A new complementary relation between classical bits and randomness in local part in simulating singlet state

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Recently Leggett's proposal of non-local model generates new interest in simulating the statistics of singlet state. Singlet state statistics can be simulated by 1 bit of classical communication without using any further nonlocal correlation. But, interestingly, singlet state statistics can also be simulated with no classical cost if a non-local box is used. In the first case, the output is completely unbiased whereas in second case outputs are completely random. We suggest a new (possibly) signaling correlation resource which successfully simulates singlet statistics and this result suggests a new complementary relation between required classical bits and randomness in local output when the classical communication is limited by 1 cbit. This result reproduces the above two models of simulation as extreme cases. This also explains why Leggett's non-local model and the model presented by Branciard et.al. should fail to reproduce the statistics of a singlet.

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Violation of Bell's inequality [1] by quantum statistics generated from singlet state implies impossibility of reproducing all quantum results by local hidden variable theory. Then Leggett proposed a non-local hidden variable model with some constraint on local statistics and showed that this model is incompatible with quantum mechanics [2, 3] . The result was further generalized by Branciard et. al [4]. All these results have generated a new interest in simulating singlet statistics by some non-local correlation. In this context, it should be mentioned that if one cbit of communication is allowed, the singlet statistics can be simulated [5]. After this work, quite interestingly, singlet statistics was simulated without communication by using the Popescu-Rorlich (P-R) Box [6]. Recently Colbeck and Renner [7] proved a general result by showing that no non-signalling non-local model can generate statistics of singlet state if the the model has non-trivial local part and this result is deeply related to the simulation problem. This result was further supported by the work of Branciard et. al [4].

Here, in this work, we suggest a general (possibly) signaling correlation which can be seen as convex combination of a correlation with communication capacity of 1 bit and a P-R box. We show that with this type of signalling correlation singlet statistics can be generated. This result suggests a complementary relation between the amount of classical communication required and randomness in the local binary output in the task of simulating singlet correlation with classical communication which is limited by 1 cbit.

To produce our result we consider the following binary input and binary output correlation hereafter designated by S^p;

$$P(ab|xy) = (xy \oplus \delta_{ab})[(a \oplus 1)p + a(1-p)] \qquad (1)$$

where $P(ab|xy)$ is the probability of outputs a and b for inputs x and y and $\frac{1}{2} \leq p \leq 1$ and $\oplus$ represents addition under modulo 2 . Interestingly for $p \neq \frac{1}{2}$, this correlation violate no-signaling condition. In particular, for $p = 1$, this correlation can be used to communicate 1 cbit from Alice to Bob. We designate this correlation by S^{1cbit}. For $p = \frac{1}{2}$, this is P-R box correlation written as P^{NL}. Then it can be easily shown that

$$S^p = (2p-1)S^{1cbit} + 2(1-p)P^{NL} \qquad (2)$$

The protocol for simulating the singlet state by S^p is same as given in [6]. For completeness we briefly describe the protocol. Alice and Bob share the correlation S^p along with shared randomness in the forms of pairs of normalized vectors $\overrightarrow{\lambda_1}$ and $\overrightarrow{\lambda_2}$, randomly and independently distributed over the Poincare sphere. $\overrightarrow{\nu_A}$ and $\overrightarrow{\nu_B}$ denote measurement directions of Alice's and Bob's measurements, respectively. The protocol runs as follows. Alice inputs

$$x = sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_1}) \oplus sgn(\overrightarrow{\nu_B}.\overrightarrow{\lambda_2}) \qquad (3)$$

into the machine (S^p correlation), where

$$sgn(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad (4)$$

She then receives the bit a out of the machine, and outputs

$$A = a \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_1}) \qquad (5)$$

as the simulated measurement outcome. Bob gives the following input into the machine;

$$y = sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_+}) \oplus sgn(\overrightarrow{\nu_B}.\overrightarrow{\lambda_-}) \qquad (6)$$

where $\overrightarrow{\lambda_\pm} = \overrightarrow{\lambda_1} \pm \overrightarrow{\lambda_2}$. After receiving the bit b from the machine, he outputs

$$B = b \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_+}) \oplus 1 \quad (7)$$

One can easily see that this strategy with the correlation S^p simulates singlet correlation given by

$$E(A \oplus B|\overrightarrow{\nu_A}, \overrightarrow{\nu_B}) = \frac{1 + \overrightarrow{\nu_A}.\overrightarrow{\nu_B}}{2} \quad (8)$$

In the same line as in [6],

$$A \oplus B = a \oplus b \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_1}) \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_+}) \oplus 1 \quad (9)$$

Using the correlation S^p we get

$$\begin{aligned} A \oplus B &= [(2p-1)xy + 2(1-p)xy] \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_1}) \\ &\oplus\ sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_+}) \oplus 1 \\ &= xy \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_1}) \oplus sgn(\overrightarrow{\nu_A}.\overrightarrow{\lambda_+}) \oplus 1 \quad (10) \end{aligned}$$

which is identical to the equation (10) in [6] and the result immediately follows.

The S^p correlation used in this model introduces biasness in the local output $R(p)$ which is quantified by Shannon entropy of the outputs for a given input,

$$R(p) = H(p) = -p \log p - (1-p) \log (1-p) \quad (11)$$

Again the amount of bits $C(p)$ that can be communicated from Alice to Bob by using this correlation S^p is quantified by the maximal mutual information between Alice's input and Bob's output (for Bob's input 1) and it can be expressed as

$$C(p) = \max_{Alice's\ \ input} I(x:b) = 1 - H(p) \quad (12)$$

where $I(x:b) = H(x) + H(b|y=1) - H(xb|y=1)$. Hence we see that in simulating singlet statistics, as communication capacity of the correlation resource increases, the randomness of local output decreases and vice-versa. The complementary relation for this model of simulation where the classical communication is limited by 1 cbit can be put as

$$\begin{aligned} &\text{Randomness in local output } + \text{ Communication} \\ &\text{capacity of the resource in use } = R(p) + C(p) = 1 \end{aligned} \quad (13)$$

Obviously one extreme point ($p=1$) generates the Toner-Bacon model [5] and the other extreme point ($P = \frac{1}{2}$) generates the model presented by Cerf et.al [6]. We must tell that in our model of simulation, the results are unbiased for every measurement. If this is not the case, $R(p)$ can be taken as average of Shannon entropy of measurement results over all possible measurements, i.e. $R =< H(a_i) >$, $H(a_i)$ being the Shannon entropy for outcome for measurement along the direction a_i on either Alice's side or Bob's side (in all the models till considered, R has been taken to be same on both side). In this context we conjecture that if there is a model for simulating statistics of singlet state with the help of classical communication limited by 1 cbit, the complementary relation $R + C = 1$ holds true. This conjecture tells that if one wants to simulate the singlet statistics by completely biased output, communication of 1 cbit is necessary.

Now we apply our result to Leggett's model [2, 3]. In Leggett's non-local hidden variable model, the local statistics for a given value of hidden variable has been considered to be same as generated by some completely polarized state and it has been shown that this model does not reproduce singlet statistics. This result has been generalized in [4] where local statistics could be generated by some mixed state. In both these models, the local randomness is not uniform and R has to be calculated by taking average of Shannon entropy of outcomes over all possible measurements performed on a pure polarized state or a mixed polarized state on either side. For a general mixed state $\rho = \frac{1}{2}[I + \lambda \overrightarrow{n}.\overrightarrow{\sigma}]$ with $0 \leq \lambda \leq 1$, the average entropy of output R over all possible polarization measurement is given by

$$\begin{aligned} R =< H(a_i) >= 1 - \frac{\log_2 e}{2\lambda}\{\frac{(1+\lambda)^2}{2}\ln(1+\lambda) \\ -\frac{(1-\lambda)^2}{2}\ln(1-\lambda) - \lambda\} \end{aligned} \quad (14)$$

From this expression one can easily check that for $\lambda \neq 0$, $R < 1$ and the complementary relation tells that both the models should fail as no classical communication is used. Still one may question why there is a successful (non-signaling) non-local model which reproduce singlet statistics for restricted choice of observable [8]. For a given pure polarized state, one can always choose the measurements in a plane of the Poincare sphere which is orthogonal to the direction of polarization and in that case $R = 1$.

Finally, a model with non-uniform biasness for the measurement outcome, the question, whether singlet statistics can be simulated with the assistance of $1 - R$ classical bit remains open.

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Note added-After we finish this work we saw a similar idea presented by Michael J.W. Hall [9].

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