2 \in L^2(\mathbb{R}^N)$ and equations (A.14) reduce to the form

$$
\begin{aligned}
A_{\sigma,\alpha,\beta}(\xi, -i\hbar\partial_\xi)\varphi_1(\xi) &= \psi_1(\xi), \\
A^\dagger_{\sigma,\alpha,\beta}(\eta, -i\hbar\partial_\eta)\varphi_2(\eta) &= \psi_2(\eta),
\end{aligned}
$$

as in $L^2(\mathbb{R}^N)$ $(A_{\sigma,\alpha,\beta}(\eta, i\hbar\partial_\eta))^* = A^\dagger_{\sigma,\alpha,\beta}(\eta, -i\hbar\partial_\eta)$.

Function Ψ can be reconstructed either from Ψ_1 or from Ψ_2. Indeed, from one side one have

$$
\Psi(x,p) = e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} e^{-\frac{i}{\hbar}(\bar{\sigma}^{-1})^j_i x^i p_j} \frac{1}{(2\pi\hbar)^{N/2}} \int \varphi_1(\bar{\sigma}^{-1}x + \sigma z)\kappa_1(z) e^{-\frac{i}{\hbar}p_i z^i} \mathrm{d}z
$$

and hence

$$
\begin{aligned}
\mathcal{F}_p^{-1}\Psi(x,y) &= \frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p) e^{\frac{i}{\hbar}p_i y^i} \mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^N} \int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_1(\bar{\sigma}^{-1}x + \sigma z)\kappa_1(z) \\
&\quad \cdot \int e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} e^{-\frac{i}{\hbar}((\bar{\sigma}^{-1})^j_i x^i + z^j)p_j} e^{-\frac{i}{\hbar}p_i z^i} \mathrm{d}p\, \mathrm{d}z \\
&= \int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_1(\bar{\sigma}^{-1}x + \sigma z)\kappa_1(z) e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j} \delta(z - (y - \bar{\sigma}^{-1}x)) \mathrm{d}z \\
&= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_1(x + \sigma y)\kappa_1(y - \bar{\sigma}^{-1}x) e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j}.
\end{aligned}
$$

From the other side one have

$$
\Psi(x,p) = e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} e^{\frac{i}{\hbar}(\sigma^{-1})^j_i x^i p_j} \frac{1}{(2\pi\hbar)^{N/2}} \int \varphi_2^*(\sigma^{-1}x - \bar{\sigma} z)\kappa_2(z) e^{-\frac{i}{\hbar}p_i z^i} \mathrm{d}z
$$

and hence

$$
\begin{aligned}
\mathcal{F}_p^{-1}\Psi(x,y) &= \frac{1}{(2\pi\hbar)^{N/2}} \int \Psi(x,p) e^{\frac{i}{\hbar}p_i y^i} \mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^N} \int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_2^*(\sigma^{-1}x + \bar{\sigma} z)\kappa_2(z) \\
&\quad \cdot \int e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}} e^{-\frac{i}{\hbar}(-(\sigma^{-1})^j_i x^i + z^j)p_j} e^{-\frac{i}{\hbar}p_i z^i} \mathrm{d}p\, \mathrm{d}z \\
&= \int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_2^*(\sigma^{-1}x + \bar{\sigma} z)\kappa_2(z) e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j} \delta(z - (y + \sigma^{-1}x)) \mathrm{d}z \\
&= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}} \varphi_2^*(x - \bar{\sigma} y)\kappa_2(y + \sigma^{-1}x) e^{-\frac{1}{2\hbar}\beta_{ij}y^i y^j}.
\end{aligned}
$$

Thus,

$$\kappa_1(y-\bar{\sigma}^{-1}x)=\varphi_2^*(x-\bar{\sigma}y), \qquad \kappa_2(y+\sigma^{-1}x)=\varphi_1(x+\sigma y)$$

and

$$\begin{aligned}\Psi(x,p) &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\frac{1}{(2\pi\hbar)^{N/2}}\int e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j}\varphi_1(x+\sigma y)\varphi_2^*(x-\bar{\sigma}y)e^{-\frac{i}{\hbar}p_iy^i}\mathrm{d}y\\ &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}}\frac{1}{(2\pi\hbar)^{N/2}}\int \varphi_1(x+\sigma y)\varphi_2^*(x-\bar{\sigma}y)e^{-\frac{i}{\hbar}p_iy^i}\mathrm{d}y\\ &= (\varphi_2\otimes_{\sigma,\alpha,\beta}\varphi_1)(x,p).\end{aligned}$$

Now,

$$\Phi_L(x,p)=e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}}e^{-\frac{i}{\hbar}(\bar{\sigma}^{-1})^j_ix^ip_j}\frac{1}{(2\pi\hbar)^{N/2}}\int \psi_1(\bar{\sigma}^{-1}x+\sigma z)\kappa_1(z)e^{-\frac{i}{\hbar}p_iz^i}\mathrm{d}z$$

and

$$\begin{aligned}\mathcal{F}_p^{-1}\Phi_L(x,y) &= \frac{1}{(2\pi\hbar)^{N/2}}\int \Phi_L(x,p)e^{\frac{i}{\hbar}p_iy^i}\mathrm{d}p\\ &= \frac{1}{(2\pi\hbar)^{N}}\int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\psi_1(\bar{\sigma}^{-1}x+\sigma z)\kappa_1(z)\\ &\quad\cdot\int e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}}e^{-\frac{i}{\hbar}((\bar{\sigma}^{-1})^j_ix^i+z^j)p_j}e^{-\frac{i}{\hbar}p_iz^i}\mathrm{d}p\,\mathrm{d}z\\ &= \int e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\psi_1(\bar{\sigma}^{-1}x+\sigma z)\kappa_1(z)e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j}\delta(z-(y-\bar{\sigma}^{-1}x))\mathrm{d}z\\ &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\psi_1(x+\sigma y)\kappa_1(y-\bar{\sigma}^{-1}x)e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j}\\ &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\psi_1(x+\sigma y)\varphi_2^*(x-\bar{\sigma}y)e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j},\end{aligned}$$

so,

$$\begin{aligned}\Phi_L(x,p) &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}\frac{1}{(2\pi\hbar)^{N/2}}\int e^{-\frac{1}{2\hbar}\beta_{ij}y^iy^j}\psi_1(x+\sigma y)\varphi_2^*(x-\bar{\sigma}y)e^{-\frac{i}{\hbar}p_iy^i}\mathrm{d}y\\ &= e^{\frac{1}{2}\hbar\alpha^{ij}\partial_{x^i}\partial_{x^j}}e^{\frac{1}{2}\hbar\beta_{ij}\partial_{p_i}\partial_{p_j}}\frac{1}{(2\pi\hbar)^{N/2}}\int \psi_1(x+\sigma y)\varphi_2^*(x-\bar{\sigma}y)e^{-\frac{i}{\hbar}p_iy^i}\mathrm{d}y\\ &= (\varphi_2\otimes_{\sigma,\alpha,\beta}\psi_1)(x,p).\end{aligned}$$

In a similar way one can show that

$$\Phi_R(x,p)=(\psi_2\otimes_{\sigma,\alpha,\beta}\varphi_1)(x,p).$$

Above calculations show that for $\Psi=\varphi_2\otimes_{\sigma,\alpha,\beta}\varphi_1$ one have

$$\begin{aligned}A_L\star_{\sigma,\alpha,\beta}\Psi &= A_{\sigma,\alpha,\beta}(\hat{q}_{\sigma,\alpha},\hat{p}_{\sigma,\beta})\Psi=\varphi_2\otimes_{\sigma,\alpha,\beta}\psi_1=\varphi_2\otimes_{\sigma,\alpha,\beta}A_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi_1,\\ A_R\star_{\sigma,\alpha,\beta}\Psi &= A_{\sigma,\alpha,\beta}(\hat{q}^*_{\bar{\sigma},\alpha},\hat{p}^*_{\bar{\sigma},\beta})\Psi=\psi_2\otimes_{\sigma,\alpha,\beta}\varphi_1=A^\dagger_{\sigma,\alpha,\beta}(\hat{q},\hat{p})\varphi_2\otimes_{\sigma,\alpha,\beta}\varphi_1.\end{aligned}$$

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