Random G-Expectations

Marcel Nutz*

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Abstract

We construct a time-consistent sublinear expectation in the setting of volatility uncertainty. This mapping extends Peng's G-expectation by allowing the range of the volatility uncertainty to be stochastic. Our construction is purely probabilistic and based on an optimal control formulation with path-dependent control sets.

Keywords G-expectation, volatility uncertainty, stochastic domain, risk measure, time-consistency

AMS 2000 Subject Classifications primary 93E20, 91B30; secondary 60H30

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1 Introduction

The so-called G-expectation as introduced by Peng [12, 13] is a dynamic nonlinear expectation which advances the notions of g-expectations (Peng [11]) and backward SDEs (Pardoux and Peng [10]). Moreover, it yields a stochastic representation for a specific PDE and a risk measure for volatility uncertainty in financial mathematics. The concept of volatility uncertainty also plays a key role in the existence theory for second order backward SDEs (Soner et al. [19]) which were introduced as representations for a large class of fully nonlinear second order parabolic PDEs (Cheridito et al. [2]).

The G-expectation is a sublinear operator defined on a class of random variables on the canonical space Ω. Intuitively, it corresponds to the "worst-case" expectation in a model where the volatility of the canonical process B is seen as uncertain, but is postulated to take values in some bounded set D. The symbol G then stands for the support function of D. If $\mathcal{P}^G$ is the set of martingale laws on Ω under which the volatility of B behaves accordingly, the G-expectation at time $t = 0$ may be expressed as the upper expectation

*ETH Zurich, Dept. of Mathematics, CH-8092 Zurich, marcel.nutz@math.ethz.ch

$\mathcal{E}_0^G(X) := \sup_{P \in \mathcal{P}^G} E^P[X]$. This description is due to Denis et al. [5]. See also Denis and Martini [6] for a general study of related capacities.

For positive times t, the G-expectation is extended to a conditional expectation $\mathcal{E}_t^G(X)$ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by B. When $X = f(B_T)$ for some sufficiently regular function f, then $\mathcal{E}_t^G(X)$ is defined via the solution of the nonlinear heat equation $\partial_t u - G(u_{xx}) = 0$ with boundary condition $u|_{t=0} = f$. The mapping $\mathcal{E}_t^G$ can be extended to random variables of the form $X = f(B_{t_1}, \dots, B_{t_n})$ by a stepwise evaluation of the PDE and finally to a suitable completion of the space of all such random variables. As a result, one obtains a family $(\mathcal{E}_t)_{t \geq 0}$ of conditional G-expectations satisfying the semigroup property $\mathcal{E}_s \circ \mathcal{E}_t = \mathcal{E}_s$ for $s \leq t$, also called time-consistency property in this context. For an exhaustive overview of G-expectations and related literature we refer to Peng's recent ICM paper [14] and survey [15].

In this paper, we develop a formulation where the set D is allowed to be path-dependent; i.e., we replace D by a set-valued process $\mathbf{D} = \{\mathbf{D}_t(\omega)\}$. Intuitively, this means that the function $G(\cdot)$ is replaced by a random function $G(t, \omega, \cdot)$ and that the a priori bounds on the volatility can be adjusted to the observed evolution of the system. The latter is highly desirable for applications. Our main result is the existence of a time-consistent family $(\mathcal{E}_t)_{t \geq 0}$ of sublinear operators corresponding to this formulation. When $\mathbf{D}$ depends on ω in a Markovian way, $\mathcal{E}_t$ can be seen as a stochastic representation for a class of state-dependent nonlinear heat equations $\partial_t u - G(x, u_{xx}) = 0$ which are not covered by [19].

At time $t = 0$, we again have a set $\mathcal{P}$ of probability measures and define $\mathcal{E}_0(X) := \sup_{P \in \mathcal{P}} E^P[X]$. For $t > 0$, we want to have

$$\mathcal{E}_t(X) \text{ ``}=\text{'' } \sup_{P \in \mathcal{P}} E^P[X|\mathcal{F}_t] \quad \text{in some sense.} \tag{1.1}$$

The main difficulty here is that the set $\mathcal{P}$ is not dominated by a finite measure. Moreover, as the resulting problem is non-Markovian in an essential way, the PDE approach lined out above seems unfeasible. We shall adopt the framework of regular conditional probability distributions and define, for each $\omega \in \Omega$, a quantity $\mathcal{E}_t(X)(\omega)$ by conditioning X and $\mathbf{D}$ (and hence $\mathcal{P}$) on the path ω up to time t,

$$\mathcal{E}_t(X)(\omega) := \sup_{P \in \mathcal{P}(t,\omega)} E^P[X^{t,\omega}], \quad \omega \in \Omega. \tag{1.2}$$

Then the right hand side is well defined since it is simply a supremum of real numbers. This approach gives direct access to the underlying measures and allows for control theoretic methods. There is no direct reference to the function G, so that G is no longer required to be finite and we can work with an unbounded domain $\mathbf{D}$. The final result is the construction of a random variable $\mathcal{E}_t(X)$ which makes (1.1) rigorous in the form

$$\mathcal{E}_t(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[X|\mathcal{F}_t] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P},$$

where $\mathcal{P}(t,P) = \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_t\}$.

The approach via (1.2) is strongly inspired by the formulation of stochastic target problems in Soner et al. [16]. There, the situation is more nonlinear in the sense that instead of taking conditional expectations on the right hand side, one solves under each P a backward SDE with terminal value X. On the other hand, those problems have (by assumption) a deterministic domain with respect to the volatility, which corresponds to a deterministic set $\mathbf{D}$ in our case, and therefore their control sets are not path-dependent.

The path-dependence of $\mathcal{P}(t,\omega)$ constitutes the main difficulty in the present paper. E.g., it is not obvious under which conditions $\omega \mapsto \mathcal{E}_t(X)(\omega)$ in (1.2) is even measurable. The main problem turns out to be the following. In our formulation, the time-consistency of $(\mathcal{E}_t)_{t\geq 0}$ takes the form of a dynamic programming principle. The proof of such a result generally relies on a pasting operation performed on controls from the various conditional problems. However, we shall see that the resulting control in general violates the constraint given by $\mathbf{D}$, when $\mathbf{D}$ is stochastic. Our construction is tailored such that we can perform the necessary pastings at least on certain well-chosen controls.

The remainder of this paper is organized as follows. Section 2 introduces the basic set up and notation. In Section 3 we formulate the control problem (1.2) for uniformly continuous random variables and introduce a regularity condition on $\mathbf{D}$. Section 4 contains the proof of the dynamic programming principle for this control problem. In Section 5 we extend $\mathcal{E}$ to a suitable completion. Section 6 concludes with a counterexample and some open problems.

2 Preliminaries

We fix a constant $T > 0$ and let $\Omega := \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous paths equipped with the uniform norm $\|\omega\|_T := \sup_{0\leq s\leq T} |\omega_s|$, where $|\cdot|$ is the Euclidean norm. We denote by B the canonical process $B_t(\omega) = \omega_t$, by P_0 the Wiener measure, and by $\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ the raw filtration generated by B. Unless otherwise stated, probabilistic notions requiring a filtration (such as adaptedness) refer to $\mathbb{F}$.

A probability measure P on Ω is called *local martingale measure* if B is a local martingale under P. We recall from Föllmer [7] that the quadratic variation process $\langle B\rangle$ can be defined *pathwise* on all ω which have well-defined quadratic variation, and we set $\langle B\rangle(\omega) \equiv 0$ otherwise. If P is a local martingale measure, B has finite quadratic variation outside a P-nullset and $\langle B\rangle$ coincides up to P-evanescence with the probabilistic quadratic variation process $\langle B\rangle^P$. As a result, $\langle B\rangle$ is the quadratic variation of B simultaneously for all such P. Taking componentwise limits, we can then define the

$\mathbb{F}$-progressively measurable process

$$\hat{a}_t(\omega) := \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\big[\langle B\rangle_t(\omega) - \langle B\rangle_{t-\varepsilon}(\omega)\big], \quad 0 < t \le T$$

taking values in the set of $d \times d$-matrices with entries in the extended real line. We also set $\hat{a}_0 = 0$.

Let $\overline{\mathcal{P}}_W$ be the set of all local martingale measures P such that $t \mapsto \langle B\rangle_t$ is absolutely continuous P-a.s. and $\hat{a}$ takes values in $\mathbb{S}_d^{>0}$ $dt \times P$-a.e., where $\mathbb{S}_d^{>0} \subset \mathbb{R}^{d\times d}$ denotes the set of strictly positive definite matrices. Note that $\hat{a}$ is then the quadratic variation density of B under any $P \in \overline{\mathcal{P}}_W$.

As in [5, 16, 19] we shall use the so-called strong formulation of volatility uncertainty in this paper; i.e., we consider a subclass of $\overline{\mathcal{P}}_W$ consisting of the laws of stochastic integrals with respect to a fixed Brownian motion. The latter is taken to be the canonical process B under P_0: we define $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$ to be the set of laws

$$P^\alpha := P_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := {}^{(P_0)}\!\!\int_0^t \alpha_s^{1/2} dB_s, \quad t \in [0,T]. \tag{2.1}$$

Here α ranges over all $\mathbb{F}$-progressively measurable processes with values in $\mathbb{S}_d^{>0}$ satisfying $\int_0^T |\alpha_t|\, dt < \infty$ P_0-a.s. The stochastic integral is the Itô integral under P_0, constructed as an $\mathbb{F}$-progressively measurable process with right-continuous and P_0-a.s. continuous paths, and in particular without passing to the augmentation of $\mathbb{F}$ (cf. Stroock and Varadhan [20, p. 97]).

2.1 Shifted Paths and Regular Conditional Distributions

We now introduce the notation for the conditional problems of our dynamic programming. Since Ω is the canonical space, we can construct for any probability measure P on Ω and any $(t,\omega) \in [0,T] \times \Omega$ the corresponding regular conditional probability distribution P_t^ω. We refer to [20, Theorem 1.3.4] for the details of the construction but stress that P_t^ω is defined for all ω, without an exceptional set. We recall that P_t^ω is a probability kernel on $\mathcal{F}_t \times \mathcal{F}_T$; i.e., it is a probability measure on $(\Omega, \mathcal{F}_T)$ for fixed ω and $\omega \mapsto P_t^\omega(A)$ is $\mathcal{F}_t$-measurable for each $A \in \mathcal{F}_T$. Moreover, the expectation under P_t^ω is the conditional expectation under P:

$$E^{P_t^\omega}[X] = E^P[X|\mathcal{F}_t](\omega) \quad P\text{-a.s.}$$

whenever X is $\mathcal{F}_T$-measurable and bounded. Finally, P_t^ω is concentrated on the set of paths that coincide with ω up to t,

$$P_t^\omega\big\{\omega' \in \Omega : \omega' = \omega \text{ on } [0,t]\big\} = 1. \tag{2.2}$$

Of course, P_t^ω is not defined uniquely by these properties; for the sequel, we choose and fix one version for each triplet (t, ω, P).

Next, we fix $0 \leq s \leq t \leq T$ and define the following shifted objects. We denote by $\Omega^t := \{\omega \in C([t,T];\mathbb{R}^d) : \omega_t = 0\}$ the shifted canonical space, by B^t the canonical process on Ω^t, by P_0^t the Wiener measure on Ω^t, and by $\mathbb{F}^t = \{\mathcal{F}_u^t\}_{t \leq u \leq T}$ the (raw) filtration generated by B^t. For $\omega \in \Omega^s$, the shifted path $\omega^t \in \Omega^t$ is defined by $\omega_u^t := \omega_u - \omega_t$ for $t \leq u \leq T$ and if furthermore $\tilde{\omega} \in \Omega^t$, then the concatenation of ω and $\tilde{\omega}$ at t is the path

$$(\omega \otimes_t \tilde{\omega})_u := \omega_u \mathbf{1}_{[s,t)}(u) + (\omega_t + \tilde{\omega}_u)\mathbf{1}_{[t,T]}(u), \quad s \leq u \leq T.$$

If $\bar{\omega} \in \Omega$, we note the associativity $\bar{\omega} \otimes_s (\omega \otimes_t \tilde{\omega}) = (\bar{\omega} \otimes_s \omega) \otimes_t \tilde{\omega}$. Given an $\mathcal{F}_T^s$-measurable random variable ξ on Ω^s and $\omega \in \Omega^s$, we define the shifted random variable $\xi^{t,\omega}$ on Ω^t by

$$\xi^{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}), \quad \tilde{\omega} \in \Omega^t.$$

Clearly $\tilde{\omega} \mapsto \xi^{t,\omega}(\tilde{\omega})$ is $\mathcal{F}_T^t$-measurable and $\xi^{t,\omega}$ depends only on the restriction of ω to $[s,t]$. For a random variable ψ on Ω, the associativity of the concatenation yields

$$(\psi^{s,\bar{\omega}})^{t,\omega} = \psi^{t,\bar{\omega}\otimes_s \omega}.$$

We note that for an $\mathbb{F}^s$-progressively measurable process $\{X_u : u \in [s,T]\}$, the shifted process $\{X_u^{t,\omega}, u \in [t,T]\}$ is $\mathbb{F}^t$-progressively measurable. If P is a probability on Ω^s, the measure $P^{t,\omega}$ on $\mathcal{F}_T^t$ defined by

$$P^{t,\omega}(A) := P_t^\omega(\omega \otimes_t A), \quad A \in \mathcal{F}_T^t, \quad \text{where } \omega \otimes_t A := \{\omega \otimes_t \tilde{\omega} : \tilde{\omega} \in A\},$$

is again a probability by (2.2). We then have

$$E^{P^{t,\omega}}[\xi^{t,\omega}] = E^{P_t^\omega}[\xi] = E^P[\xi|\mathcal{F}_t^s](\omega) \quad P\text{-a.s.}$$

In analogy to the above, we also introduce the set $\overline{\mathcal{P}}_W^t$ of martingale measures on Ω^t under which the quadratic variation density process $\hat{a}^t$ of B^t is well defined with values in $\mathbb{S}_d^{>0}$ and the subset $\overline{\mathcal{P}}_S^t \subseteq \overline{\mathcal{P}}_W^t$ induced by (P_0^t, B^t)-stochastic integrals of $\mathbb{F}^t$-progressively measurable integrands. (By convention, $\overline{\mathcal{P}}_S^T = \overline{\mathcal{P}}_W^T$ consists of the unique probability on $\Omega^T = \{0\}$.) Finally, we denote by $\Omega_t^s := \{\omega|_{[s,t]} : \omega \in \Omega^s\}$ the restriction of Ω^s to $[s,t]$ and note that Ω_t^s can be identified with $\{\omega \in \Omega^s : \omega_u = \omega_t \text{ for } u \in [t,T]\}$.

3 Formulation of the Control Problem

We start with a closed set-valued process $\mathbf{D} : \Omega \times [0,T] \to 2^{\mathbb{S}_d^+}$ taking values in the positive semidefinite matrices; i.e., $\mathbf{D}_t(\omega)$ is a closed set of matrices for each $(t,\omega) \in [0,T] \times \Omega$. We assume that $\mathbf{D}$ is progressively measurable in the sense that for every compact $K \subset \mathbb{S}_d^+$ the lower inverse image $\{(t,\omega) : \mathbf{D}_t(\omega) \cap K \neq \emptyset\}$ is a progressively measurable subset of

$[0,T] \times \Omega$. In particular, the value of $\mathbf{D}_t(\omega)$ depends only on the restriction of ω to $[0,t]$.

In our setting with a nondominated set of probabilities, it is crucial to introduce topological regularity. As a first step to obtain some stability, we consider laws under which the quadratic variation density of B takes values in a uniform interior of $\mathbf{D}$. For a set $D \subseteq \mathbb{S}_d^+$ and $\delta > 0$ we define δ-interior $\operatorname{Int}^\delta D := \{x \in D : B_\delta(x) \subseteq D\}$, where $B_\delta(x)$ denotes the open ball of radius δ.

Definition 3.1. Given $(t,\omega) \in [0,T] \times \Omega$, we define $\mathcal{P}(t,\omega)$ to be the collection of all $P \in \overline{\mathcal{P}}_S^t$ for which there exists $\delta = \delta(t,\omega,P) > 0$ such that

$$\hat{a}_s^t(\tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \quad \text{for } ds \times P\text{-a.e. } (s,\tilde{\omega}) \in [t,T] \times \Omega^t.$$

Furthermore, if δ^* denotes the supremum of all such δ, we define the positive quantity $\deg(t,\omega,P) := \delta^*/2$. We note that $\mathcal{P}(0,\omega)$ does not depend on ω and denote this set by $\mathcal{P}$.

The following is the main regularity condition in this paper.

Definition 3.2. We say that $\mathbf{D}$ is *uniformly continuous* if for all $\delta > 0$ and $(t,\omega) \in [0,T] \times \Omega$ there exists $\varepsilon = \varepsilon(t,\omega,\delta) > 0$ such that $\|\omega - \omega'\|_t \leq \varepsilon$ implies

$$\operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \subseteq \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \quad \text{for all } (s,\tilde{\omega}) \in [t,T] \times \Omega^t.$$

Assumption 3.3. We assume throughout that $\mathbf{D}$ is uniformly continuous and such that $\mathcal{P} \neq \emptyset$.

This assumption is in force for the entire paper. We remark that $\mathcal{P} \neq \emptyset$ implies that $\mathcal{P}(t,\omega) \neq \emptyset$ for all $(t,\omega) \in [0,T] \times \Omega$; indeed, we shall see in Lemma 4.1 below that $P^{t,\omega} \in \mathcal{P}(t,\omega)$ for any $P \in \mathcal{P}$.

We now introduce the value function which will play the role of the sublinear (conditional) expectation. We denote by $\mathrm{UC}_b(\Omega)$ the space of bounded uniformly continuous functions on Ω.

Definition 3.4. Given $\xi \in \mathrm{UC}_b(\Omega)$, we define for each $t \in [0,T]$ the value function

$$V_t(\omega) := V_t(\xi)(\omega) := \sup_{P \in \mathcal{P}(t,\omega)} E^P[\xi^{t,\omega}], \quad \omega \in \Omega.$$

Until Section 5, *the function ξ is fixed* and often suppressed in the notation. The following result will guarantee enough separability to prove the dynamic programming principle; it is a direct consequence of the preceding definitions.

Lemma 3.5. *Let $(t,\omega) \in [0,T] \times \Omega$ and $P \in \mathcal{P}(t,\omega)$. Then there exists $\varepsilon = \varepsilon(t,\omega,P) > 0$ such that $P \in \mathcal{P}(t,\omega')$ and $\deg(t,\omega',P) \geq \varepsilon$ whenever $\|\omega - \omega'\|_t \leq \varepsilon$.*

Proof. With $\delta := \deg(t, \omega, P)$ we have

$$\hat{a}_s^t(\tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \quad \text{for } ds \times P\text{-a.e. } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

Let $\varepsilon = \varepsilon(t, \omega, \delta)$ be as in Definition 3.2 and ω' such that $\|\omega - \omega'\|_t \leq \varepsilon$, then $\operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) \subseteq \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega})$ by Assumption 3.3 and hence

$$\hat{a}_s^t(\tilde{\omega}) \in \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \quad \text{for } ds \times P\text{-a.e. } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

That is, $P \in \mathcal{P}(t, \omega')$ and $\deg(t, \omega', P) \geq \varepsilon/2$. □

A first consequence of the preceding lemma is the measurability of V_t. We denote $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega_s|$.

Corollary 3.6. *Let $\xi \in \mathrm{UC}_b(\Omega)$. The value function $\omega \mapsto V_t(\xi)(\omega)$ is lower semicontinuous for $\|\cdot\|_t$ and in particular $\mathcal{F}_t$-measurable.*

Proof. Fix $\omega \in \Omega$ and $P \in \mathcal{P}(t, \omega)$. Since ξ is uniformly continuous, there exists a modulus of continuity $\rho^{(\xi)}$,

$$|\xi(\omega) - \xi(\omega')| \leq \rho^{(\xi)}(\|\omega - \omega'\|_T) \quad \text{for all } \omega, \omega' \in \Omega.$$

It follows that for all $\tilde{\omega} \in \Omega^t$,

$$\begin{aligned} |\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega}(\tilde{\omega})| &= |\xi(\omega \otimes_t \tilde{\omega}) - \xi(\omega' \otimes_t \tilde{\omega})| \\ &\leq \rho^{(\xi)}(\|\omega \otimes_t \tilde{\omega} - \omega' \otimes_t \tilde{\omega}\|_T) \\ &= \rho^{(\xi)}(\|\omega - \omega'\|_t). \end{aligned} \tag{3.1}$$

Consider a sequence (ω^n) such that $\|\omega - \omega^n\|_t \to 0$. The preceding lemma shows that $P \in \mathcal{P}(t, \omega^n)$ for all $n \geq n_0 = n_0(t, \omega, P)$ and thus

$$\begin{aligned} \liminf_{n \to \infty} V_t(\omega^n) &= \liminf_{n \to \infty} \sup_{P' \in \mathcal{P}(t, \omega^n)} E^{P'}[\xi^{t,\omega^n}] \\ &\geq \liminf_{n \to \infty} \left[\sup_{P' \in \mathcal{P}(t, \omega^n)} E^{P'}[\xi^{t,\omega}] - \rho^{(\xi)}(\|\omega - \omega^n\|_t) \right] \\ &= \liminf_{n \to \infty} \sup_{P' \in \mathcal{P}(t, \omega^n)} E^{P'}[\xi^{t,\omega}] \\ &\geq E^P[\xi^{t,\omega}]. \end{aligned}$$

As $P \in \mathcal{P}(t, \omega)$ was arbitrary, we conclude that $\liminf_n V_t(\omega^n) \geq V_t(\omega)$. □

We note that the obtained regularity of V_t is significantly weaker than the uniform continuity of ξ; this is a consequence of the state-dependence in our problem. Indeed, the above proof shows that if $\mathcal{P}(t, \omega)$ is independent of ω, then V_t is again uniformly continuous with the same modulus of continuity as ξ (see also [16]). Similarly, in Peng's construction of the G-expectation, the preservation of Lipschitz-constants arises because the nonlinearity in the underlying PDE has no state-dependence.

Remark 3.7. Since ξ is bounded and continuous, the value function $V_t(\xi)$ remains unchanged if $\mathcal{P}(t,\omega)$ is replaced by its weak closure (in the sense of weak convergence of probability measures). As an application, we show that we retrieve Peng's G-expectation under a nondegeneracy condition.

Given G, we recall from [5, Section 3] that there exists a compact and convex set $D \subset \mathbb{S}_d^+$ such that G is the support function of D and such that $\mathcal{E}_0^G(\psi) = \sup_{P\in\mathcal{P}^G} E^P[\psi]$ for sufficiently regular ψ, where

$$\mathcal{P}^G := \big\{P^\alpha \in \overline{\mathcal{P}}_S :\ \alpha_t(\omega) \in D \quad \text{for } dt \times P_0\text{-a.e. } (t,\omega) \in [0,T]\times\Omega\big\}.$$

We make the additional assumption that D has nonempty interior $\operatorname{Int} D$. In the scalar case $d = 1$, this precisely rules out the trivial case where $\mathcal{E}_0^G$ is an expectation in the usual sense.

We then choose $\mathbf{D} := D$. In this deterministic situation, our formulation boils down to

$$\mathcal{P} = \bigcup_{\delta>0} \big\{P^\alpha \in \overline{\mathcal{P}}_S :\ \alpha_t(\omega) \in \operatorname{Int}^\delta D \quad \text{for } dt \times P_0\text{-a.e. } (t,\omega) \in [0,T]\times\Omega\big\}.$$

Clearly $\mathcal{P} \subset \mathcal{P}^G$, so it remains to show that $\mathcal{P}$ is dense. To this end, fix a point $\alpha^* \in \operatorname{Int} D$ and let $P^\alpha \in \mathcal{P}^G$; i.e., α takes values in D. Then for $0 < \varepsilon < 1$, the process $\alpha^\varepsilon := \varepsilon\alpha^* + (1-\varepsilon)\alpha$ takes values in $\operatorname{Int}^\delta D$ for some $\delta > 0$, due to the fact that the disjoint sets ∂D and $\{\varepsilon\alpha^* + (1-\varepsilon)x :\ x \in D\}$ have positive distance by compactness. We have $P^{\alpha^\varepsilon} \in \mathcal{P}$ and it follows by dominated convergence for stochastic integrals that $P^{\alpha^\varepsilon} \to P^\alpha$ for $\varepsilon \to 0$.

While this shows that we can indeed recover the G-expectation, we should mention that if one wants to treat only deterministic sets $\mathbf{D}$, one can use a much simpler construction than in this paper, and in particular there is no need to use the sets $\operatorname{Int}^\delta \mathbf{D}$ at all.

We close this section by an example where our continuity assumption on $\mathbf{D}$ is satisfied.

Example 3.8. We consider the scalar case $d = 1$. Let $a, b : [0,T]\times\Omega \to \mathbb{R}$ be progressively measurable processes satisfying $0 \le a < b$. Assume that a is uniformly continuous in ω, uniformly in time; i.e., that for all $\delta > 0$ there exists $\varepsilon > 0$ such that

$$\|\omega - \omega'\|_T \le \varepsilon \quad \text{implies} \quad \sup_{0\le s\le T} |a_s(\omega) - a_s(\omega')| \le \delta.$$

Assume that b is uniformly continuous in the same sense. Then the random interval

$$\mathbf{D}_t(\omega) := [a_t(\omega), b_t(\omega)]$$

is uniformly continuous. Indeed, given $\delta > 0$, there exists $\varepsilon' = \varepsilon'(\delta) > 0$ such that $|a_s(\omega) - a_s(\omega')| < \delta/2$ for all $0 \le s \le T$ whenever $\|\omega - \omega'\|_T \le \varepsilon'$, and the same for b. We set $\varepsilon := \varepsilon' \wedge \delta/2$. Then for ω, ω' such that $\|\omega - \omega'\|_t \leq \varepsilon$ we have that $\|\omega \otimes_t \tilde{\omega} - \omega' \otimes_t \tilde{\omega}\|_T = \|\omega - \omega'\|_t \leq \varepsilon$ and hence

$$\begin{aligned}\operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(\tilde{\omega}) &= \big[a_s(\omega \otimes_t \tilde{\omega}) + \delta\,,\, b_s(\omega \otimes_t \tilde{\omega}) - \delta\big] \\ &\subseteq \big[a_s(\omega' \otimes_t \tilde{\omega}) + \varepsilon\,,\, b_s(\omega' \otimes_t \tilde{\omega}) - \varepsilon\big] \\ &= \operatorname{Int}^\varepsilon \mathbf{D}_s^{t,\omega'}(\tilde{\omega}) \quad \text{for all } (s, \tilde{\omega}) \in [t, T] \times \Omega^t.\end{aligned}$$

We note the special case where a and b are functions of the current state; i.e., $a_s(\omega) = \tilde{a}(\omega_s)$ and $b_s(\omega) = \tilde{b}(\omega_s)$ for some uniformly continuous functions $\tilde{a}, \tilde{b} : \mathbb{R} \to \mathbb{R}$. This case corresponds to the state-dependent nonlinear heat equation $\partial_t u - G(x, u_{xx}) = 0$ for $G(x, q) := \sup_{\tilde{a}(x) \leq p \leq \tilde{b}(x)} pq/2$ as mentioned in the introduction.

4 Dynamic Programming

The main goal of this section is to prove the dynamic programming principle for $V_t(\xi)$, which corresponds to the time-consistency property of our sublinear expectation. For the case where $\mathbf{D}$ is deterministic and $V_t(\xi) \in \mathrm{UC}_b(\Omega)$, the relevant arguments were previously given in [16].

4.1 Shifting and Pasting of Measures

As usual, one inequality in the dynamic programming principle will be the consequence of an invariance property of the control sets.

Lemma 4.1 (Invariance). *Let $0 \leq s \leq t \leq T$ and $\bar{\omega} \in \Omega$. If $P \in \mathcal{P}(s, \bar{\omega})$, then*

$$P^{t,\omega} \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega^s.$$

Proof. It is shown in [16, Lemma 4.1] that $P^{t,\omega} \in \overline{\mathcal{P}}_S^t$ and that under $P^{t,\omega}$, the quadratic variation density of B^t coincides with the shift of $\hat{a}^s$:

$$\hat{a}_u^t(\tilde{\omega}) = (\hat{a}_u^s)^{t,\omega}(\tilde{\omega}) \quad \text{for } du \times P^{t,\omega}\text{-a.e. } (u, \tilde{\omega}) \in [t, T] \times \Omega^t \tag{4.1}$$

and P-a.e. $\omega \in \Omega^s$. Let $\delta := \deg(s, \bar{\omega}, P)$, then

$$\hat{a}_u^s(\omega') \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}(\omega') \quad \text{for } du \times P\text{-a.e. } (u, \omega') \in [s, T] \times \Omega^s$$

and hence

$$\hat{a}_u^s(\omega \otimes_t \tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}(\omega \otimes_t \tilde{\omega}) \quad \text{for } du \times P^{t,\omega}\text{-a.e. } (u, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

Now (4.1) shows that for $du \times P^{t,\omega}$-a.e. $(u, \tilde{\omega}) \in [t, T] \times \Omega^t$ we have

$$\hat{a}_u^t(\tilde{\omega}) = (\hat{a}_u^s)^{t,\omega}(\tilde{\omega}) = \hat{a}_u^s(\omega \otimes_t \tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}_u^{s,\bar{\omega}}(\omega \otimes_t \tilde{\omega}) = \operatorname{Int}^\delta \mathbf{D}_u^{t,\bar{\omega} \otimes_s \omega}(\tilde{\omega})$$

for P-a.e. $\omega \in \Omega^s$; i.e., that $P^{t,\omega} \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega)$. □

The dynamic programming principle is intimately related to a stability property of the control sets under a pasting operation. More precisely, it is necessary to collect ε-optimizers from the conditional problems over $\mathcal{P}(t,\omega)$ and construct from them a control in $\mathcal{P}$ (if $s=0$). As a first step, we give a tractable criterion for the admissibility of a control. We recall the process X^α from (2.1) and note that since it has continuous paths P_0-a.s., X^α can be seen as a transformation of the canonical space under the Wiener measure.

Lemma 4.2. *Let $(t,\omega)\in[0,T]\times\Omega$ and $P=P^\alpha\in\overline{\mathcal{P}}_S^t$. Then $P\in\mathcal{P}(t,\omega)$ if and only if there exists $\delta>0$ such that*

$$\alpha_s(\tilde\omega)\in \operatorname{Int}^\delta \mathbf{D}_s^{t,\omega}(X^\alpha(\tilde\omega))\quad \text{for } ds\times P_0^t\text{-a.e. } (s,\tilde\omega)\in[t,T]\times\Omega^t.$$

Proof. We first note that

$$\langle B^t\rangle=\int_t^\cdot \hat a_u^t(B^t)\,du\quad P^\alpha\text{-a.s.}\quad\text{and}\quad d\langle X^\alpha\rangle=\int_t^\cdot \alpha_u(B^t)\,du\quad P_0^t\text{-a.s.}$$

Recalling that $P^\alpha=P_0^t\circ(X^\alpha)^{-1}$, we have that the P^α-distribution of $\big(B^t,\int_t^\cdot \hat a^t(B^t)\,du\big)$ coincides with the P_0^t-distribution of $\big(X^\alpha,\int_t^\cdot \alpha(B^t)\,du\big)$. By definition, $P^\alpha\in\mathcal{P}(t,\omega)$ if and only if there exists $\delta>0$ such that

$$\hat a^t(B^t)\in\operatorname{Int}^\delta\mathbf{D}^{t,\omega}(B^t)\quad ds\times P^\alpha\text{-a.e. on } [t,T]\times\Omega^t,$$

and by the above this is further equivalent to

$$\alpha(B^t)\in\operatorname{Int}^\delta\mathbf{D}^{t,\omega}(X^\alpha)\quad ds\times P_0^t\text{-a.e. on } [t,T]\times\Omega^t.$$

This was the claim. □

To motivate the steps below, we first consider the admissibility of pastings in general. We can paste given measures $P=P^\alpha\in\overline{\mathcal{P}}_S$ and $\hat P=P^{\hat\alpha}\in\overline{\mathcal{P}}_S^t$ at time t to obtain a measure $\bar P$ on Ω and we shall see that $\bar P=P^{\bar\alpha}$ for

$$\bar\alpha_u(\omega)=\mathbf{1}_{[0,t)}(u)\alpha_u(\omega)+\mathbf{1}_{[t,T]}(u)\hat\alpha_u(X^\alpha(\omega)^t).$$

Now assume that $P\in\mathcal{P}$ and $\hat P\in\mathcal{P}(t,\hat\omega)$. By the previous lemma, these constraints may be formulated as $\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^\alpha)$ and $\hat\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^{\hat\alpha})^{t,\hat\omega}$, respectively. If $\mathbf{D}$ is deterministic, we immediately see that $\bar\alpha(\omega)\in\operatorname{Int}^\delta\mathbf{D}$ for all $\omega\in\Omega$ and therefore $\bar P\in\mathcal{P}$. However, in the stochastic case we merely obtain that the constraint on $\bar\alpha(\omega)$ is satisfied for ω such that $X^\alpha(\omega)^t=\hat\omega$. Therefore, we typically have $\bar P\notin\mathcal{P}$.

The idea to circumvent this difficulty is that, due to the formulation chosen in the previous section, there exists a neighborhood $B(\hat\omega)$ of $\hat\omega$ such that $\hat P\in\mathcal{P}(t,\omega')$ for all $\omega'\in B(\hat\omega)$. Therefore, the constraint $\bar\alpha\in\operatorname{Int}^\delta\mathbf{D}(X^{\bar\alpha})$ is verified on the preimage of $B(\hat\omega)$ under X^α. In the next lemma, we exploit the separability of Ω to construct a sequence of $\hat P$'s such that the corresponding neighborhoods cover the space Ω, and in Proposition 4.4 below we shall see how to obtain an admissible pasting from this sequence. We denote $\|\omega\|_{[s,t]}:=\sup_{s\le u\le t}|\omega_u|$.

Lemma 4.3 (**Separability**). *Let $0 \leq s \leq t \leq T$ and $\bar\omega \in \Omega$. Given $\varepsilon > 0$, there exist a sequence $(\hat\omega^i)_{i\geq 1}$ in Ω^s, an $\mathcal{F}^s_t$-measurable partition $(E^i)_{i\geq 1}$ of Ω^s, and a sequence $(P^i)_{i\geq 1}$ in $\overline{\mathcal{P}}^t_S$ such that*

(i) $\|\omega - \hat\omega^i\|_{[s,t]} \leq \varepsilon$ *for all* $\omega \in E^i$,

(ii) $P^i \in \mathcal{P}(t, \bar\omega \otimes_s \omega)$ *for all* $\omega \in E^i$ *and* $\inf_{\omega \in E^i} \deg(t, \bar\omega \otimes_s \omega, P^i) > 0$,

(iii) $V_t(\bar\omega \otimes_s \hat\omega^i) \leq E^{P^i}[\xi^{t,\bar\omega\otimes_s\hat\omega^i}] + \varepsilon$.

Proof. Fix $\varepsilon > 0$ and let $\hat\omega \in \Omega$. By definition of $V_t(\bar\omega \otimes_s \hat\omega)$ there exists $P(\hat\omega) \in \mathcal{P}(t, \bar\omega \otimes_s \hat\omega)$ such that

$$V_t(\hat\omega) \leq E^{P(\hat\omega)}[\xi^{t,\bar\omega\otimes_s\hat\omega}] + \varepsilon.$$

Furthermore, by Lemma 3.5, there exists $\varepsilon(\hat\omega) = \varepsilon(t, \bar\omega \otimes_s \hat\omega, P(\hat\omega)) > 0$ such that $P(\hat\omega) \in \mathcal{P}(t, \bar\omega \otimes_s \omega')$ and $\deg(t, \bar\omega \otimes_s \omega', P(\hat\omega)) \geq \varepsilon(\hat\omega)$ for all $\omega' \in B(\varepsilon(\hat\omega), \hat\omega) \subseteq \Omega^s$. Here $B(\varepsilon, \hat\omega) := \{\omega' \in \Omega^s : \|\hat\omega - \hat\omega'\|_{[s,t]} < \varepsilon\}$ denotes the open $\|\cdot\|_{[s,t]}$-ball. By replacing $\varepsilon(\hat\omega)$ with $\varepsilon(\hat\omega) \wedge \varepsilon$ we may assume that $\varepsilon(\hat\omega) \leq \varepsilon$.

As the above holds for all $\hat\omega \in \Omega^s$, the collection $\{B(\varepsilon(\hat\omega), \hat\omega) : \hat\omega \in \Omega^s\}$ forms an open cover of Ω^s. Since the (quasi-)metric space $(\Omega^s, \|\cdot\|_{[s,t]})$ is separable and therefore Lindelöf, there exists a countable subcover $(B^i)_{i\geq 1}$, where $B^i := B(\varepsilon(\hat\omega^i), \hat\omega^i)$. As a $\|\cdot\|_{[s,t]}$-open set, each B^i is $\mathcal{F}^s_t$-measurable and

$$E^1 := B^1, \quad E^{i+1} := B^{i+1} \setminus (E^1 \cup \cdots \cup E^i), \quad i \geq 1$$

defines a partition of Ω^s. It remains to set $P^i := P(\hat\omega^i)$ and note that $\inf_{\omega \in E^i} \deg(t, \bar\omega \otimes_s \omega, P^i) \geq \varepsilon(\hat\omega^i) > 0$ for each $i \geq 1$. □

For $A \in \mathcal{F}^s_T$ we denote $A^{t,\omega} = \{\tilde\omega \in \Omega^t : \omega \otimes_t \tilde\omega \in A\}$.

Proposition 4.4 (**Pasting**). *Let $0 \leq s \leq t \leq T$, $\bar\omega \in \Omega$ and $P \in \mathcal{P}(s, \bar\omega)$. Let $(E^i)_{0\leq i\leq N}$ be a finite $\mathcal{F}^s_t$-measurable partition of Ω^s. For $1 \leq i \leq N$, assume that $P^i \in \overline{\mathcal{P}}^t_S$ are such that $P^i \in \mathcal{P}(t, \bar\omega \otimes_s \omega)$ for all $\omega \in E^i$ and $\inf_{\omega \in E^i} \deg(t, \bar\omega \otimes_s \omega, P^i) > 0$. Then*

$$\bar P(A) := P(A \cap E^0) + \sum_{i=1}^N E^P\big[P^i(A^{t,\omega})\mathbf{1}_{E^i}(\omega)\big], \quad A \in \mathcal{F}^s_T$$

defines an element of $\mathcal{P}(s, \bar\omega)$. Furthermore,

(i) $\bar P = P$ *on* $\mathcal{F}^s_t$,

(ii) $\bar P^{t,\omega} = P^{t,\omega}$ *for P-a.e.* $\omega \in E^0$,

(iii) $\bar P^{t,\omega} = P^i$ *for P-a.e.* $\omega \in E^i$ *and* $1 \leq i \leq N$.

Proof. We first show that $\bar P \in \mathcal{P}(s, \bar\omega)$. The proof that $\bar P \in \overline{\mathcal{P}}^s_S$ is the same as in [16, Appendix, "Proof of Claim (4.18)"]; the observation made there is that if α, α^i are the $\mathbb{F}^s$- resp. $\mathbb{F}^t$-progressively measurable processes such that $P = P^\alpha$ and $P^i = P^{\alpha^i}$, then $\bar{P} = P^{\bar{\alpha}}$ for $\bar{\alpha}$ defined by

$$\bar{\alpha}_u(\omega) := \mathbf{1}_{[s,t)}(u)\alpha(\omega) + \mathbf{1}_{[t,T]}(u)\Big[\alpha_u(\omega)\mathbf{1}_{E^0}(X^\alpha(\omega)) + \sum_{i=1}^N \alpha^i_u(\omega^t)\mathbf{1}_{E^i}(X^\alpha(\omega))\Big]$$

for $(u,\omega) \in [s,T] \times \Omega^s$. To show that $\bar{P} \in \mathcal{P}(s,\bar{\omega})$, it remains to check that

$$\hat{a}^s_u(\omega) \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar{\omega}}_u(\omega) \quad \text{for } du \times \bar{P}\text{-a.e. } (u,\omega) \in [s,T] \times \Omega^s$$

for some $\delta > 0$. Indeed, this is clear for $s \leq u \leq t$ since both sides are adapted and $\bar{P} = P$ on $\mathcal{F}^s_t$ by (i), which is proved below. In view of Lemma 4.2 it remains to show that

$$\bar{\alpha}_u(\omega) \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar{\omega}}_u(X^{\bar{\alpha}}(\omega)) \quad \text{for } du \times P^s_0\text{-a.e. } (u,\omega) \in [t,T] \times \Omega^s. \tag{4.2}$$

Let $A^i := \{X^\alpha \in E^i\} \in \mathcal{F}^s_t$ for $0 \leq i \leq N$. Note that A^i is defined up to a P^s_0-nullset since X^α is defined as an Itô integral under P^s_0. Let $\omega \in A^0$, then $X^\alpha(\omega) \in E^0$ and thus $\bar{\alpha}_u(\omega) = \alpha_u(\omega)$ for $t \leq u \leq T$. With $\delta^0 := \deg(s,\bar{\omega},P)$, Lemma 4.2 shows that

$$\bar{\alpha}_u(\omega) = \alpha_u(\omega) \in \operatorname{Int}^{\delta^0} \mathbf{D}^{s,\bar{\omega}}_u(X^\alpha(\omega)) = \operatorname{Int}^{\delta^0} \mathbf{D}^{s,\bar{\omega}}_u(X^{\bar{\alpha}}(\omega))$$
$$\text{for } du \times P^s_0\text{-a.e. } (u,\omega) \in [t,T] \times A^0.$$

Next, consider $1 \leq i \leq N$ and $\omega^i \in E^i$. By assumption, $P^i \in \mathcal{P}(t, \bar{\omega} \otimes_s \omega^i)$ and

$$\deg(t, \bar{\omega} \otimes_s \omega^i, P^i) \geq \delta^i := \inf_{\omega \in E^i} \deg(t, \bar{\omega} \otimes_s \omega, P^i) > 0.$$

We set $\delta := \min\{\delta^0, \dots, \delta^N\} > 0$, then Lemma 4.2 yields

$$\alpha^i_u(\tilde{\omega}) \in \operatorname{Int}^\delta \mathbf{D}^{t,\bar{\omega}\otimes_s\omega^i}_u(X^{\alpha^i}(\tilde{\omega})) = \operatorname{Int}^\delta \mathbf{D}^{s,\bar{\omega}}_u(\omega^i \otimes_t X^{\alpha^i}(\tilde{\omega}))$$
$$\text{for } du \times P^t_0\text{-a.e. } (u,\tilde{\omega}) \in [t,T] \times \Omega^t.$$

Now let $\omega \in A^i$ for some $1 \leq i \leq N$. Applying the previous observation with $\omega^i := X^\alpha(\omega) \in E^i$, we deduce that

$$\bar{\alpha}_u(\omega) = \alpha^i_u(\omega^t) \in \operatorname{Int}^\delta \mathbf{D}^{s,\bar{\omega}}_u(X^\alpha(\omega) \otimes_t X^{\alpha^i}(\omega^t)) = \operatorname{Int}^\delta \mathbf{D}^{s,\bar{\omega}}_u(X^{\bar{\alpha}}(\omega))$$
$$\text{for } du \times P^s_0\text{-a.e. } (u,\omega) \in [t,T] \times A^i.$$

More precisely, we have used here the following two facts. Firstly, to pass from $du \times P^t_0$-nullsets to $du \times P^s_0$-nullsets, we have used that if $G \subset \Omega^t$ is a P^t_0-nullset, then $P^s_0\{\omega \in \Omega^s : \omega^t \in G\} = P^t_0(G) = 0$ since the canonical process B^s has P^s_0-independent increments. Secondly, we have used that

$\psi(\omega) := X^{\alpha}(\omega) \otimes_t X^{\alpha^i}(\omega^t) = X^{\bar{\alpha}}(\omega)$ for $\omega \in A^i$. Indeed, for $s \le u < t$ we have $\psi_u(\omega) = X^{\alpha}_u(\omega) = X^{\bar{\alpha}}_u(\omega)$ while for $t \le u \le T$, $\psi_u(\omega)$ equals

$$\int_s^t \!\!{}^{(P_0^s)} \alpha^{1/2}\, dB(\omega) + \int_t^u \!\!{}^{(P_0^t)} (\alpha^i)^{1/2}\, dB^t(\omega^t) = \int_s^u \!\!{}^{(P_0^s)} (\bar{\alpha})^{1/2}\, dB(\omega) = X^{\bar{\alpha}}_u(\omega).$$

As $P_0^s\big[\bigcup_{i=0}^N A^i\big] = 1$, we have proved (4.2) therefore $\bar{P} \in \mathcal{P}(s, \bar{\omega})$.

It remains to show (i)–(iii). These assertions are fairly standard; we include the proofs for completeness.

(i) Let $A \in \mathcal{F}_t^s$, we show that $\bar{P}(A) = P(A)$. Indeed, for $\omega \in \Omega$, the question whether $\omega \in A$ depends only on the restriction of ω to $[s, t]$. Therefore,

$$P^i(A^{t,\omega}) = P^i\{\tilde{\omega} : \omega \otimes_t \tilde{\omega} \in A\} = \mathbf{1}_A(\omega), \quad 1 \le i \le N$$

and thus $\bar{P}(A) = \sum_{i=0}^N E^P[A \cap E^i] = P(A)$.

(ii), (iii) Let $F \in \Omega^t$, we show that

$$\bar{P}^{t,\omega}(F) = P^{t,\omega}(F)\mathbf{1}_{E^0}(\omega) + \sum_{i=1}^N P^i(F)\mathbf{1}_{E^i}(\omega) \quad P\text{-a.s.}$$

Using the definition of conditional expectation and (i), this is equivalent to the following equality for all $\Lambda \in \mathcal{F}_t^s$,

$$\bar{P}\{\omega \in \Lambda : \omega^t \in F\} = P\{\omega \in \Lambda \cap E^0 : \omega^t \in F\} + \sum_{i=1}^N P^i(F)P(\Lambda \cap E^i).$$

For $A := \{\omega \in \Lambda : \omega^t \in F\}$ we have $A^{t,\omega} = \{\tilde{\omega} \in F : \omega \otimes_t \tilde{\omega} \in \Lambda\}$ and since $\Lambda \in \mathcal{F}_t^s$, $A^{t,\omega}$ equals F if $\omega \in \Lambda$ and is empty otherwise. Thus the definition of $\bar{P}$ yields $\bar{P}(A) = P(A \cap E^0) + \sum_{i=1}^N E^P[P^i(F)\mathbf{1}_\Lambda(\omega)\mathbf{1}_{E^i}(\omega)] = P(A \cap E^0) + \sum_{i=1}^N P^i(F)P(\Lambda \cap E^i)$ as desired. □

We remark that the above arguments apply also to a countably infinite partition $(E^i)_{i \ge 1}$, provided that $\inf_{i\ge1} \inf_{\omega \in E^i} \deg(t, \omega, P^i) > 0$. However, this condition is difficult to guarantee. A second observation is that the results of this subsection are based on the regularity property of $\omega \mapsto \mathcal{P}(t, \omega)$ stated in Lemma 3.5, but make no use of the continuity of ξ or the measurability of $V_t(\xi)$.

4.2 Dynamic Programming Principle

We can now prove the key result of this paper. We recall the value function $V_t = V_t(\xi)$ from Definition 3.4 and denote by $\operatorname{ess\,sup}^{(P,\mathcal{F}_s)}$ the essential supremum of a family of $\mathcal{F}_s$-measurable random variables with respect to the collection of $(P, \mathcal{F}_s)$-nullsets.

Theorem 4.5. *Let $0 \leq s \leq t \leq T$. Then*

$$V_s(\omega) = \sup_{P \in \mathcal{P}(s,\omega)} E^P\big[(V_t)^{s,\omega}\big] \quad \text{for all } \omega \in \Omega. \tag{4.3}$$

With $\mathcal{P}(s,P) := \{P' \in \mathcal{P} : P' = P \text{ on } \mathcal{F}_s\}$, we also have

$$V_s = \operatorname*{ess\,sup}^{(P,\mathcal{F}_s)}_{P' \in \mathcal{P}(s,P)} E^{P'}[V_t|\mathcal{F}_s] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P} \tag{4.4}$$

and in particular

$$V_s = \operatorname*{ess\,sup}^{(P,\mathcal{F}_s)}_{P' \in \mathcal{P}(s,P)} E^{P'}[\xi|\mathcal{F}_s] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P}. \tag{4.5}$$

Proof. (i) We first show the inequality "$\leq$" in (4.3). Fix $\bar\omega \in \Omega$ as well as $P \in \mathcal{P}(s,\bar\omega)$. Lemma 4.1 shows that $P^{t,\omega} \in \mathcal{P}(t, \bar\omega \otimes_s \omega)$ for P-a.e. $\omega \in \Omega^s$, yielding the inequality in

$$\begin{aligned} E^{P^{t,\omega}}\big[(\xi^{s,\bar\omega})^{t,\omega}\big] &= E^{P^{t,\omega}}\big[\xi^{t,\bar\omega \otimes_s \omega}\big] \\ &\leq \sup_{P' \in \mathcal{P}(t,\bar\omega\otimes_s\omega)} E^{P'}\big[\xi^{t,\bar\omega\otimes_s\omega}\big] \\ &= V_t(\bar\omega \otimes_s \omega) \\ &= V_t^{s,\bar\omega}(\omega) \quad \text{for } P\text{-a.e. } \omega \in \Omega^s, \end{aligned}$$

where $V_t^{s,\bar\omega} := (V_t)^{s,\bar\omega}$. Since V_t is measurable by Corollary 3.6, we can take $P(d\omega)$-expectations on both sides to obtain that

$$E^P\big[\xi^{s,\bar\omega}\big] = E^P\Big[E^{P^{t,\omega}}\big[(\xi^{s,\bar\omega})^{t,\omega}\big]\Big] \leq E^P\big[V_t^{s,\bar\omega}\big].$$

Thus taking supremum over $P \in \mathcal{P}(s,\bar\omega)$ yields the claim.

(ii) We now show the inequality "$\geq$" in (4.3). Fix $\bar\omega \in \Omega$ and $P \in \mathcal{P}(s,\bar\omega)$ and let $\delta > 0$. We start with a preparatory step.

(ii.a) We claim that there exists a $\|\cdot\|_{[s,t]}$-compact set $E \in \mathcal{F}_t^s$ with $P(E) > 1 - \delta$ such that the restriction

$$V_t^{s,\bar\omega}(\cdot)|_E \quad \text{is uniformly continuous for} \quad \|\cdot\|_{[s,t]}.$$

In particular, there exists then a modulus of continuity $\rho^{(V_t^{s,\bar\omega}|E)}$ such that

$$|V_t^{s,\bar\omega}(\omega) - V_t^{s,\bar\omega}(\omega')| \leq \rho^{(V_t^{s,\bar\omega}|E)}\big(\|\omega - \omega'\|_{[s,t]}\big) \quad \text{for all } \omega, \omega' \in E.$$

Indeed, since P is a Borel measure on the Polish space Ω_t^s, there exists a compact set $K = K(P,\delta) \subset \Omega_t^s$ such that $P(K) > 1 - \delta/2$. As $V_t^{s,\bar\omega}$ is $\mathcal{F}_t^s$-measurable (and thus Borel-measurable as a function on Ω_t^s), there exists by Lusin's theorem a closed set $\Lambda = \Lambda(P,\delta) \subseteq \Omega_t^s$ such that $P(\Lambda) > 1 - \delta/2$ and such that $V_t^{s,\bar\omega}|_\Lambda$ is $\|\cdot\|_{[s,t]}$-continuous. Then $E' := K \cap \Lambda \subset \Omega_t^s$ is compact and hence the restriction of $V_t^{s,\bar\omega}$ to E' is even uniformly continuous. It remains to set $E := \{\omega \in \Omega^s : \omega|_{[s,t]} \in E'\}$.

(ii.b) Let $\varepsilon > 0$. We apply Lemma 4.3 to E (instead of Ω^s) and obtain a sequence $(\hat\omega^i)$ in E, an $\mathcal{F}_t^s$-measurable partition (E^i) of E, and a sequence (P^i) in $\overline{\mathcal{P}}_S^t$ such that

(a) $\|\omega - \hat\omega^i\|_{[s,t]} \leq \varepsilon$ for all $\omega \in E^i$,

(b) $P^i \in \mathcal{P}(t, \bar\omega \otimes_s \omega)$ for all $\omega \in E^i$ and $\inf_{\omega\in E^i} \deg(t, \bar\omega \otimes_s \omega, P^i) > 0$,

(c) $V_t(\bar\omega \otimes_s \hat\omega^i) \leq E^{P^i}[\xi^{t,\bar\omega\otimes_s\hat\omega^i}] + \varepsilon$.

Let $A_N := E^1 \cup \dots \cup E^N$ for $N \geq 1$. In view of (a)–(c), we can apply Proposition 4.4 to the finite partition $\{A_N^c, E^1, \dots, E^N\}$ of Ω^s and obtain a measure $\bar P = \bar P_N \in \mathcal{P}(s, \bar\omega)$ such that

$$\bar P = P \text{ on } \mathcal{F}_t^s \quad \text{and} \quad \bar P^{t,\omega} = \begin{cases} P^{t,\omega} & \text{for } \omega \in A_N^c, \\ P^i & \text{for } \omega \in E^i,\ 1 \leq i \leq N. \end{cases}$$

Since ξ is uniformly continuous, we obtain similarly as in (3.1) that there exists a modulus of continuity $\rho^{(\xi)}$ such that

$$|\xi^{t,\bar\omega\otimes_s\omega} - \xi^{t,\bar\omega\otimes_s\omega'}| \leq \rho^{(\xi)}(\|\omega - \omega'\|_{[s,t]}).$$

Let $\omega \in E^i \subset \Omega^s$ for some $1 \leq i \leq N$. Then using (a) and (c),

$$\begin{aligned} V_t^{s,\bar\omega}(\omega) &\leq V_t^{s,\bar\omega}(\hat\omega^i) + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \\ &\leq E^{P^i}\big[\xi^{t,\bar\omega\otimes_s\hat\omega^i}\big] + \varepsilon + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \\ &\leq E^{P^i}\big[\xi^{t,\bar\omega\otimes_s\omega}\big] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \\ &= E^{\bar P^{t,\omega}}\big[\xi^{t,\bar\omega\otimes_s\omega}\big] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \\ &= E^{\bar P^{t,\omega}}\big[(\xi^{s,\bar\omega})^{t,\omega}\big] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \\ &= E^{\bar P}\big[\xi^{s,\bar\omega}\big|\mathcal{F}_t^s\big](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|E)}(\varepsilon) \end{aligned}$$

for $\bar P$-a.e. (and thus P-a.e.) $\omega \in E^i$. This holds for all $1 \leq i \leq N$. As $P = \bar P$ on $\mathcal{F}_t^s$, taking P-expectations yields

$$E^P[V_t^{s,\bar\omega}\mathbf{1}_{A_N}] \leq E^{\bar P}[\xi^{s,\bar\omega}\mathbf{1}_{A_N}] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|\Lambda^c)}(\varepsilon).$$

Recall that $\bar P = \bar P_N$. Using dominated convergence on the left hand side, and on the right hand side that $\bar P_N(E \setminus A_N) = P(E \setminus A_N) \to 0$ as $N \to \infty$ and that

$$\begin{aligned} E^{\bar P_N}[\xi^{s,\bar\omega}\mathbf{1}_{A_N}] &= E^{\bar P_N}[\xi^{s,\bar\omega}\mathbf{1}_E] - E^{\bar P_N}[\xi^{s,\bar\omega}\mathbf{1}_{E\setminus A_N}] \\ &\leq E^{\bar P_N}[\xi^{s,\bar\omega}\mathbf{1}_E] + \|\xi\|_\infty P_N(E \setminus A_N), \end{aligned} \tag{4.6}$$

we conclude that

$$
\begin{aligned}
E^P[V_t^{s,\bar\omega}\mathbf{1}_E] &\le \limsup_{N\to\infty} E^{\bar P_N}[\xi^{s,\bar\omega}\mathbf{1}_E] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|\Lambda^c)}(\varepsilon)\\
&\le \sup_{P'\in\mathcal{P}(s,\bar\omega,t,P)} E^{P'}[\xi^{s,\bar\omega}\mathbf{1}_E] + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t^{s,\bar\omega}|\Lambda^c)}(\varepsilon),
\end{aligned}
$$

where $\mathcal{P}(s,\bar\omega,t,P) := \{P' \in \mathcal{P}(s,\bar\omega) : P' = P \text{ on } \mathcal{F}_t^s\}$. As $\varepsilon > 0$ was arbitrary, this shows that

$$
E^P[V_t^{s,\bar\omega}\mathbf{1}_E] \le \sup_{P'\in\mathcal{P}(s,\bar\omega,t,P)} E^{P'}[\xi^{s,\bar\omega}\mathbf{1}_E].
$$

Finally, since $P'(E) = P(E) > 1-\delta$ for all $P' \in \mathcal{P}(s,\bar\omega,t,P)$ and $\delta > 0$ was arbitrary, we obtain by an argument similar to (4.6) that

$$
E^P[V_t^{s,\bar\omega}] \le \sup_{P'\in\mathcal{P}(s,\bar\omega,t,P)} E^{P'}[\xi^{s,\bar\omega}] \le \sup_{P'\in\mathcal{P}(s,\bar\omega)} E^{P'}[\xi^{s,\bar\omega}] = V_s(\bar\omega).
$$

The claim follows as $P \in \mathcal{P}(s,\bar\omega)$ was arbitrary. This completes the proof of (4.3).

(iii) The next step is to prove that

$$
V_t \le \operatorname*{ess\,sup}_{P'\in\mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[\xi|\mathcal{F}_t] \quad P\text{-a.s.} \quad \text{for all } P\in\mathcal{P}. \tag{4.7}
$$

Fix $P \in \mathcal{P}$. We use the previous step (ii) for the special case $s = 0$ and obtain that for given $\varepsilon > 0$ there exists for each $N \ge 1$ a measure $\bar P_N \in \mathcal{P}(t,P)$ such that

$$
V_t(\omega) \le E^{\bar P_N}[\xi|\mathcal{F}_t](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t|E)}(\varepsilon) \quad \text{for } P\text{-a.s. } \omega \in E^1\cup\dots\cup E^N.
$$

Therefore, since $E = \bigcup_{i\ge 1} E^i$, we have

$$
V_t(\omega) \le \sup_{N\ge 1} E^{\bar P_N}[\xi|\mathcal{F}_t](\omega) + \rho^{(\xi)}(\varepsilon) + \varepsilon + \rho^{(V_t|E)}(\varepsilon) \quad \text{for } P\text{-a.s. } \omega \in E.
$$

We recall that the set E depends on δ, but not on ε. Thus letting $\varepsilon \to 0$ yields

$$
V_t\mathbf{1}_E \le \operatorname*{ess\,sup}_{P'\in\mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} \Big(E^{P'}[\xi|\mathcal{F}_t]\mathbf{1}_E\Big) = \bigg(\operatorname*{ess\,sup}_{P'\in\mathcal{P}(t,P)}{}^{(P,\mathcal{F}_t)} E^{P'}[\xi|\mathcal{F}_t]\bigg)\mathbf{1}_E \quad P\text{-a.s.},
$$

where we have used that $E \in \mathcal{F}_t$. In view of $P(E) > 1-\delta$, the claim follows by taking the limit $\delta \to 0$.

(iv) We now prove the inequality "$\le$" in (4.4); we shall reduce this claim to its special case (4.7). Fix $P \in \mathcal{P}$. For any $P' \in \mathcal{P}(s,P)$ we have that $(P')^{t,\omega} \in \mathcal{P}(t,\omega)$ for P'-a.s. $\omega \in \Omega$ by Lemma 4.1. Thus we can infer from (4.3), applied with $s := t$ and $t := T$, that

$$
V_t(\omega) \ge E^{(P')^{t,\omega}}[\xi^{t,\omega}] = E^{P'}[\xi|\mathcal{F}_t](\omega) \quad P'\text{-a.s.}
$$

and in particular that $E^{P'}[V_t|\mathcal{F}_s] \geq E^{P'}[\xi|\mathcal{F}_s]$ P'-a.s. on $\mathcal{F}_s$, hence also P-a.s. This shows that

$$\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[V_t|\mathcal{F}_s] \geq \operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\xi|\mathcal{F}_s] \quad P\text{-a.s.}$$

But (4.7), applied with s instead of t, yields that the right hand side P-a.s. dominates V_s. This proves the claim.

(v) It remains to show the inequality "$\geq$" in (4.4). Fix $P \in \mathcal{P}$ and $P' \in \mathcal{P}(s, P)$. Since $(P')^{s,\omega} \in \mathcal{P}(s, \omega)$ for P'-a.s. $\omega \in \Omega$ by Lemma 4.1, (4.3) yields

$$V_s(\omega) \geq E^{(P')^{s,\omega}}[V_t^{s,\omega}] = E^{P'}[V_t|\mathcal{F}_s](\omega)$$

P'-a.s. on $\mathcal{F}_s$ and hence also P-a.s. The claim follows as $P' \in \mathcal{P}(s, P)$ was arbitrary. □

5 Extension to the Completion

So far, we have studied the value function $V_t = V_t(\xi)$ for $\xi \in \mathrm{UC}_b(\Omega)$. We now set $\mathcal{E}_t(\xi) := V_t$ and extend this operator to a completion of $\mathrm{UC}_b(\Omega)$ by the usual procedure. The main result in this section is that the dynamic programming principle carries over to the extension.

Given $p \in [1, \infty)$ and $t \in [0, T]$, we define $L^p_{\mathcal{P}}(\mathcal{F}_t)$ to be the space of $\mathcal{F}_t$-measurable random variables X satisfying

$$\|X\|_{L^p_{\mathcal{P}}} := \sup_{P\in\mathcal{P}} \|X\|_{L^p(P)} < \infty,$$

where $\|X\|^p_{L^p(P)} := E[|X|^p]$. More precisely, we take equivalences classes with respect to $\mathcal{P}$-quasi-sure equality so that $L^p_{\mathcal{P}}(\mathcal{F}_t)$ becomes a Banach space. (Two functions are equal $\mathcal{P}$-quasi-surely, $\mathcal{P}$-q.s. for short, if they are equal P-a.s. for all $P \in \mathcal{P}$.) Furthermore,

$$\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t) \text{ is defined as the } \|\cdot\|_{L^p_{\mathcal{P}}}\text{-closure of } \mathrm{UC}_b(\Omega_t) \subseteq L^p_{\mathcal{P}}(\mathcal{F}_t).$$

For brevity, we shall sometimes write $\mathbb{L}^p_{\mathcal{P}}$ for $\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_T)$ and $L^p_{\mathcal{P}}$ for $L^p_{\mathcal{P}}(\mathcal{F}_T)$.

Lemma 5.1. *Let $p \in [1, \infty)$. The mapping $\mathcal{E}_t$ on $\mathrm{UC}_b(\Omega)$ is 1-Lipschitz for the norm $\|\cdot\|_{L^p_{\mathcal{P}}}$,*

$$\|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)\|_{L^p_{\mathcal{P}}} \leq \|\xi - \psi\|_{L^p_{\mathcal{P}}} \quad \text{for all } \xi, \psi \in \mathrm{UC}_b(\Omega).$$

As a consequence, $\mathcal{E}_t$ uniquely extends to a Lipschitz-continuous mapping

$$\mathcal{E}_t : \mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_T) \to L^p_{\mathcal{P}}(\mathcal{F}_t).$$

Proof. Note that $|\xi - \psi|^p$ is again in $\mathrm{UC}_b(\Omega)$. The definition of $\mathcal{E}_t$ and Jensen's inequality imply that $|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)|^p \leq \mathcal{E}_t(|\xi - \psi|)^p \leq \mathcal{E}_t(|\xi - \psi|^p)$. Therefore,

$$\|\mathcal{E}_t(\xi) - \mathcal{E}_t(\psi)\|_{L^p_{\mathcal{P}}} \leq \sup_{P \in \mathcal{P}} E^P\big[\mathcal{E}_t(|\xi - \psi|^p)\big]^{1/p} = \sup_{P \in \mathcal{P}} E^P[|\xi - \psi|^p]^{1/p},$$

where the equality is due to (4.3). □

Since we shall use $\mathbb{L}^p_{\mathcal{P}}$ as the domain of $\mathcal{E}_t$, we also give an explicit description of this space. We say that (an equivalence class) $X \in L^1_{\mathcal{P}}$ is $\mathcal{P}$*-quasi uniformly continuous* if X has a representative X' with the property that for all $\varepsilon > 0$ there exists an open set $G \subset \Omega$ such that $P(G) < \varepsilon$ for all $P \in \mathcal{P}$ and such that the restriction $X'|_{\Omega \setminus G}$ is uniformly continuous. We define $\mathcal{P}$-quasi continuity in an analogous way and denote by $C_b(\Omega)$ the space of bounded continuous functions on Ω. The following is very similar to the results in [5].

Proposition 5.2. *Let $p \in [1, \infty)$. The space $\mathbb{L}^p_{\mathcal{P}}$ consists of all $X \in L^p_{\mathcal{P}}$ such that X is $\mathcal{P}$-quasi uniformly continuous and $\lim_{n\to\infty} \|X\mathbf{1}_{\{|X| \geq n\}}\|_{L^p_{\mathcal{P}}} = 0$.*

If $\mathbf{D}$ is uniformly bounded, then $\mathbb{L}^p_{\mathcal{P}}$ coincides with the $\|\cdot\|_{L^p_{\mathcal{P}}}$-closure of $C_b(\Omega) \subset L^p_{\mathcal{P}}$ and "uniformly continuous" can be replaced by "continuous".

Proof. For the first part, it suffices to go through the proof of [5, Theorem 25] and replace continuity by uniform continuity everywhere. The only difference is that one has to use a refined version of Tietze's extension theorem which yields uniformly continuous extensions (cf. Mandelkern [9]).

If $\mathbf{D}$ is uniformly bounded, $\mathcal{P}$ is a set of laws of continuous martingales with uniformly bounded quadratic variation density and therefore $\mathcal{P}$ is tight. Together with the aforementioned extension theorem we derive that $C_b(\Omega)$ is contained in $\mathbb{L}^p_{\mathcal{P}}$ and now the result follows from [5, Theorem 25]. □

Before extending the dynamic programming principle, we prove the following auxiliary result which shows in particular that the essential suprema in Theorem 4.5 can be represented as increasing limits. This is a consequence of a standard pasting argument which involves only controls with the same "history" and hence there are no problems of admissibility as in Section 4.

Lemma 5.3. *Let τ be an $\mathbb{F}$-stopping time and $X \in L^1_{\mathcal{P}}(\mathcal{F}_T)$. For each $P \in \mathcal{P}$ there exists a sequence $P_n \in \mathcal{P}(\tau, P)$ such that*

$$\operatorname*{ess\,sup}^{(P,\mathcal{F}_\tau)}_{P' \in \mathcal{P}(\tau,P)} E^{P'}[X|\mathcal{F}_\tau] = \lim_{n\to\infty} E^{P_n}[X|\mathcal{F}_\tau] \quad P\text{-a.s.},$$

where the limit is increasing and $\mathcal{P}(\tau, P) := \{P' \in \mathcal{P} :\, P' = P \text{ on } \mathcal{F}_\tau\}$.

Proof. It suffices to show that the set $\{E^{P'}[X|\mathcal{F}_\tau] : P' \in \mathcal{P}(\tau, P)\}$ is P-a.s. upward filtering. Indeed, we prove that for $\Lambda \in \mathcal{F}_\tau$ and $P_1, P_2 \in \mathcal{P}(\tau, P)$ there exists $\bar{P} \in \mathcal{P}(\tau, P)$ such that

$$E^{\bar{P}}[X|\mathcal{F}_\tau] = E^{P_1}[X|\mathcal{F}_\tau]\mathbf{1}_\Lambda + E^{P_2}[X|\mathcal{F}_\tau]\mathbf{1}_{\Lambda^c} \quad P\text{-a.s.},$$

then the claim follows by letting $\Lambda := \{E^{P_1}[X|\mathcal{F}_\tau] > E^{P_2}[X|\mathcal{F}_\tau]\}$. Similarly as in Proposition 4.4, we define

$$\bar{P}(A) := E^P\big[P^1(A|\mathcal{F}_\tau)\mathbf{1}_\Lambda + P^2(A|\mathcal{F}_\tau)\mathbf{1}_{\Lambda^c}\big], \quad A \in \mathcal{F}_T. \tag{5.1}$$

Let $\alpha, \alpha^1, \alpha^2$ be such that $P^\alpha = P$, $P^{\alpha^1} = P_1$ and $P^{\alpha^2} = P_2$. The fact that $P = P^1 = P^2$ on $\mathcal{F}_\tau$ translates to $\alpha = \alpha^1 = \alpha^2$ $du \times P_0$-a.e. on $[\![0, \tau(X^\alpha)[\![$ and with this observation we have as in Proposition 4.4 that $\bar{P} = P^{\bar{\alpha}} \in \overline{\mathcal{P}}_S$ for the $\mathbb{F}$-progressively measurable process

$$\bar{\alpha}_u(\omega) := \mathbf{1}_{[\![0,\tau(X^\alpha)[\![}(u)\alpha(\omega) + \mathbf{1}_{[\![\tau(X^\alpha),T]\!]}(u)\Big[\alpha^1_u(\omega)\mathbf{1}_\Lambda(X^\alpha(\omega)) + \alpha^2_u(\omega)\mathbf{1}_{\Lambda^c}(X^\alpha(\omega))\Big].$$

Since $P, P^1, P^2 \in \mathcal{P}$, Lemma 4.2 yields that $\bar{P} \in \mathcal{P}$. Moreover, we have $\bar{P} = P$ on $\mathcal{F}_\tau$ and $\bar{P}^{\tau(\omega),\omega} = P_1^{\tau(\omega),\omega}$ for $\omega \in \Lambda$ and $\bar{P}^{\tau(\omega),\omega} = P_2^{\tau(\omega),\omega}$ for $\omega \in \Lambda^c$. Thus $\bar{P}$ has the required properties. □

We now show that the extension $\mathcal{E}_t$ from Lemma 5.1 again satisfies the dynamic programming principle.

Theorem 5.4. *Let $0 \le s \le t \le T$ and $X \in \mathbb{L}^1_{\mathcal{P}}$. Then*

$$\mathcal{E}_s(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\mathcal{E}_t(X)|\mathcal{F}_s] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P} \tag{5.2}$$

and in particular

$$\mathcal{E}_s(X) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s] \quad P\text{-a.s.} \quad \text{for all } P \in \mathcal{P}. \tag{5.3}$$

Proof. Fix $P \in \mathcal{P}$. Given $\varepsilon > 0$, there exists $\psi \in \mathrm{UC}_b(\Omega)$ such that

$$\|\mathcal{E}_s(X) - \mathcal{E}_s(\psi)\|_{L^1_{\mathcal{P}}} \le \|X - \psi\|_{L^1_{\mathcal{P}}} \le \varepsilon.$$

For any $P' \in \mathcal{P}(s, P)$, we also note the trivial identity

$$\begin{aligned} &E^{P'}[X|\mathcal{F}_s] - \mathcal{E}_s(X) \\ &\quad = E^{P'}[X - \psi|\mathcal{F}_s] + \big(E^{P'}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big) + \big(\mathcal{E}_s(\psi) - \mathcal{E}_s(X)\big) \quad P\text{-a.s.} \end{aligned} \tag{5.4}$$

(i) We first prove the inequality "$\le$" in (5.3). By (4.5) and Lemma 5.3 there exists a sequence (P_n) in $\mathcal{P}(s, P)$ such that

$$\mathcal{E}_s(\psi) = \operatorname*{ess\,sup}_{P' \in \mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[\psi|\mathcal{F}_s] = \lim_{n\to\infty} E^{P_n}[\psi|\mathcal{F}_s] \quad P\text{-a.s.} \tag{5.5}$$

Using (5.4) with $P' := P_n$ and taking $L^1(P)$-norms we find that

$$\begin{aligned}
&\big\|E^{P_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\big\|_{L^1(P)} \\
&\quad\leq \|X-\psi\|_{L^1(P_n)} + \big\|E^{P_n}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big\|_{L^1(P)} + \big\|\mathcal{E}_s(\psi) - \mathcal{E}_s(X)\big\|_{L^1(P)} \\
&\quad\leq \big\|E^{P_n}[\psi|\mathcal{F}_s] - \mathcal{E}_s(\psi)\big\|_{L^1(P)} + 2\varepsilon.
\end{aligned}$$

Now bounded convergence and (5.5) yield that

$$\limsup_{n\to\infty} \big\|E^{P_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\big\|_{L^1(P)} \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this implies that there is a sequence $\tilde{P}_n \in \mathcal{P}(s,P)$ such that $E^{\tilde{P}_n}[X|\mathcal{F}_s] \to \mathcal{E}_s(X)$ P-a.s. In particular, we have proved the claimed inequality.

(ii) We now complete the proof of (5.3). By Lemma 5.3 we can choose a sequence (P'_n) in $\mathcal{P}(s,P)$ such that

$$\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s] = \lim_{n\to\infty} E^{P'_n}[X|\mathcal{F}_s] \quad P\text{-a.s.},$$

with an increasing limit. Let $A_n := \{E^{P'_n}[X|\mathcal{F}_s] \geq \mathcal{E}_s(X)\}$. As a result of Step (i), the sets A_n increase to Ω P-a.s. Moreover,

$$0 \leq \big(E^{P'_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\big)\mathbf{1}_{A_n} \nearrow \operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s] - \mathcal{E}_s(X) \quad P\text{-a.s.}$$

By (5.4) with $P' := P'_n$ and by the first equality in (5.5), we also have that

$$E^{P'_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X) \leq E^{P'_n}[X-\psi|\mathcal{F}_s] + \mathcal{E}_s(\psi) - \mathcal{E}_s(X) \quad P\text{-a.s.}$$

Taking $L^1(P)$-norms and using monotone convergence, we deduce that

$$\begin{aligned}
&\left\|\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\right\|_{L^1(P)} \\
&\qquad\qquad = \lim_{n\to\infty} \left\|\big(E^{P'_n}[X|\mathcal{F}_s] - \mathcal{E}_s(X)\big)\mathbf{1}_{A_n}\right\|_{L^1(P)} \\
&\qquad\qquad \leq \limsup_{n\to\infty} \|X-\psi\|_{L^1(P'_n)} + \big\|\mathcal{E}_s(\psi) - \mathcal{E}_s(X)\big\|_{L^1(P)} \\
&\qquad\qquad \leq 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves (5.3).

(iii) It remains to show (5.2) for a given $P \in \mathcal{P}$. In view of (5.3), it suffices to prove that

$$\begin{aligned}
&\operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}[X|\mathcal{F}_s] \\
&\qquad = \operatorname*{ess\,sup}_{P'\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'}\left[\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)} E^{P''}[X|\mathcal{F}_t]\,\middle|\,\mathcal{F}_s\right] \quad P\text{-a.s.}
\end{aligned}$$

The inequality "$\leq$" is obtained by considering $P'' := P' \in \mathcal{P}(t, P')$ on the right hand side. To see the converse inequality, fix $P' \in \mathcal{P}(s, P)$ and choose by Lemma 5.3 a sequence (P''_n) in $\mathcal{P}(t, P')$ such that

$$\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)} E^{P''}[X|\mathcal{F}_t] = \lim_{n\to\infty} E^{P''_n}[X|\mathcal{F}_t] \quad P'\text{-a.s.},$$

with an increasing limit. Then monotone convergence and the observation that $\mathcal{P}(t, P') \subseteq \mathcal{P}(s, P)$ yield

$$\begin{aligned} E^{P'}\bigg[\operatorname*{ess\,sup}_{P''\in\mathcal{P}(t,P')}{}^{(P',\mathcal{F}_t)} E^{P''}[X|\mathcal{F}_t]\bigg|\mathcal{F}_s\bigg] &= \lim_{n\to\infty} E^{P''_n}[X|\mathcal{F}_s] \\ &\leq \operatorname*{ess\,sup}_{P'''\in\mathcal{P}(s,P)}{}^{(P,\mathcal{F}_s)} E^{P'''}[X|\mathcal{F}_s] \quad P\text{-a.s.} \end{aligned}$$

As $P' \in \mathcal{P}(s, P)$ was arbitrary, this proves the claim. □

We note that (5.3) determines $\mathcal{E}_s(X)$ $\mathcal{P}$-q.s. and can therefore be used as an alternative definition. For most purposes, it is not necessary to go back to the construction. Relation (5.2) expresses the time-consistency property of $\mathcal{E}_t$. With a mild abuse of notation, it can also be stated as

$$\mathcal{E}_s(\mathcal{E}_t(X)) = \mathcal{E}_s(X), \quad 0 \leq s \leq t \leq T, \quad X \in \mathbb{L}^1_{\mathcal{P}};$$

indeed, the domain of $\mathcal{E}_s$ has to be slightly enlarged for this statement as in general we do not know whether $\mathcal{E}_t(X) \in \mathbb{L}^1_{\mathcal{P}}$. However, it turns out that $\mathcal{E}_t(X)$ is necessarily contained in $\mathbb{L}^1_{\mathcal{P}}$ under an additional assumption.

Proposition 5.5. *Let $p \in [1, \infty)$ and $t \in [0, T]$. If $\mathbf{D}$ is uniformly bounded, $\mathcal{E}_t$ maps $\mathbb{L}^p_{\mathcal{P}}$ into $\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t)$.*

Proof. It follows from Lemma 5.1 that $\mathcal{E}_t(\mathbb{L}^p_{\mathcal{P}})$ is contained in the $L^p_{\mathcal{P}}$-closure of $\mathcal{E}_t(\mathrm{UC}_b(\Omega))$ and we know from Corollary 3.6 that $\mathcal{E}_t(X)$ is lower semicontinuous (and obviously bounded) for $X \in \mathrm{UC}_b(\Omega)$. Hence it suffices to show that any bounded $\|\cdot\|_t$-lower semicontinuous function f is contained in $\mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t)$.

By Proposition 5.2 and its proof, the assumption implies that $\mathcal{P}$ is tight and that $C_b(\Omega_t) \subset \mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t)$. Since Ω_t is a Polish space, we can find a uniformly bounded sequence of functions $f_n \in C_b(\Omega_t)$ such that $f_n(\omega)$ increases to $f(\omega)$ for all $\omega \in \Omega$. As the upper expectation of a tight family is continuous from above (e.g., [5, Theorem 12]), $\|f - f_n\|^p_{L^p_{\mathcal{P}}} = \sup_{P\in\mathcal{P}} E^P[(f - f_n)^p] \to 0$ and hence $f \in \mathbb{L}^p_{\mathcal{P}}(\mathcal{F}_t)$. □

We close the section by summarizing some of the basic properties of $\mathcal{E}_t$.

Proposition 5.6. *Let $X, X' \in \mathbb{L}^p_{\mathcal{P}}$ for some $p \in [1,\infty)$ and let $t \in [0,T]$. Then the following relations hold $\mathcal{P}$-q.s.*

(i) $\mathcal{E}_t(X) \geq \mathcal{E}_t(X')$ *if* $X \geq X'$,

(ii) $\mathcal{E}_t(X + X') = \mathcal{E}_t(X) + X'$ *if* X' *is* $\mathcal{F}_t$*-measurable,*

(iii) $\mathcal{E}_t(\eta X) = \eta^+ \mathcal{E}_t(X) + \eta^- \mathcal{E}_t(-X)$ *if* η *is* $\mathcal{F}_t$*-measurable and* $\eta X \in \mathbb{L}^1_{\mathcal{P}}$,

(iv) $\mathcal{E}_t(X) - \mathcal{E}_t(X') \leq \mathcal{E}_t(X - X')$,

(v) $\mathcal{E}_t(X + X') = \mathcal{E}_t(X) + \mathcal{E}_t(X')$ *if* $\mathcal{E}_t(-X') = -\mathcal{E}_t(X')$,

(vi) $\|\mathcal{E}_t(X) - \mathcal{E}_t(X')\|_{L^p_{\mathcal{P}}} \leq \|X - X'\|_{L^p_{\mathcal{P}}}$.

Proof. Statements (i)–(iv) follow directly from (5.2). The argument for (v) is as in [15, Proposition III.2.8]: we have $\mathcal{E}_t(X + X') - \mathcal{E}_t(X') \leq \mathcal{E}_t(X)$ by (iv) while $\mathcal{E}(X + X') \geq \mathcal{E}_t(X) - \mathcal{E}_t(-X') = \mathcal{E}_t(X) + \mathcal{E}_t(X')$ by (iv) and the assumption on X'. Of course, (vi) is contained in Lemma 5.1. □

6 Closing Remarks

In this final section we present some further remarks and open problems.

6.1 Strong and Weak Formulation

In our formulation of the control problem, we have chosen to work only with measures from the set $\overline{\mathcal{P}}_S$; cf. (2.1). Alternatively, Definition 3.1 could also be stated with the larger set $\overline{\mathcal{P}}_W$. It is open whether this would lead to the same value function. For the G-expectation, the answer is known to be positive (cf. Soner et al. [17, Proposition 3.4]), but the argument given there makes direct use of the underlying PDE.

6.2 Constraint in the Strong Formulation

We have worked with the set-valued process $\mathbf{D}$ as a constraint on the density $\hat{a}$ of the quadratic variation of the canonical process. Since the measures $P^\alpha \in \overline{\mathcal{P}}_S$ are also parametrized by their integrands, we could alternatively have constrained α to take values in a set-valued process $\mathbf{D}^*$ considered under P_0. For a deterministic set $\mathbf{D} = \mathbf{D}^*$, Lemma 4.2 shows that these two options yield the same result and we remark that the latter formulation is used in [5]. The situation is quite different in the stochastic case; in particular, the invariance property stated in Lemma 4.1 fails in the formulation via $\mathbf{D}^*$ and it is open whether a time-consistent expectation can be constructed in this way.

6.3 Regularity of D

Recall that we have constructed the operator $\mathcal{E} = \mathcal{E}^{\mathbf{D}}$ under the assumption that the set-valued process $\mathbf{D}$ is uniformly continuous (Definition 3.2). While it seems difficult to provide a construction without topological regularity of $\mathbf{D}$, this has an evident drawback: Since $\mathbf{D}_t$ models the a priori bounds for the volatility after t, natural examples of $\mathbf{D}$ should take into account the observed volatility on $[0,t]$. But as the quadratic variation $\omega \mapsto \langle B\rangle(\omega)$ is not uniformly continuous, this may conflict with our assumption.

For practical applications, this is not really an issue since the volatility is given by an estimator comprising of discrete observations and which is indeed a uniformly continuous function of ω. Mathematically, however, the situation is dissatisfactory. While it is beyond the scope of this paper to investigate this in detail, we note that there is some room for generalizations by approximation. For instance, it is obvious that $\mathcal{E}^{\mathbf{D}}$ is monotone in $\mathbf{D}$. Hence, if we consider a sequence of processes $(\mathbf{D}^n)$ such that each $\mathbf{D}^n$ is uniformly continuous and $\mathbf{D}^n \subseteq \mathbf{D}^{n+1}$, we could define $\mathcal{E}_t^{\mathbf{D}}(X) := \lim_n \mathcal{E}_t^{\mathbf{D}^n}(X)$ for the set-valued process $\mathbf{D} := \bigcup_{n\geq 1} \mathbf{D}^n$ which need not be uniformly continuous.

6.4 On the General Structure of Sublinear Expectations

We first recall some representation results from the theory of time-consistent sublinear expectations under a given reference probability P_*; here one is interested in drift uncertainty rather than volatility uncertainty. It was shown by Delbaen [4, Theorem 2] that if there is a continuous P_*-martingale M having the martingale representation property for the given filtration $\mathbb{G}$, the class of all such expectations can be parametrized by set-valued processes. More precisely, each expectation is characterized by a set $\mathcal{P}$ of measures $Q \ll P_*$ and, using the representation property, one identifies $\{Q \in \mathcal{P} : Q \sim P_*\}$ with a set of stochastic exponentials $\{\mathscr{E}(\int \alpha^Q \, dM) = E^{P_*}[dQ/dP_*|\mathbb{G}]\}$. Then the set-valued process is chosen such that its selectors correspond to the integrands α^Q.

A second stream of results for Brownian settings states that if a time-consistent sublinear expectation has certain continuity properties, it can be represented as a g-expectation for a random function g; i.e., as the solution of a backward SDE with coefficient g (see, e.g., Coquet et al. [3]).

Finally, we mention that for each of the measures $P \in \overline{\mathcal{P}}_S$ used in the present paper, it is known that the augmented filtration $\overline{\mathbb{F}}^P$ is in fact a Brownian filtration and that the canonical process B has the martingale representation property (Soner et al. [18, Lemma 8.1]).

In view of these result and various analogies between the g- and the G-expectations, it has been conjectured that all time-consistent sublinear expectations in our setting of volatility uncertainty should be "random G"-expectations of the type constructed in the present paper, at least in a wide sense and up to technical assumptions. However, the following example seems to point in a different direction.

Example 6.1. We denote by P^1 the Wiener measure, by P^2 the measure P^α with $\alpha \equiv 2$, and we let

$$A_t^i := \big\{\langle B\rangle_s = si \text{ for all } 0 \leq s \leq t\big\}, \quad i = 1, 2$$

for $t > 0$. Then $A_t^1 \cap A_t^2 = \emptyset$ and $P^1(A_t^1) = P^2(A_t^2) = 1$. For any bounded and measurable random variable X we set

$$\mathcal{E}_t(X) := \begin{cases} E^{P^1}[X] \vee E^{P^2}[X], & t = 0, \\ E^{P^1}[X|\mathcal{F}_t]\mathbf{1}_{A_t^1} + E^{P^2}[X|\mathcal{F}_t]\mathbf{1}_{A_t^2}, & t > 0. \end{cases}$$

More precisely, we define the conditional expectations on the entire space Ω by choosing and fixing versions of the corresponding regular conditional distributions. As a consequence of $P^1(A_t^2) = P^2(A_t^1) = 0$, we have the time-consistency property $\mathcal{E}_s(\mathcal{E}_t(X)) = \mathcal{E}_s(X)$ $\mathcal{P}$-q.s. for all $0 \leq s \leq t$.

The set $\mathcal{P} = \{P^1, P^2\}$ in this example does not seem to be given by a set-valued process in any sense; note in particular that $\mathcal{P}$ is much smaller than the set $\{P^\alpha \in \overline{\mathcal{P}}_S : \alpha \text{ takes values in } \{1, 2\}\}$. There are of course dominating measures for $\mathcal{P}$ in this example; however, they necessarily have undesirable features. Indeed, the set $A := \bigcap_{n\geq 1} A_{1/n}^1$ satisfies $A \in \mathcal{F}_{0+}$ and if P is any probability dominating P^1 and P^2 then $P^1(A) = P^2(A^c) = 1$ implies that $0 < P(A) < 1$. In particular, the Blumenthal 0/1 law fails for P and B cannot have the martingale representation property in $\overline{\mathbb{F}}^P$.

We remark that the random variables $\mathcal{E}_t(X)$ in Example 6.1 are quite irregular from a topological point of view. Therefore, the example does not necessarily entail that the conjecture cannot be formulated in a valid way under additional assumptions.

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