On the emergence of critical behavior in evolving financial networks

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We introduce a simple model for addressing the controversy in the study of financial systems, sometimes taken as brownian-like processes and other as critical systems with fluctuations of arbitrary magnitude. The model considers a collection of economical agents which establish trade connections among them according to basic economical principles properly translated into physical properties and interaction. With our model we are able to reproduce the evolution of macroscopic quantities (indices) and to correctly retrieve the common exponent value characterizing several indices in financial markets, relating it to the underlying topology of connections.

The introduction of the theory for brownian motion in the analysis of financial data in 1973 by Black, Scholes and Merton[1] boosted a new field of research in economy, explaining the price evolution in an organized market. For that, an analogy between the fluctuating forces that a Brownian particle experiences in its interaction with the surrounding environment was introduced corroborating previous studies on portfolio management[2] and establishing a theoretical framework that was extendend from interest rate[3] to credit risk[4] evaluation. More recently, a procedure was introduced[5, 6] for quantitatively describing financial data, namely by explicitly deriving a Fokker-Planck equation for the empirical probability distributions which catches the typical non-Gaussian heavy tails of financial time-series. Such description supported a cascade model for the probability density function deformation accross scales. However, as Mandelbrodt already pointed out in the 60's[7], other typical features are observed in the variation of prices, namely scale invariance behavior, which cannot be explained by means of a cascade model. Therefore, the modelling of financial index dynamics and other complex phenomena taking into account non-Gaussianity and scale invariant behavior has been indicated as an appealing problem to address in the scope of finance analysis and statistical physics[8].

In this Letter we propose a model based on first principles borrowed from Economy which are straightforwardly translated to the physical context, dropping the assumption for the system to be in equilibrium, differently from the Brownian particle approach. Since the main concern in risk management deals with the distribution of drops on financial market indices, we restrict ourselfs to that distribution. We show that using such first principles, we are able to retrieve the typical behaviour of financial indices drops and furthermore to derive quantitatively the exponent characterizing the critical behavior in the system. We also show that this exponent depends linearly on the underlying complex network exponent.

Our model is based in the paradigm of self-organized criticality (SOC)[9] where each local avalanche represents one local economical crash. Such local crashes appear at no particular moment and may cascade into an avalanche of arbitrary

size, i.e. a succession of an arbitrary large number of local economical crashes, taken as a global economical crisis.

The model considers a collection of agents, labelled i and j in Fig. 1, which establish among them bi-directed connections. When agent i delivers labor W_i to agent j (labor connection) it receives in return a payment proportional that labor, $E_j = \alpha_{ij}W_i$ (energy connection). Since the payment E_j is in itself a part of the labor produced by agent j, both E_j and W_i have units of energy.

The factor α_{ij} is a (adimensional) measure of the labor price, defined as

$$\alpha_{ij} = \frac{2}{1 + e^{-(k_{out,i} - k_{in,j})}} \qquad (1)$$

where $k_{out,i}$ and $k_{in,j}$ are the outgoing and incoming 'labor' connections of agent i respectively. With such defini-
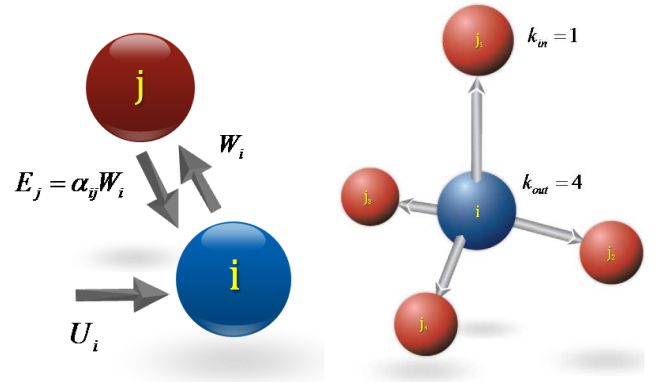

FIG. 1: **Left:** Illustration of an economical connection between two agents i and j. Agent i transfer labor W_i to agent j receiving an energy $E_j = \alpha_{ij}W_i$ where α_{ij} measures how well the labor is rewarded. This interaction attributes to agent i an amount of "internal energy" $U_i = W_i - E_j$ that can be accumulated by the agent up to a threshold U_{th} beyond which it is distributed among the neighbors (see text). **Right:** The transfer of labor is done according to a preferential attachment scheme: the agent prefers to work for agents which have already a large number of labor connections to them.

tion $\alpha_{ij} \in [0,2]$, where the middle value $\alpha_{ij} = 1$ occurs when the labor is payed with the same amount of energy. For $\alpha_{ij} > 1$ the labor of agent i is payed above the amount of energy W_i it delivers, i.e. agent i profits from the trade, whereas for $\alpha_{ij} < 1$ the opposite occurs. According to basic economy principles[12] such over- and underpayments are ruled according to the demand of and supply from a specific agent, which is properly incorporated in our definition in Eq. (1): a large (small) $k_{in,j}$ indicates a large (small) supply for agent i and a large (small) $k_{out,i}$ indicates a large (small) demand of agent i. Thus, the difference $k_{out,i} - k_{in,j}$ measures the balance between the demand of an agent with the supply it has. For the two limit cases of a very large supply and a very large demand, i.e. $k_{in,j} \gg k_{out,i}$ and $k_{in,j} \ll k_{out,i}$, the price α_{ij} approaches the two limiting values, 0 and 2 respectively.

At each agent i, the energy balance implies the introduction of an additional internal energy defined as $U_i = \sum_{out,i}(W_i - E_j) + \sum_{in,i}(E_i - W_j)$. Here, we assume a mean-field approach where all α_{ij} in the system are substituted by the average value $\alpha = \langle \alpha_{ij} \rangle$ yielding for each agent

$$U_i = (1 - \alpha)(k_{out,i} - k_{in,i}). \tag{2}$$

These rules are incorporated in a system of N agents, initially with the agents having a fixed number k_{out} of outgoing connections according to a preferential attachment scheme[16]. Initially the outgoing connections follow a δ-distribution $P_{out}(k) = \delta(k - k_{out})$ and the incoming connections follow a scale-free distribution $P_{in}(k) = k^{-\gamma}$. Behind this topology underlies the empirical observation in economic-like systems that agents are more likely to deliver their work to agents receiving already significantly amount of work[17].

When the system evolves, the number of agents remains constant but at each event-time n one new connection joining two agents is introduced. Both agents are independently chosen according to preferential attachment schemes as above. Consequently, the joint distribution of outgoing and incoming connections follows $P(k_{out}, k_{in}) = k_{out}^{-\gamma_{out}} k_{in}^{-\gamma_{in}}$.

In a single trade, the internal energy of an agent i is either $W_i - E_j$ or $E_i - W_j$ depending if the agent is delivering or receiving labor. Thus, for $u_i < 0$ (resp. $u_i > 0$) the agent increases its asset (resp. liability). Since new connections are being introduced, the price values (α_{ij}) change during the system's evolution, eventually promoting the accumulation of debt in a specific agent. Such accumulation can only exist up to a certain threshold $U_{th,i}$ that depends on how much debt the agent can afford. The ability to accumulate debt is related[18] to how much influence the agent has in the economy. Such influence can be measured by the turnover $T_i = k_{out,i} + k_{in,i}$. Thus, we fixe a threshold $d_{th} = U_{th,i}/T_i$, assumed constant for each agent, and during evolution keep track of the quantity $d_i = U_i/T_i$, we call percentual deficit. One sees that the smaller the percentual deficit of one agent the larger its consumption in comparison to what it produces.

Under these assumption we consider that when $d_i < d_{th}$ the agent collapses and an avalanche takes place. This collapse
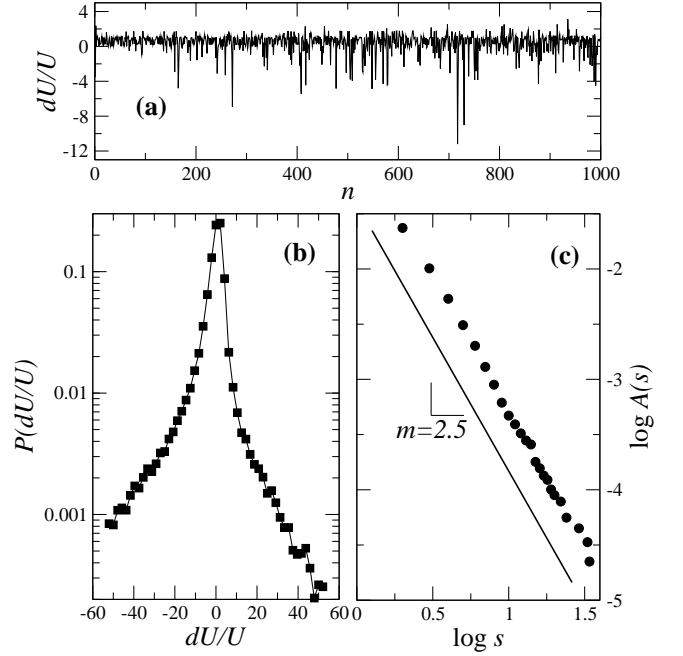

FIG. 2: **(a)** Evolution of the variation of the total internal energy, which shows **(b)** probability density function and **(c)** avalanche size distribution similar to the ones observed for empirical data (compare with Fig. 1).

induces the removal of all the $k_{in,i}$ consumption connections of agent i from the system. Consequently,

$$U_i \rightarrow U_i + (1 - \alpha)k_{in,i} \tag{3a}$$
$$T_i \rightarrow T_i - k_{in,i} \tag{3b}$$
$$U_j \rightarrow U_j - (1 - \alpha) \tag{3c}$$
$$T_j \rightarrow T_j - 1, \tag{3d}$$

where j labels each neighbor of agent i. If the avalanche stops here one has an avalanche of size $s = 1$. If the collapse of the first agent induces neighboring agents to collapse either the size of the avalanche increases.

Figure 2a shows a sketch of the evolution of a typical time-series for the logaritmic returns of the total internal energy, dU/U. As can be seen from Fig. 2b the distribution of the logarithmic returns is non-Gaussian with the heavy tails observed in empirical data[8]. In Fig. 2c the cummulative distribution $A(s)$ of the avalanche size s is plotted showing a power-law whose fit yields $A(s) \sim s^{-m}$ with an exponent $m = 2.51$ ($R^2 = 0.99$) which does not change for systems with different number of agents. To be comparable to empirical series (see below) we consider in our analysis a sampling of data which takes one measure of the original series from the system each five iterations.

The condition for the system to remain at the critical state can be derived as follows. Using the method of Barabási-Albert-Jeong[16] for the iterative procedure of preferential attachment, we can derive the degree distribution as $P(k) \sim$
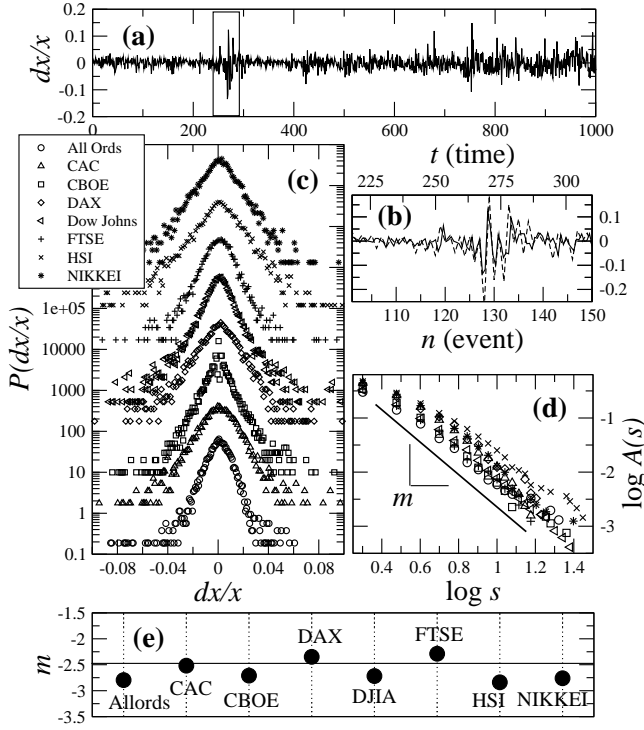
FIG. 3: **(a)** Time evolution of the logaritmic returns of the DJIA index (partial). The box is zoomed in **(b)** to emphasize the constrast between real data (solid line) throught time t and the succession of events (dashed line) where the variation changes sign. **(c)** Probability density functions for some important financial indices, including interest rate options (CBOE). **(d)** Distribution of avalanche sizes detected throught the evolution of each financial index showing critical behavior with **(e)** an exponent m approximatelly invariant and similar to the one obtained for branching processes and to our model (see tex).

$K_0 k^{-\gamma}$, where k is the degree, γ the exponent of the degree distribution and K_0 is the initial number of outgoing connections a node has (here $K_0 = 1$) valid for both the consumption and production networks. Since the system is in a critical state whenever the expected number of neighbors that experience collapse ($d_i < d_{th}$) is one or larger, one needs to derive the probability for a neighboring node to collapse. From Eqs. (3) a collapsing node i has all its consumption connections removed from its $k_{in,i}$ neighbors, producing a reduction in the total number $k_{out,i}$ of production connections for its neighbors. Thus, the collapse of one neighboring node fulfills $k_{out,i} - k_{in,i} > d_{th}(k_{out,i} + k_{in,i})$ and $k_{out,i} - 1 - k_{in,i} \leq d_{th}(k_{out,i} - 1 + k_{in,i})$, yielding

$$\omega k_{in,i} < k_{out,i} \leq \omega k_{in,i} + 1 \qquad (4)$$

with $\omega = \frac{1+d_{th}}{1-d_{th}}$. Taking a collapsing node, the probability P_{br} for a neighbor to also collapse is the probability for the above condition to be fulfilled, i.e. $P_{br} = P(\omega k_{in,i} < k_{out,i} \leq \omega k_{in,i} + 1)$, and since $k_{out,i}$ is an integer one obtains $P_{br} \approx K_0(\omega k_{in,i})^{-\gamma}$. Therefore, assuming that the system cannot consume an infinite quantity of energy, the expected number of collapsing nodes from a starting one is given by the condition

$$\sum_{k_{in,i}=1}^{\infty} k_{in,i} P(k_{in,i}) P_{br} = 1 \qquad (5)$$

which yields $\sum_{k_{in,i}=1}^{\infty} k_{in,i}^{-\gamma} = \left(\frac{\omega}{K_0}\right)^2$, i.e. the system remains in the critical state[15] as

$$\omega^2 = K_0^2 \zeta(\gamma + 1) \qquad (6)$$

where ζ is the Riemann zeta-function. Condition (6) close our model, since ω depends only on d_{th} and $K_0 = 1$ is our 'grain unit'.

The results obtained from our model (Fig. 2) do agree with the analysis done on eight main financial indices, as shown in Fig. 3. Here, the time-series of the logarithmic returns (Fig. 3a) must first be mapped in an series of events as shown in Fig. 3b. The non-Gaussian distribution of the logarithmic returns (Fig. 3c) were extracted from the logarithm returns of the original series of each index, similarly to what is done in Ref. [8]. The characteristic heavy tails observed in [8] are observed for short time lag (hours or smaller), wheres in Fig. 3c the daily closure values are considered. The power-law behavior of the avalanche size (Fig. 3d) is indeed similar to the simulated results. Moreover the exponents m have all approximate values, plotted in Fig. 3e, around the simulated value $m = 2.51$ (solid line).

This exponent can be derived analytically as follows. Being r the number of nodes in the multiplicative process that represents the avalanche and taking the probability P_{br} for a node to be part of it one can write $r = LK_0(\omega k_{in})^{-\gamma}$ where L is the total number of nodes in the system. From r one gets the total number k_{in} of connections involved in the avalanche:

$$k_{in} = \frac{1}{\omega}\left(\frac{r}{K_0 L}\right)^\gamma \qquad (7)$$

From Otter's theorem[11] for branching processes the distribution for r is $P(r) \propto r^{-\frac{3}{2}}$ which leads to a degree distribution $P(k) \propto k^{-\frac{3}{2}\gamma}$ and therefore the avalanche size distribution reads

$$P(k \geq s) \propto \int_s^{+\infty} k^{-\frac{3}{2}\gamma} dk \propto s^{-\frac{3}{2}\gamma+1} \equiv s^{-m}. \qquad (8)$$

Equation (8) relates the exponent characterizing the network topology with the exponent taken from the avalanches, using first principles in economy theory. Since, typically $\gamma \in [2.1, 2.7]$ one arrives to the values of m observed in Fig. (3) and in our model (Fig. 2c).

Figure 4 shows how the overall index dynamic emerges from the underlying network mechanics. As agents connect each other by preferential attachment, the topology of the system is pushed to a power law degree distribution. On the other hand, avalanches push the system away from it. Thus, the system undergoes a structural fluctuation that generates a fat tail
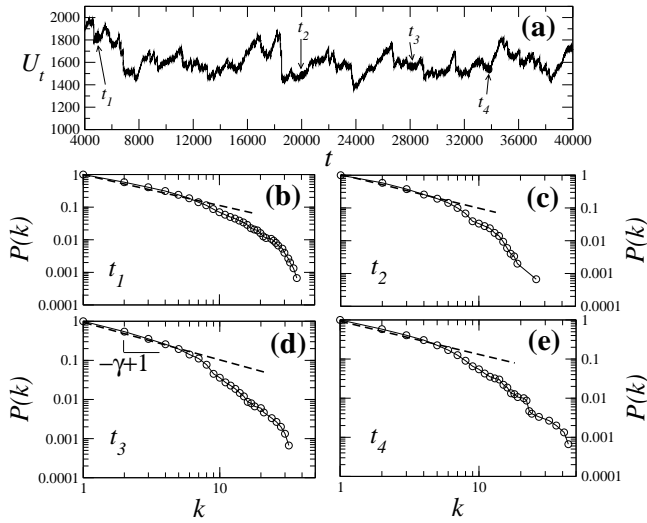
FIG. 4: **(a)** Time series of the total internal energy U_t of the agents. For the four instantes t_1, t_2, t_3 and t_4 we show the observed cummulative degree distribution $P(k^* \geq k) \propto k^{-\gamma+1}$ yielding exponents **(b)** $\gamma = 2.14$, **(c)** $\gamma = 2.25$, **(d)** $\gamma = 2.30$, **(e)** $\gamma = 2.11$, all of them according to the theoretical prediction (check Eq. (8)). The deviations from the power-law for large k are due to the avalanches (crisis) in the system (see text).

distribution of the index, expressed by Eq. (8) and shown in Fig. 3, rather then a gaussian one. Figure 4a shows a typical set of successive U_t values taken from our model. The cummulative degree distribution $P(k)$ at the marked instants t_1-t_4 are shown in Fig. 4b-4e. The dashed lines guide the eye for the scaling behavior observed at low degree. For large degree k the distribution deviates from the power-law, due to the drops of connections for agents experiencing an economical crash. Nonetheless, in every case the slope $-\gamma+1$ of the dashed line yields values in the predicted range, namely [2.1,2.7].

We have showed that, based only on first principles of economic theory and assuming that agents form an open system of economic connections organized by preferential attachment mechanisms, one reaches the distribution of drops observed in financial markets indices, including stocks and interest rate options. Assuming that the preferential attachment mechanism is part of the growing of economic connections, the resulting self-similar topology allow us to assume that all economy has the same topological structure as the sub-economy around financial markets and, thus, market indices are good proxies for the all economy. Since the exponent of the degree distribution is reflected directly on the exponent of the drop magnitude distribution, the diferences between the different indices can be explained by the different social-economic realities beneath them, dispite the fact that, in general, they all have the same behaviour. Our numerical model was based on the same principles to simulate drops on energy and reproduces the drops on the financial indices with an error smaller than the differences between real indices. Interestingly, the exponent $m = 2.5$ appears in other contexts and physical systems, e.g. in Fiber bundle models[19], which raises the question if the financial systems belong to some broader universality class. Further, while the local crashes appear randomly in time, one can argue that with such a model predictions of broad avalanches and economic crisis may be possible, similarly to what was recently demonstrated in granular systems[10].

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[1] F. Black, M. Scholes, J. of Polit. Econ. **81**, 637 (1973); R.C. Merton, Bell J. of Econ. Manag. Sci. **4** (1973).
[2] H.M. Markowitz, The Journal of Finance **7**, 77 (1952).
[3] O. Vasicek, J. of Fin. Economics. **5**, 177 (1977).
[4] O. Vasicek, "Distribution of Loan Portfolio Value" available at http://www.moodyskmv.com/conf04/pdf/papers/dist_loan_port_val.pdf.
[5] R. Friedrich, J. Peinke, and Ch. Renner, Phys. Rev. Lett. **84**, 5224 (2000).
[6] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, Y. Dodge, Nature (London) **381**, 767 (1996).
[7] B. Mandelbrot, R. Hudson, *The (mis)Behavior of Markets - A Fractal View of Risk, Ruin, and Reward*, (Basic Books, USA, 2004).
[8] K. Kiyono, Z.R. Struzik, and Y. Yamamoto, Phys. Rev. Lett. **96**, 068701 (2006).
[9] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59** 381 (1987).
[10] O. Ramos, E. Altshuler, and K.J. Młøy Phys. Rev. Lett. **102**, 078701 (2009).
[11] R. Otter, Annals of Mathematical Statistics **49** 206 (1949).
[12] R. Lipsey and P. Steyner *Economics* (Harper & Row, USA, 1981).
[13] J. von Neumann, *The Rev. Econ. Studies* **13**, 1 (1945).
[14] P. Wilmott, *Quantitative Finance* (John Wiley, England, 2007).
[15] T.E. Harris, *The Theory of Branching Processes* (Springer-Verlag, Germany, 1963).
[16] A.-L. Barabási, R. Albert, Rev. Mod. Phys. **74** 1 (2002).
[17] A. Broder et al., Comp. Networks **33** 309 (2000).
[18] R.C.- Merton, J. Finance **44** 1352 (1973)
[19] M.J. Alava, P.K.V.V. Nukala and S. Zapperi, Adv. Phys. **55** 349 (2006).