Entanglement evolution of non-trace-preserving maps

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We study the entanglement evolution of non-trace-preserving one-sided maps for 2×2 quantum states. We present an expression for the maximum entanglement of the output state of a given non-trace preserving map, and also an explicit equation of the initial state which maximizes the entanglement of the output state. An experiment is proposed and it is numerically simulated. We also present the averaged entanglement factorization equation in a multi-outcome process.

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Introduction.— Quantum entanglement plays a central role in quantum information and also in the foundations of quantum physics. Thus, it has been extensively studied (see, e.g., [1–6]).

In practice, one often needs to manipulate quantum entanglement with measurements. Moreover, systems are exposed to the disturbance of a noisy environment. For example, there is no perfect channel for entanglement distribution, i.e., the channel can be regarded as a one-sided noisy channel. In teleporting quantum entanglement [7], the pre-shared entanglement is often imperfect and Bell measurements are involved. Most generally, the channel is noisy and *non*-trace-preserving. Even though the entanglement dynamics in open systems has been extensively studied, there are almost no general results on entanglement evolution independent of the detailed dynamical process.

Recently, Konrad et al [8] proposed a striking factorization law for entanglement evolution of one-sided maps. They [8] concludes that the output entanglement depends on the channel and the entanglement of the initial state, but it is independent of the specific form of the initial state. This conclusion has been experimentally tested [9] with a trace-preserving map.

Here, we study the entanglement evolution of *non*-trace-preserving maps. We emphasize that the results presented here, though closely related to, are not subsumed by Ref. [8]. As shown below, by a concrete counter example, the entanglement factorization law presented in Ref. [8] does not apply to non-trace-preserving maps.

Here we show that, given a non-trace-preserving one-sided map, $I \otimes \$$, if we know how the map changes the maximized entangled state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we can always know how the map changes any other pure bipartite state. More importantly, we present an explicit expression for the largest possible output entanglement over all input states. Moreover, we derive the explicit initial state which achieves such maximum output entanglement.

Multi-outcome processes, conditional output states, and non-trace-preserving maps.— Any quantum state evolution can be described by a completely positive map. If there are measurements during the evolution, then we need a non-trace preserving map. Following Ref. [10], we consider the intuitive picture in Fig. 1. Initially, a

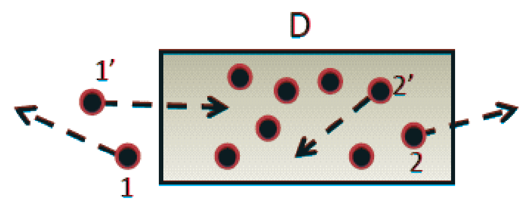

FIG. 1: (color on line) An intuitive description of a non-trace preserving map. Let mode $1'$ be the incident mode. Then a certain process P, which contains measurements, takes place. If certain specific measurement outcome i appears in process P, we regard the outgoing state on qubit 2 as the output state of the non-trace preserving map for the conditional process $P|i$. If we regard the initial bipartite state of qubit 1 and 1' as the input state, a conditional process corresponds to a one-sided map and the output state is a bipartite state on qubit 1 and qubit 2.

single-mode state $\rho_{\rm in}$ on mode $1'$ is sent to box D. Process P then takes place inside the box, where there can be many qubits and complicated interactions, as well as measurements with known results. We regard P as a multi-outcome process, if there are different possible measurement outcomes at the measurement stages. After process P is run, a qubit labeled by 2 comes out of the box. Instead of studying the averaged output state over all possible measurement outcomes $\{i\}$, we shall study the output state conditional on the appearance the specific outcome i. Namely, we want to study the *single-shot* output state only when the measurement outcome is i. Equivalently, if outcome i is obtained for the measurements, we say that the conditional process $P|i$ happens, and the final state $\rho_{\rm out}$ on qubit 2 outgoing from the box is the state we study. We focus on the output state of the conditional process $P|i$ rather than the averaged output state of process P.

To fully identify the conditional process $P|i$, one can prepare a maximized two-mode entangled state $|\phi^+\rangle$ (on qubits 1 and 1' in Fig. 1) and let one mode (qubit 1') of the bipartite state evolve under the conditional process $P|i$. The final two-mode state $\rho_\$$ on qubit 1 and qubit 2 fully characterizes the conditional process [11]. Such a conditional process can be described by a non-trace preserving (one-sided) map.

If the initial state is a two-mode state on modes 1 and $1'$, and only mode $1'$ is sent to box D, such a conditional process $P|i$ is regarded as a one sided map of $I\otimes\$$, and the output state is also a two-mode state on mode 1 and 2. This map is fully characterized by $\rho_\$$, which is the output state on modes 1 and 2, given the maximally-entangled state $|\phi^+\rangle\langle\phi^+|$ as the input state [11]. We assume

$$\begin{aligned} I\otimes\$(|\phi^+\rangle\langle|\phi^+|) &= f\rho_\$ \\ I\otimes\$(|\psi\rangle\langle\psi|) &= f'\rho_\psi \ , \end{aligned} \tag{1}$$

where $f = \mathrm{tr}\,[I\otimes\$(|\phi^+\rangle\langle|\phi^+|)]$, $f' = \mathrm{tr}\,[I\otimes\$(|\psi\rangle\langle\psi|)]$. We wish to derive an explicit expression for the largest possible output entanglement among all initial input pure states $\{|\psi\rangle\}$, and also to seek the specific initial state $|\psi\rangle$ which leads to the maximum output entanglement given $\rho_\$$ which characterizes the map.

No entanglement factorization law for non-trace preserving map.—The explicit formula [8] for the factorization law is $C(\rho_\psi) = C(|\psi\rangle\langle\psi|)C(\rho_\$)$, where $C(X)$ is the entanglement concurrence for density matrix X. This shows that the output entanglement concurrence is dependent on the input entanglement but independent of the input state. Also the output entanglement is an *increasing* function of the entanglement of the input state. This factorization for $C(\rho_\psi)$ does not apply to non-trace-preserving maps, as shown by the following counter example.

We now present a counter example with a *non*-trace-preserving map. Consider the following specific map

$$I\otimes\$(\rho_{\rm in}) = I\otimes\hat{M}(\tilde{a},\tilde{b})\rho_{\rm in} I\otimes\hat{M}(\tilde{a},\tilde{b}) \tag{2}$$

where $\hat{M}(\tilde{a},\tilde{b}) = \tilde{a}|0\rangle\langle 0| + \tilde{b}|1\rangle\langle 1|$, and $|\tilde{a}|^2 + |\tilde{b}|^2 = 1$. Given a normalized input state $\rho_{\rm in} = |\chi\rangle\langle\chi|$ and $|\chi\rangle = a|00\rangle + b|11\rangle$, we have

$$I\otimes\$(\rho_{\rm in}) = \gamma|\chi'\rangle\langle\chi'| \tag{3}$$

and $\gamma = \sqrt{|a\tilde{a}|^2 + |b\tilde{b}|^2}$, $|\chi'\rangle = \frac{a\tilde{a}}{\gamma}|00\rangle + \frac{b\tilde{b}}{\gamma}|11\rangle$. The entanglement concurrence of the outcome state is

$$C(|\chi'\rangle\langle\chi'|) = \frac{2|a\tilde{a}b\tilde{b}|}{|a\tilde{a}|^2 + |b\tilde{b}|^2}. \tag{4}$$

Setting $|a| = |\tilde{b}|$ and $|b| = |\tilde{a}|$ for the input state, we shall obtain the maximum output entanglement concurrence, $C = 1$. Clearly, the entanglement of the output state is *dependent* on the *input state* rather than the *entanglement* of the input state, and, in general, is *not* an increasing function of the initial entanglement. For example, setting $\tilde{a} = \sqrt{\frac{1}{3}}$ and $\tilde{b} = \sqrt{\frac{2}{3}}$, the input state $\sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$ will have the maximum output entanglement concurrence $C = 1$, while the input state $|\phi^+\rangle$ would only have an output entanglement concurrence of $C = \frac{2}{3}$, less than 1.

The map given by Eq. (3) corresponds to the following conditional process in Fig. 1: Prior to the process, there is a bipartite channel state $\tilde{a}|00\rangle + \tilde{b}|11\rangle$ for modes $2'$ and 2 inside the box. The initial input bipartite state is prepared on modes 1 and $1'$. Mode $1'$ is then sent to the box D and a Bell measurement is taken on the input mode and mode $2'$. If $|\phi^+\rangle\langle\phi^+|$ is observed, the conditional process $P|i$ happens, and the bipartite state of mode 1 and mode 2 is the output state.

Entanglement evolution maximization over non-trace-preserving one-sided maps.— A 2×2 pure state $|\chi\rangle = a|00\rangle + b|11\rangle$ can be rewritten in the form $|\chi\rangle\langle\chi| = 2\hat{M}(a,b)\otimes I(|\phi^+\rangle\langle\phi^+|)\hat{M}(a,b)\otimes I$. From Eq. (1) we have $I\otimes\$(|\chi\rangle\langle\chi|) = 2f\hat{M}(a,b)\otimes I\rho_\$\hat{M}(a,b)\otimes I = f'\rho_\chi$. We emphasize here that even though $\rho_\$$ is normalized, the operator $2\hat{M}(a,b)\otimes I\rho_\$\hat{M}(a,b)\otimes I$ is not necessarily normalized. Moreover, the factor g here is dependent on the parameters in both $|\chi\rangle$ (the initial state) and $\rho_\$$ (which characterizes the map $I\otimes\$$). We shall present the explicit formula for the relation between the entanglement of $\rho_\$$ and ρ_χ. Based on this we show which input state $|\chi\rangle$ will lead to the maximum entanglement concurrence of ρ_χ, given the channel $\rho_\$$.

For clarity, we define the following function C of an arbitrary non-negative definite 4×4 matrix (operator) N

$$C(N) = \max\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \xi_4\} \tag{5}$$

where the ξ_is are the eigenvalues of $N\cdot\tilde{N}$, in descending order, with $\tilde{N} = \sigma_y\otimes\sigma_y N^*\sigma_y\otimes\sigma_y$, with N^* the complex conjugate of N. If N is a density matrix of a 2×2 system, $C(N)$ is just the entanglement concurrence of the system [5]. In particular, $C(|\chi\rangle\langle\chi|) = 2|ab|$. With this definition of C, we can summarize the major result, equation (5) in Ref. [8] as follows:

Lemma 1 Given any density matrix $\rho_\$$, if $N = 2\hat{M}(a,b)\otimes I\rho_\$\hat{M}^\dagger(a,b)\otimes I$, then

$$C(N) = C(|\chi\rangle\langle\chi|)\cdot C(\rho_\$) = 2|ab|C(\rho_\$). \tag{6}$$

However, this is not the entanglement concurrence of ρ_χ because N is not necessarily normalized, even though $\rho_\$$ is. Now denote $N = g\rho_\chi$, and $g = \mathrm{tr}N$. According to the definition of C:

$$C(\rho_\chi) = C(N)/g = 2|ab|C(\rho_\$)/g. \tag{7}$$

Note that the so-called factorization law as presented in [8] is the function C of the operator $N = 2\hat{M}(a,b) \otimes I\rho_\$ \hat{M}^\dagger(a,b) \otimes I$, rather than, the entanglement concurrence of the density operator ρ_χ. For any trace-preserving map, we always have $N = \rho_\chi$, and therefore the major conclusion in [8] applies. However, for a *non*-trace-preserving map, in general $\rho_\$ \neq N$, because the factor g depends on *both* $\rho_\$$ *and* $|\chi\rangle$.

The remaining task now is to derive an explicit formula for g in terms of $\rho_\$$ and $|\chi\rangle$. According to its definition, $g = \mathrm{tr}N = 2\mathrm{tr}\hat{M}(a,b)\otimes I\rho_\$\hat{M}^\dagger(a,b)\otimes I$. To avoid meaningless results, we assume $C(\rho_\$) > 0$ throughout this paper. Suppose the density matrix of mode 1 of $\rho_\$$ is $\rho_0 = \mathrm{tr}_2\rho_\$ = \begin{pmatrix} c_1 & \alpha \\ \alpha^* & c_2 \end{pmatrix} = K_0$, where tr_2 is the partial trace over the subspace of the second qubit and

$$c_1 = \langle 0|\mathrm{tr}_2\rho_\$|0\rangle,\ c_2 = \langle 1|\mathrm{tr}_2\rho_\$|1\rangle. \tag{8}$$

Consequently, We have

$$g = 2\mathrm{tr}M(a,b)\rho_0 M^\dagger(a,b) = 2|a|^2c_1 + 2|b|^2c_2 \tag{9}$$

where $c_1 = \langle 0|\mathrm{tr}_2\rho_\$|0\rangle$ and $c_2 = \langle 1|\mathrm{tr}_2\rho_\$|1\rangle$. Since $\rho_\$$ itself is normalized, we have $c_1 + c_2 = 1$. Therefore, when $|a| = \sqrt{c_2}$, $|b| = \sqrt{c_1}$, the value $C(\rho_\chi)$ in Eq. (7) is maximized and

$$C(\rho_\chi) = \frac{C(\rho_\$)}{2\sqrt{c_1c_2}} \tag{10}$$

More generally, the initial pure state can be

$$|\psi\rangle = I\otimes U|\chi\rangle = \sqrt{2}\hat{M}(a,b)\otimes U|\phi\rangle \tag{11}$$

where U is an arbitrary unitary operator. Given the fact that $U\otimes U|\phi^+\rangle = |\phi^+\rangle$ for any unitary U, we have

$$\sqrt{2}\hat{M}(a,b)\otimes U|\phi^+\rangle = \sqrt{2}\hat{M}(a,b)U^\dagger\otimes I|\phi^+\rangle. \tag{12}$$

In such a case, we have

$$C(\rho_\psi) = 2|ab|\cdot C(\rho_\$)/g' \tag{13}$$

and $g' = \mathrm{tr}\left[\hat{M}(a,b)U^\dagger\otimes I\rho_\$\hat{U}M(a,b)\otimes I\right]$. To maximize $C(\rho_\psi)$, we first fix U and maximize it with a,b. Suppose $U^\dagger K_0 U = \begin{pmatrix} c_1' & \alpha' \\ \alpha'^* & c_2' \end{pmatrix}$. The largest value for $C(\rho_\psi)$ is $\frac{C(\rho_\$)}{2\sqrt{c_1'c_2'}}$ as shown already. To maximize the value over all U, we only need to minimize $c_1'c_2'$. Since U is unitary, $\det(U^\dagger K_0 U) = \det K_0$. Therefore $c_1'c_2' = \det K_0 + |\alpha'|^2$, which is minimized when $\alpha' = 0$. Namely, $C(\rho_\psi)$ is maximized when $U^\dagger K_0 U$ is diagonalized. Therefore, we have the following formula for the maximized output entanglement concurrence for a given map $\rho_\$$:

$$\mathrm{Max}\{C(\rho_\psi)\} = \frac{C(\rho_\$)}{2\sqrt{\det\left[\mathrm{tr}_2\rho_\$\right]}} \tag{14}$$

when

$$|\psi\rangle = I\otimes U(a|00\rangle + b|11\rangle) \tag{15}$$

and U is the unitary operator that diagonalizes $\mathrm{tr}_2\rho_\$$ in $U^\dagger\mathrm{tr}_2\rho_\$ U = \mathrm{diag}(c_1', c_2')$, $|a| = \sqrt{c_1'}; |b| = \sqrt{c_2'}$. Eq. (14-15) are the main results of this paper.

Proposed experiment and numerical simulation.—We propose to test Eq. (14) with imperfect entanglement and a Bell measurement. The proposed conditional process consists of two stages: (*i*) Disturbance of the incident qubit, mode $1'$. Assume now a probability $(1-\epsilon)$ of not affecting the mode and a probability ϵ to make a rotation $\hat{R}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$; (*ii*) Initially there is a bipartite mixed state $\tilde{\rho}$ for qubits $2'$ and 2, inside the box. After stage (*i*), a Bell measurement on qubits $1'$ and $2'$ takes place and $|\phi^+\rangle\langle\phi^+|$ is observed as the measurement outcome.

In an experiment, we assume there is no a priori knowledge of the disturbance and $\tilde{\rho}$. One needs to know $\rho_\$$ by testing the channel with the maximally entangled state $|\phi^+\rangle$ as the input state; and then calculate the largest possible entanglement output of the conditional process and the corresponding input state, according to our Eq. (14). Later on, the channel should be tested with many different bipartite input states to verify whether the largest entanglement output and its corresponding input states are in agreement with the theoretical prediction from Eq. (14).

Now we check Eq. (14) numerically. The output state on qubits 1 and 2 of the conditional process can be written in the following input-output form with the map $I\otimes\$$:

$$\begin{aligned} I\otimes\$(|\psi\rangle\langle\psi|) &= \langle\phi^+|_{1'2'}\left[\Omega_{11'}(\psi)\otimes\tilde{\rho}_{2'2}\right]|\phi^+\rangle_{1'2'} \\ &= \mathcal{N}\left[(1-\epsilon)\tilde{\rho}_{12} + \epsilon\hat{R}^\dagger(\theta)\otimes I\tilde{\rho}_{12}\hat{R}(\theta)\otimes I\right] \end{aligned} \tag{16}$$

where $\mathcal{N}$ is the normalization factor and $\Omega_{11'}(\psi) = (1-\epsilon)|\psi\rangle\langle\psi| + \epsilon I\otimes\hat{R}(\theta)|\psi\rangle\langle\psi|I\otimes\hat{R}^\dagger(\theta)$. The map is characterized by the output state $\rho_\$$, which can be calculated with Eq. (16) by setting $\psi = \phi^+$. Suppose $\epsilon = 0.06$, $\theta = 2\pi/5$, $\tilde{\rho} = 0.1|\varphi'\rangle\langle\varphi'| + 0.12|\varphi''\rangle\langle\varphi''| + 0.78|\phi^+\rangle\langle\phi^+|$, where $|\varphi'\rangle = \hat{R}(\theta_1)\otimes I|00\rangle$, $|\varphi''\rangle = \hat{R}(\theta_2)\otimes I|11\rangle$ with $\theta_1 = \pi/5, \theta_2 = -3\pi/10$. We find $\mathrm{tr}_2\tilde{\rho} = \begin{pmatrix} 0.5340 & 0.1046 \\ 0.1046 & 0.4660 \end{pmatrix}$ which can be diagonalized via $U^\dagger\mathrm{tr}_2\tilde{\rho}U = \mathrm{diag}[0.39, 0.61]$ and $U = \begin{pmatrix} 0.5878 & -0.8090 \\ -0.8090 & -0.5878 \end{pmatrix}$. According to Eq. (14), the largest output entanglement is $C(\tilde{\rho})/(2\sqrt{\det(\mathrm{tr}_2\tilde{\rho})}) = 0.7036$. This value is reached when setting the initial input state to be $|\psi\rangle = 0.4591|00\rangle - 0.6318|01\rangle - 0.5053|10\rangle - 0.3671|11\rangle$. Numerical tests in Figs. 2-4 have indeed confirmed this. By changing the input state $|\psi\rangle$ with many different values of a and θ, we first calculate the output state by Eq. (16). After normalization, we

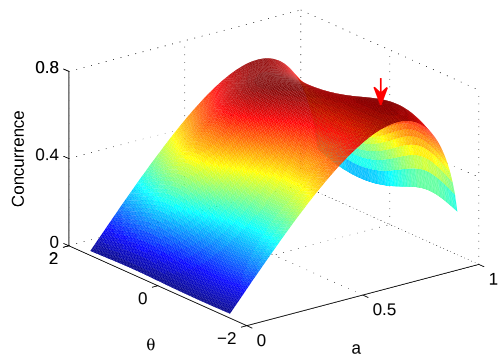

FIG. 2: (color online) The concurrence C versus a and θ. The arrow on top indicates the maximum concurrence 0.7036 with $a = 0.7810$ and $\theta = -0.9424$.

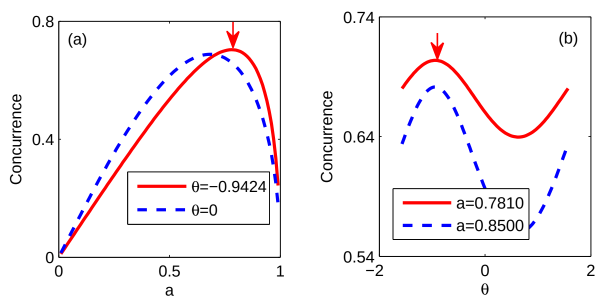

FIG. 3: (color online) The concurrence C versus a [in (a)] and θ [in (b)].

calculate its entanglement concurrence directly by using Wooters's formula as Eq. (5).

Factorization law for the averaged entanglement of a multi-outcome process.— Consider now a multi-outcome process P. The conditional process $P|i$ produces the output states, $\rho_{\$_i}$ and $\rho_{i\psi}$, for the input states $|\phi^+\rangle$ and $|\psi\rangle$, respectively. For simplicity, we first assume $|\psi\rangle = |\chi\rangle = a|00\rangle + b|11\rangle$. Denote: $\$_i$ as the map of process $P|i$, $p_{i\$}$ and $p_{i\psi}$ the probabilities that the conditional process $P|i$ takes place in applying P to initial input states $|\phi^+\rangle$ or $|\psi\rangle$, respectively. We now have $p_{i\$}/p_{i\psi} = f_i/f_i'$, where $f_i = \mathrm{tr}\,[I \otimes \$_i(|\phi^+\rangle\langle\phi^+|)]$ and $f_i' = \mathrm{tr}\,[I \otimes \$_i(|\psi^+\rangle\langle\psi^+|)]$. Using $|\psi\rangle\langle\psi| = 2\hat{M}(a,b) \otimes I|\phi^+\rangle\langle\phi^+|\hat{M}^\dagger(a,b) \otimes I$, we have $g_i = f_i'/f_i = p_{i\psi}/p_{\$_i}$, where $g_i = \mathrm{tr}N$ and $N = \left[2\hat{M}(a,b) \otimes I\rho_{\$_i}\hat{M}(a,b) \otimes I\right]$. Given a multi-outcome process P, Lemma 1 actually says $C(\rho_{\$_i}) = C(|\psi\rangle\langle\psi|)C(g_i\rho_{i\psi})$, which is equivalent to $p_{\$_i}C(\rho_{\$_i}) = p_{i\psi}C(|\psi\rangle\langle\psi|)C(\rho_{i\psi})$. This leads to

$$\langle C_\psi\rangle = C(|\psi\rangle\langle\psi|)\langle C_{\phi^+}\rangle \tag{17}$$

where $\langle C_\psi\rangle = \sum_i p_{i\psi}C(\rho_{i\psi})$ and $\langle C_{\phi^+}\rangle = \sum_i p_{\$_i}C(\rho_{\$_i})$. This provides the factorization law for the averaged entanglement in a multi-outcome process. When there is only one outcome in the process, it is just the factorization equation for the trace-preserving map.

Eq. (17) actually holds for any initial state $|\psi\rangle = U'\otimes U|\chi\rangle = \mathcal{U}\otimes I|\tilde{\psi}\rangle$, where $\mathcal{U} = U^{-1}U'^{-1}$, $|\tilde{\psi}\rangle = a|\tilde{0}\tilde{0}\rangle + b|\tilde{1}\tilde{1}\rangle$, and $|\tilde{x}\rangle = U|x\rangle$, $x = 0,1$. Since the entanglement evolution does not change under any local unitary transformation of mode 1, we have $\langle C_\psi\rangle = \langle C_{\tilde{\psi}}\rangle$. Since Eq. (17) does not depend on any specific basis, it holds if $|0\rangle$ and $|1\rangle$ are replaced by $|\tilde{0}\rangle$ and $|\tilde{1}\rangle$ for all states. Hence $\langle C_{\tilde{\psi}}\rangle = C(|\tilde{\psi}\rangle\langle\tilde{\psi}|)\langle C_{\tilde{\phi}^+}\rangle = C(|\psi\rangle\langle\psi|)\langle C_{\tilde{\phi}^+}\rangle$, and $C_{\tilde{\phi}^+}$ is the averaged entanglement output given the initial input state $|\tilde{\phi}^+\rangle = \frac{1}{\sqrt{2}}(|\tilde{0}\tilde{0}\rangle + |\tilde{1}\tilde{1}\rangle)$. Obviously, $|\tilde{\phi}^+\rangle = |\phi^+\rangle$. This proves that Eq. (17) holds for any initial state $|\psi\rangle$.

In summary, we have studied the entanglement evolution under non-trace-preserving maps. The formulas for the maximum output entanglement and its corresponding initial state are presented. We also show that the factorization equation of Ref. [8] also applies to the averaged entanglement evolution in a multi-outcome process.

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