General theory of assisted entanglement distillation

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Abstract

We evaluate the one-shot entanglement of assistance for an arbitrary bipartite state. This yields another interesting result, namely a characterization of the one-shot distillable entanglement of a bipartite pure state. This result is shown to be stronger than that obtained by specializing the one-shot hashing bound to pure states. Finally, we show how the one-shot result yields the operational interpretation of the asymptotic entanglement of assistance proved in [Smolin et al. Phys. Rev. A **72**, 052317 (2005)].

1 Introduction

One of the most basic and widely studied entanglement measures for bipartite quantum states is the *entanglement of formation* (EoF) [1], a quantity so named because it was intended to quantify the resources needed to create (or form) a given bipartite entangled state. The EoF of any bipartite pure state is quantified by the entropy of entanglement, which is equal to the von Neumann entropy of the reduced state of a subsystem. The EoF of a bipartite mixed state ρ_{AB}, is then defined via the convex roof extension, that is, as the minimum average entanglement of an ensemble of pure states that represents ρ_{AB}:

$$E_F(\rho_{AB}) := \min_{\mathfrak{E}} \sum_i p_i S(\rho_A^i), \tag{1}$$

where $\mathfrak{E} = \{p_i, |\psi_{AB}^i\rangle\}$ is an ensemble of pure biparite states such that $\sum_i p_i |\psi^i\rangle\langle\psi^i| = \rho_{AB}$, and $S(\rho_A^i)$ is the von Neumann entropy of the reduced state $\rho_A^i = \mathrm{Tr}_B\, |\psi^i\rangle\langle\psi^i|_{AB}$. The popularity of the EoF is partly due to its formal elegance and the many nice properties it enjoys [29, 30], and

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perhaps also due to its connections with the additivity problem in quantum information theory [31].

From the operational point of view, the EoF is associated with the entanglement manipulation protocol by which two distant parties, say Alice and Bob, prepare a given bipartite quantum state, starting from an initial entangled state which they share, by using only local operations and classical communication (LOCC). It turns out that the optimal (i.e., minimum) rate, at which entanglement has to be consumed in order for Alice and Bob to create multiple copies of the state with asymptotically vanishing error, is given by the regularized EoF of the state [4].

Soon after the introduction of the EoF, another quantity, namely the *entanglement of assistance* (EoA) [3], was introduced as its "dual". It is defined analogously to EoF but with the minimisation over ensembles replaced by a maximisation, i.e.,

$$E_A(\rho_{AB}) := \max_{\mathfrak{E}} \sum_i p_i S(\rho_A^i). \tag{2}$$

Unlike the EoF, the EoA is not an entanglement monotone and hence it can in general increase under local operations and classical communication [2]. However, like the EoF, the EoA too can be associated with an entanglement manipulation protocol, namely the one by which Alice and Bob distill entanglement from an initial mixed bipartite state which they share, when a third party (say Charlie), who holds the purification of the state, assists them. Charlie is allowed to do local operations on his share of the tripartite pure state, and his assistance is in the form of one-way classical communication to Alice and Bob. This is the sort of scenario which occurs, for example, in the case of *environment-assisted quantum error correction* [5, 6, 7, 8, 9, 10], in which errors, incurred from sending quantum information through a noisy environment, are corrected by using classical information obtained from a measurement on the environment. In this case the tripartite structure Alice-Bob-Charlie is mirrored by the structure sender-receiver-environment, and the assistance from Charlie is replaced by the ability to perform measurements on the environment and to exploit the resulting information for error correction.

Another area in which the EoA arises naturally, is in the study of *localizable entanglement* in spin systems [11, 12, 13, 14]. The scenario here is as follows: a pure state of a system of $n \gg 1$ interacting spins is given, and the goal is to localize (or "focus") as much entanglement as possible between two arbitrarily chosen spins, by performing a suitable measurement on the remaining $n-2$ spins. In this case, the assisting party is actually divided into many subsystems (which are the $n-2$ spins) and so it is natural to ask what happens when the assisting measurements are restricted to be *local* in each subsystem. The amount of entanglement that can be focussed in this case is referred to as the localizable entanglement, and it is always at most

as much as the EoA. In fact, it is equal to the EoA when the assisting party is allowed to act *globally* on all the constituent subsystems.

In the literature, one encounters cases in which the EoA is used to characterize operational tasks of assisted distillation studied in the generic scenario, where no assumptions are made on the state to be distilled. This is often referred to as the "one-shot" scenario. However, the definition of the EoA given in eq. (2) has been shown to have an operational relevance only in the asymptotic regime, i.e., when asymptotically many copies of the same state are available for assisted distillation [7]. This points to an apparent mismatch between the operational task and the quantity used to characterize it. In order to remedy this problem, one should start from the operational task itself, and from it, *evaluate* an expression quantifying the amount of entanglement that can be distilled under assistance. This leads to a *one-shot* EoA, which, by its very construction, has a direct operational interpretation.

In Section 2 we introduce the necessary notation and definitions. In Section 3 we evaluate the one-shot distillable entanglement of a pure bipartite state. The one-shot entanglement of assistance is introduced in Section 4 and evaluated in Section 5. Section 6 deals with the asymptotic scenario, where some previous results are recovered. Finally, Section 7 concludes the paper with a summary and an open question.

2 Notation and definitions

2.1 Mathematical preliminaries

Let $\mathcal{B}(\mathscr{H})$ denote the algebra of linear operators acting on a finite–dimensional Hilbert space $\mathscr{H}$ and let $\mathfrak{S}(\mathscr{H}) \subset \mathcal{B}(\mathscr{H})$ denote the subset of positive operators of unit trace (states). Further, let $\mathbb{1} \in \mathcal{B}(\mathscr{H})$ denote the identity operator. Throughout this paper we restrict our considerations to finite-dimensional Hilbert spaces, and we take the logarithm to base 2. For any given pure state $|\phi\rangle$, we denote the projector $|\phi\rangle\langle\phi|$ simply as ϕ. Moreover, for any state ρ, we define Π_ρ to be the projector onto the support of ρ.

For a state $\rho \in \mathfrak{S}(\mathscr{H})$, the von Neumann entropy is defined as $S(\rho) := -\operatorname{Tr}\rho\log\rho$. Further, for a state ρ and a positive operator σ such that $\operatorname{supp}\rho \subseteq \operatorname{supp}\sigma$, the quantum relative entropy is defined as $S(\rho||\sigma) = \operatorname{Tr}\rho\log\rho - \rho\log\sigma$, whereas the relative Rényi entropy of order $\alpha \in (0,1)$ is defined as

$$S_\alpha(\rho||\sigma) := \frac{1}{\alpha-1}\log\big[\operatorname{Tr}(\rho^\alpha\sigma^{1-\alpha})\big]. \tag{3}$$

For given orthonormal bases $\{|i_A\rangle\}_{i=1}^d$ and $\{|i_B\rangle\}_{i=1}^d$ in isomorphic Hilbert spaces $\mathscr{H}_A \simeq \mathscr{H}_B$ of dimension d, we define a maximally entangled state

(MES) of rank $M \leq d$ to be

$$|\Psi_{AB}^{M}\rangle = \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |i_A\rangle \otimes |i_B\rangle. \tag{4}$$

In order to measure how close two states are, we will use the fidelity, defined as

$$F(\rho, \sigma) := \operatorname{Tr} \sqrt{\sqrt{\rho}\sigma\sqrt{\rho}} = \left\| \sqrt{\rho}\sqrt{\sigma} \right\|_1, \tag{5}$$

and the trace distance

$$\|\rho - \sigma\|_1 := \operatorname{Tr} |\rho - \sigma|. \tag{6}$$

The trace distance between two states ρ and σ is related to the fidelity $F(\rho, \sigma)$ as follows (see e. g. [18]):

$$1 - F(\rho, \sigma) \leqslant \frac{1}{2} \|\rho - \sigma\|_1 \leqslant \sqrt{1 - F^2(\rho, \sigma)}, \tag{7}$$

where we use the notation $F^2(\rho, \sigma) = \big(F(\rho, \sigma)\big)^2$

The following lemmas will prove useful.

Lemma 1 (Gentle measurement lemma [26, 27])**.** *For a state $\rho \in \mathfrak{S}(\mathscr{H})$ and operator $0 \leq \Lambda \leq \mathbb{1}$, if $\operatorname{Tr}(\rho\ \Lambda) \geq 1 - \delta$, then*

$$\left\| \rho - \sqrt{\Lambda}\rho\sqrt{\Lambda} \right\|_1 \leq 2\sqrt{\delta}.$$

The same holds if ρ is a subnormalized density operator.

Lemma 2. *For any pure state $|\phi\rangle$ and any given $\varepsilon \geq 0$, if $0 \leq P \leq \mathbb{1}$ is an operator such that $\operatorname{Tr}(P\phi) \geq 1 - \varepsilon$, then*

$$F(\sqrt{P}|\phi\rangle, |\phi\rangle) \geq 1 - \sqrt{\varepsilon}. \tag{8}$$

Proof. Since, $\operatorname{Tr}(P\phi) \geq 1 - \varepsilon$, by Lemma 1 we have that

$$\|\sqrt{P}\phi\sqrt{P} - \phi\|_1 \leq 2\sqrt{\varepsilon}.$$

The lower bound on the trace distance in (7) then yields

$$F(\sqrt{P}|\phi\rangle, |\phi\rangle) \equiv F(\sqrt{P}\phi\sqrt{P}, \phi) \geq 1 - \sqrt{\varepsilon}. \tag{9}$$

∎

Lemma 3. *For any normalized state ρ and any $0 \leq P \leq \mathbb{1}$, if $\operatorname{Tr}[P\rho] \geq 1-\varepsilon$, then*

$$F(\omega, \rho) \geq 1 - 2\sqrt{\varepsilon}, \tag{10}$$

where $\omega := \frac{\sqrt{P}\rho\sqrt{P}}{\operatorname{Tr}[P\rho]}$.

Proof. The condition $\mathrm{Tr}[P\rho] \geq 1-\varepsilon$ implies that $\left\| \sqrt{P}\rho\sqrt{P} - \rho \right\|_1 \leq 2\sqrt{\varepsilon}$, [26, 27]. Let us define $\tilde{\omega} := \sqrt{P}\rho\sqrt{P}$. Due to Lemma 11 in [28], we have that

$$\begin{aligned} F(\tilde{\omega}, \rho) :&= \left\| \sqrt{\tilde{\omega}}\sqrt{\rho} \right\|_1 \\ &\geq \frac{\mathrm{Tr}[P\rho] + 1}{2} - \frac{1}{2} \left\| \tilde{\omega} - \rho \right\|_1 \\ &\geq 1 - \frac{\varepsilon}{2} - \sqrt{\varepsilon} \\ &\geq 1 - 2\sqrt{\varepsilon}. \end{aligned} \tag{11}$$

Let ω be a normalized state defined as $\omega := \frac{\tilde{\omega}}{\mathrm{Tr}(\tilde{\omega})}$. Since $F(\omega, \rho) \geq F(\tilde{\omega}, \rho)$, we obtain the statement of the lemma. ∎

In this paper we consider entanglement distillation under LOCC transformations. In this context, a result by Lo and Popescu [17] on entanglement manipulation of bipartite *pure states* plays a crucial role. They proved that any LOCC transformation ($AB \mapsto A'B'$) on a bipartite pure state $|\phi_{AB}\rangle$, shared between two distant parties Alice and Bob, is equivalent to a LOCC transformation with only one-way classical communication, which can be represented as follows:

$$\Lambda(\phi_{AB}) = \sum_k (U_k \otimes E_k)\phi_{AB}(U_k \otimes E_k)^\dagger, \tag{12}$$

where the operators U_k are unitary and the operators E_k satisfy the relation $\sum_k E_k^\dagger E_k = \mathbb{1}_B$. Henceforth, we say that an LOCC transformation is of the *Lo-Popescu form* if it can be expressed as in (12). Consequently, for a map Λ of the Lo-Popescu form, we have

$$\begin{aligned} \Lambda(\mathbb{1}_A \otimes \sigma_B) &= \sum_k U_k U_k^\dagger \otimes E_k \sigma_B E_k^\dagger, \\ &= \mathbb{1}_{A'} \otimes \tau_{B'}, \end{aligned} \tag{13}$$

where $\tau_{B'} := \sum_k E_k \sigma_B E_k^\dagger$. .

2.2 Entropies and coherent information

Optimal rates of the entanglement distillation protocols considered in this paper are expressible in terms of the following entropic quantities:

For any $\rho, \sigma \geq 0$, any $0 \leq P \leq \mathbb{1}$, and any $\alpha \in (0, \infty)\backslash\{1\}$, we define the following entropic function (introduced in [22])

$$S_\alpha^P(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \mathrm{Tr}[\sqrt{P}\rho^\alpha\sqrt{P}\sigma^{1-\alpha}]. \tag{14}$$

Notice that, for $P = \mathbb{1}$, the function defined above reduces to relative Rényi entropy of order α given by (3).

In this paper, we are in particular interested in the quantity,

$$S_0^P(\rho\|\sigma) := \lim_{\alpha \searrow 0} S_\alpha^P(\rho\|\sigma) = -\log \mathrm{Tr}[\sqrt{P}\Pi_\rho\sqrt{P}\ \sigma], \tag{15}$$

where Π_ρ denotes the projector onto the support of ρ.

Note that

$$S_0^{\mathbb{1}}(\rho\|\sigma) = S_0(\rho\|\sigma) := -\log(\mathrm{Tr}\,\Pi_\rho\sigma), \tag{16}$$

which is the relative Rényi entropy of order zero. This quantity acts as a parent quantity for the *zero-coherent information*, defined as follows:

$$I_0^{A\to B}(\rho_{AB}) := \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S_0(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B), \tag{17}$$

the nomenclature arising from its analogy with the ordinary coherent information $I^{A\to B}(\rho_{AB})$, which is expressible in a similar manner, when the zero-relative Rényi entropy is replaced by the ordinary relative entropy:

$$\begin{aligned} I^{A\to B}(\rho_{AB}) &:= S(\rho_B) - S(\rho_{AB}) &(18)\\ &\equiv \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B). &(19)\end{aligned}$$

If Ψ^ρ_{ABE} is a purification of the state ρ_{AB}, then

$$I^{A\to B}(\rho_{AB}) = -I^{A\to E}(\rho_{AE}), \tag{20}$$

where $\rho_{AE} = \mathrm{Tr}_B\,\Psi^\rho_{ABE}$.

Note in particular that for a MES of rank M, as defined by (4),

$$I_0^{A\to B}(\Psi_{AB}^M) = I^{A\to B}(\Psi_{AB}^M) = \log M. \tag{21}$$

Another entropic quantity of relevance in this paper is the *min-entropy of a state*, which is defined for any state ρ as follows:

$$S_{\min}(\rho) = -\log\big[\lambda_{\max}(\rho)\big], \tag{22}$$

where $\lambda_{\max}(\rho)$ denotes the maximum eigenvalue of the state ρ.

For one-shot entanglement distillation protocols it is natural to allow for a finite accuracy, i.e., a non-zero error (say $\varepsilon \geq 0$), in the extraction of singlets from a given state. In this case the optimal rates of the protocols are given by "smoothed versions" of the entropic quantities introduced above. In order to define them we consider the following sets of positive operators for any normalized state ρ, and any $\varepsilon > 0$:

$$\mathfrak{b}(\rho;\varepsilon) := \left\{\sigma : \sigma \geq 0,\ \mathrm{Tr}[\sigma] = 1,\ F^2(\rho,\sigma) \geq 1-\varepsilon^2\right\}, \tag{23}$$

$$\mathfrak{p}(\rho;\varepsilon) := \{P : 0 \leq P \leq \mathbb{1},\ \mathrm{Tr}[P\rho] \geq 1-\varepsilon\}\,. \tag{24}$$

Henceforth we shall refer to $\mathfrak{b}(\rho;\varepsilon)$ as the $\varepsilon-$ball around the state ρ.

Further, by restricting the states σ in (23) to be pure states, we obtain the subset

$$\mathfrak{b}_*(\rho;\varepsilon) := \{|\varphi\rangle :\ \varphi \in \mathfrak{b}(\rho;\varepsilon)\}\,. \tag{25}$$

It was proved in [25] that for a bipartite pure state $|\phi_{AB}\rangle$, for any $\varepsilon \geq 0$,

$$\{\mathrm{Tr}_A[\varphi_{AB}] : \varphi_{AB} \in \mathfrak{b}_*(\phi_{AB};\varepsilon)\} = \mathfrak{b}(\rho_B^\phi;\varepsilon), \tag{26}$$

where $\rho_\phi^B := \mathrm{Tr}_A\,\phi_{AB}$.

The relevant smoothed entropic quantities are then defined as follows:

Definition 1. For any given $\varepsilon \geq 0$ the *smoothed* min-entropy of a state ρ is defined as

$$S_{\min}^{\varepsilon}(\rho) := \max_{\bar{\rho}\in\mathfrak{b}(\rho;\varepsilon)} S_{\min}(\bar{\rho}). \tag{27}$$

We consider two different smoothed versions of the zero-coherent information, defined as follows:

Definition 2. The state-smoothed zero-coherent information is given by

$$I_{0,\varepsilon}^{A\to B}(\rho_{AB}) := \max_{\bar{\rho}_{AB}\in\mathfrak{b}(\rho_{AB};\varepsilon)} \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S_0(\bar{\rho}_{AB}\|\mathbb{1}_A\otimes\sigma_B), \tag{28}$$

and the operator-smoothed zero-coherent information is given by

$$\widetilde{I}_{0,\varepsilon}^{A\to B}(\rho_{AB}) := \max_{P\in\mathfrak{p}(\rho_{AB};\varepsilon)} \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S_0^P(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B). \tag{29}$$

The following technical lemmas involving the operator-smoothed coherent information are used in proving some of our main results.

Lemma 4. *If for a bipartite state ρ_{AB} and a pure state $|\psi_{AB}\rangle$, for any given $\varepsilon \geq 0$,*

$$F^2(\rho_{AB},\psi_{AB}) \equiv \mathrm{Tr}[\rho_{AB}\psi_{AB}] \geq 1-\varepsilon, \tag{30}$$

then

$$\widetilde{I}_{0,\varepsilon}^{A\to B}(\rho_{AB}) \geq I_0^{A\to B}(\psi_{AB}). \tag{31}$$

Proof. From (30) it follows that $\psi_{AB} \in \mathfrak{p}(\rho_{AB};\varepsilon)$. Using this fact, (29) and (14), we obtain

$$\begin{aligned}
\widetilde{I}_{0,\varepsilon}^{A\to B}(\rho_{AB}) &\geq \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)}\Big[-\log\mathrm{Tr}\big(\sqrt{\psi_{AB}}\Pi_{\rho_{AB}}\sqrt{\psi_{AB}}(\mathbb{1}_A\otimes\sigma_B)\big)\Big] \\
&\geq \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)}\Big[-\log\mathrm{Tr}\big(\psi_{AB}(\mathbb{1}_A\otimes\sigma_B)\big)\Big] \\
&= I_0^{A\to B}(\psi_{AB}).
\end{aligned} \tag{32}$$

where the second inequality follows from the fact that $\sqrt{\psi_{AB}}\Pi_{\rho_{AB}}\sqrt{\psi_{AB}} \leq \psi_{AB}$. ∎

Lemma 5. *For any bipartite pure state $|\phi_{AB}\rangle$, any LOCC map $\Lambda : AB \mapsto A'B'$, and any $\varepsilon \geq 0$,*

$$\widetilde{I}^{A\to B}_{0,2\sqrt{\varepsilon}}(\phi_{AB}) \geq \widetilde{I}^{A'\to B'}_{0,\varepsilon}(\Lambda(\phi_{AB})). \tag{33}$$

Proof. Since the LOCC map Λ acts on a pure state, without loss of generality we can assume it to be of the Lo-Popescu form (12). Defining $\omega_{A'B'} := \Lambda(\phi_{AB})$, we have, starting from (29),

$$\begin{aligned}
\widetilde{I}^{A'\to B'}_{0,\varepsilon}(\Lambda(\phi_{AB})) &= \max_{P\in\mathfrak{p}(\omega_{A'B'};\varepsilon)} \min_{\sigma_{B'}\in\mathfrak{S}(\mathscr{H}_{B'})} \left\{ -\log \mathrm{Tr}\left[\sqrt{P}\Pi_{\omega_{A'B'}}\sqrt{P}\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\} \\
&= \min_{\sigma_{B'}\in\mathfrak{S}(\mathscr{H}_{B'})} \left\{ -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\} \\
&\leq -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ (\mathbb{1}_{A'}\otimes\tilde{\sigma}_{B'})\right] \\
&= -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ \Lambda(\mathbb{1}_A\otimes\sigma_B)\right] \\
&= -\log \mathrm{Tr}\left[\Lambda^*\left(\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\right)\ (\mathbb{1}_A\otimes\sigma_B)\right],
\end{aligned} \tag{34}$$

for any state $\sigma_B \in \mathfrak{S}(\mathscr{H}_B)$. In the above, P_0 is the operator in $\mathfrak{p}(\omega_{A'B'};\varepsilon)$ for which the maximum in the first line is achieved; $\tilde{\sigma}_{B'}$ is a state in $\mathfrak{S}(\mathscr{H}_{B'})$ such that $\tilde{\sigma}_{B'} = \Lambda(\mathbb{1}_A\otimes\sigma_B)$, and Λ^* denotes the dual map of Λ, defined, for any operator X and state ρ, as $\mathrm{Tr}[X\Lambda(\rho)] = \mathrm{Tr}[\Lambda^*(X)\rho]$.

Let us now define $\widetilde{Q}_{AB} := \Lambda^*(\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0})$. Then, continuing from equation (34), we obtain

$$\begin{aligned}
\widetilde{I}^{A'\to B'}_{0,\varepsilon}(\Lambda(\phi_{AB})) &\leq -\log \mathrm{Tr}\left[\widetilde{Q}_{AB}\ (\mathbb{1}_A\otimes\sigma_B)\right] \\
&\leq -\log \mathrm{Tr}\left[\sqrt{\widetilde{Q}_{AB}}\ \phi_{AB}\ \sqrt{\widetilde{Q}_{AB}}\ (\mathbb{1}_A\otimes\sigma_B)\right],
\end{aligned} \tag{35}$$

for any state σ_B and any pure state ϕ_{AB}, since $\widetilde{Q}_{AB} \geq \sqrt{\widetilde{Q}_{AB}}\phi_{AB}\sqrt{\widetilde{Q}_{AB}}$. Let us now choose σ_B to be the state $\widetilde{\sigma}_B$ achieving the minimum in the second line of (35), i. e.

$$\begin{aligned}
&\min_{\sigma_B}\left\{ -\log \mathrm{Tr}\left[\sqrt{\widetilde{Q}_{AB}}\ \phi_{AB}\ \sqrt{\widetilde{Q}_{AB}}\ (\mathbb{1}_A\otimes\sigma_B)\right]\right\} \\
&\quad = -\log \mathrm{Tr}\left[\sqrt{\widetilde{Q}_{AB}}\ \phi_{AB}\ \sqrt{\widetilde{Q}_{AB}}\ (\mathbb{1}_A\otimes\widetilde{\sigma}_B)\right],
\end{aligned} \tag{36}$$

so that

$$\widetilde{I}^{A'\to B'}_{0,\varepsilon}(\Lambda(\phi_{AB})) \leq \min_{\sigma_B}\left\{ -\log \mathrm{Tr}\left[\sqrt{\widetilde{Q}_{AB}}\ \phi_{AB}\ \sqrt{\widetilde{Q}_{AB}}\ (\mathbb{1}_A\otimes\sigma_B)\right]\right\} \tag{37}$$

We next prove that $\widetilde{Q}_{AB} \in \mathfrak{p}(\phi_{AB}; 2\sqrt{\varepsilon})$. In fact, since $P_0 \in \mathfrak{p}(\omega_{A'B'}; \varepsilon)$, by the Gentle Measurement Lemma,

$$\left\| \Lambda(\phi_{AB}) - \sqrt{P_0}\Lambda(\phi_{AB})\sqrt{P_0} \right\|_1 \leq 2\sqrt{\varepsilon}. \tag{38}$$

We therefore have, by definition of $\widetilde{Q}_{AB}$,

$$\begin{aligned}
\operatorname{Tr}\left[\widetilde{Q}_{AB}\phi_{AB}\right] &= \operatorname{Tr}\left[\sqrt{P_0}\Pi_{\Lambda(\phi_{AB})}\sqrt{P_0}\ \Lambda(\phi_{AB})\right] \\
&= \operatorname{Tr}\left[\Pi_{\Lambda(\phi_{AB})}\sqrt{P_0}\Lambda(\phi_{AB})\sqrt{P_0}\right] \\
&= \operatorname{Tr}\left[\Pi_{\Lambda(\phi_{AB})}\Lambda(\phi_{AB})\right] \\
&\qquad + \operatorname{Tr}\left[\Pi_{\Lambda(\phi_{AB})}\big(\sqrt{P_0}\ \Lambda(\phi_{AB})\sqrt{P_0} - \Lambda(\phi_{AB})\big)\right] \\
&\geq 1 - \|\sqrt{P_0}\Lambda(\phi_{AB})\sqrt{P_0} - \Lambda(\phi_{AB})\|_1 \\
&\geq 1 - 2\sqrt{\varepsilon},
\end{aligned} \tag{39}$$

where the second line follows from the cyclicity of the trace, and the last inequality follows from (38). This implies that $\widetilde{Q}_{AB} \in \mathfrak{p}(\phi_{AB}; 2\sqrt{\varepsilon})$. Hence, we have from (37)

$$\begin{aligned}
\widetilde{I}_{0,\varepsilon}^{A'\to B'}(\Lambda(\phi_{AB})) &\leq \min_{\sigma_B}\left\{ -\log \operatorname{Tr}\left[\sqrt{\widetilde{Q}_{AB}}\ \phi_{AB}\ \sqrt{\widetilde{Q}_{AB}}\ (\mathbb{1}_A \otimes \sigma_B)\right]\right\} \\
&\leq \max_{P\in\mathfrak{p}(\phi_{AB};2\sqrt{\varepsilon})} \min_{\sigma_B} -\log \operatorname{Tr}\left[\sqrt{P}\ \phi_{AB}\ \sqrt{P}\ (\mathbb{1}_A \otimes \sigma_B)\right] \\
&\equiv \widetilde{I}_{0,2\sqrt{\varepsilon}}^{A\to B}(\phi_{AB}),
\end{aligned} \tag{40}$$

which completes the proof. ∎

Lemma 6. *For any bipartite pure state $|\phi_{AB}\rangle$ and any $\varepsilon \geq 0$,*

$$I_{0,\varepsilon}^{A\to B}(\phi_{AB}) \geq S_{\min}^{\varepsilon}(\rho_A^{\phi}), \tag{41}$$

where $\rho_A^{\phi} := \operatorname{Tr}_B \phi_{AB}$. Further,

$$\widetilde{I}_{0,\varepsilon}^{A\to B}(\phi_{AB}) \leq S_{\min}^{2\sqrt{\varepsilon}}(\rho_A^{\phi}) - \log(1-\varepsilon). \tag{42}$$

Proof. We first prove (41). Starting from (28) we have:

$$\begin{aligned} I_{0,\varepsilon}^{A\to B}(\phi_{AB}) :&= \max_{\bar\rho_{AB}\in\mathfrak{b}(\phi_{AB};\varepsilon)} \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S_0(\bar\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B) \\ &\geq \max_{\bar\varphi_{AB}\in\mathfrak{b}_*(\phi_{AB};\varepsilon)} \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} S_0(\bar\varphi_{AB}\|\mathbb{1}_A\otimes\sigma_B) \\ &= \max_{\bar\varphi_{AB}\in\mathfrak{b}_*(\phi_{AB};\varepsilon)} \min_{\sigma_B\in\mathfrak{S}(\mathscr{H}_B)} \{-\log \mathrm{Tr}\left[\bar\varphi_{AB}(\mathbb{1}_A\otimes\sigma_B)\right]\} \\ &= \max_{\bar\varphi_{AB}\in\mathfrak{b}_*(\phi_{AB};\varepsilon)} \left\{-\log\lambda_{\max}(\rho_B^{\bar\varphi})\right\} \\ &= \max_{\bar\rho_B\in\mathfrak{b}(\rho_B^\phi;\varepsilon)} S_{\min}(\bar\rho_B) \\ &= S_{\min}^{\varepsilon}(\rho_B^\phi), \end{aligned} \tag{43}$$

where in the fifth line we made use of (26).

Next, we prove (42). By Lemma 2 for any $P\in\mathfrak{p}(\phi;\varepsilon)$, the state $\sqrt{P}|\phi\rangle$ is a pure state such that $F^2\left(\sqrt{P}|\phi\rangle,|\phi\rangle\right)\geq 1-2\sqrt{\varepsilon}$. Let us define the following two sets, for any given bipartite pure state ϕ_{AB} and any $\varepsilon'=2\sqrt{\varepsilon}$:

$$\mathscr{A}_1^{\varepsilon'}(\phi_{AB}) := \left\{ |\varphi_{AB}\rangle\in\mathscr{H}_A\otimes\mathscr{H}_B : |\varphi_{AB}\rangle = \frac{\sqrt{P}|\phi_{AB}\rangle}{\sqrt{\mathrm{Tr}[P\phi_{AB}]}}, P\in\mathfrak{p}(\phi_{AB};\varepsilon')\right\}. \tag{44}$$

Obviously, $\mathscr{A}_1^{\varepsilon'}(\phi_{AB})\subseteq\mathfrak{b}_*(\phi_{AB};\varepsilon')$, with the set $\mathfrak{b}_*(\phi_{AB};\varepsilon')$ being defined by (25). Then,

$$\begin{aligned} \widetilde{I}_{0,\varepsilon}^{A\to B}(\phi_{AB}) &= \max_{P\in\mathfrak{p}(\phi_{AB};\varepsilon)} \min_{\sigma_B}\Big[-\log\mathrm{Tr}\big(\sqrt{P}\phi_{AB}\sqrt{P}(\mathbb{1}\otimes\sigma_B)\big)\Big] \\ &\leq \max_{|\varphi_{AB}\rangle\in\mathscr{A}_1^{\varepsilon'}(\phi_{AB})} \min_{\sigma_B}\Big[-\log\mathrm{Tr}\big(\varphi_{AB}(\mathbb{1}\otimes\sigma_B)\big)\Big] - \log(1-\varepsilon) \\ &\leq \max_{|\varphi_{AB}\rangle\in\mathfrak{b}_*(\phi_{AB};\varepsilon')} \min_{\sigma_B}\Big[-\log\mathrm{Tr}\big(\varphi_{AB}(\mathbb{1}\otimes\sigma_B)\big)\Big] - \log(1-\varepsilon), \\ &= \max_{\bar\rho_B\in\mathfrak{b}(\rho_\phi^B;\varepsilon')} \min_{\sigma_B}\Big[-\log\mathrm{Tr}\big(\bar\rho_B\sigma_B\big)\Big] - \log(1-\varepsilon), \\ &= \max_{\bar\rho_B\in\mathfrak{b}(\rho_B;\varepsilon')}\big[-\log\lambda_{\max}(\bar\rho_B)\big] - \log(1-\varepsilon) \\ &= S_{\min}^{\varepsilon'}(\rho_B^\phi) - \log(1-\varepsilon) \qquad (45) \\ &= S_{\min}^{2\sqrt{\varepsilon}}(\rho_A^\phi) - \log(1-\varepsilon), \qquad (46) \end{aligned}$$

where $\rho_B^\phi := \mathrm{Tr}_A\,\phi_{AB}$ and $\rho_\phi^A := \mathrm{Tr}_B\,\phi_{AB}$. In the above, the second inequality follows from the fact that $\mathscr{A}_1^{\varepsilon'}(\phi_{AB})\subseteq\mathfrak{b}_*(\phi_{AB};\varepsilon')$, the third inequality follows from the fact that $\mathfrak{b}_*(\phi_{AB};\varepsilon')=\mathfrak{b}(\rho_B^\phi;\varepsilon')$ as stated in (26), and the last identity holds because ϕ_{AB} is a pure state. ∎

3 Distillable entanglement of a single pure state

In order to approach the problem of quantifying the one-shot EoA of an arbitrary bipartite mixed state, we start from the simple but insightful case in which two distant parties, say Alice and Bob, initially share a single copy of a *pure state* $|\phi_{AB}\rangle$. Their aim is to distill entanglement from this shared state (i.e., convert the state to a maximally entangled state) using local operations and classical communication (LOCC) only. For sake of generality, we consider the situation where, for any given $\varepsilon \geq 0$, the final state of the protocol is ε-close to a maximally entangled state, with respect to a suitable distance measure. More precisely, we require the fidelity (5) between the final state of the protocol and a maximally entangled state to be $\geq 1 - \varepsilon$.

Definition 3 (ε-achievable distillation rates for pure states[1])**.** For any given $\varepsilon \geq 0$, a real number $R \geq 0$ is said to be an *ε-achievable rate* for one-shot entanglement distillation of a pure state $\phi_{AB} := |\phi_{AB}\rangle\langle\phi_{AB}|$, if there exists an integer $M \geq 2^R$ and a maximally entangled state $\Psi^M_{A'B'}$ such that

$$F^2\left(\Lambda(\phi_{AB}), \Psi^M_{A'B'}\right) \geq 1 - \varepsilon, \tag{47}$$

for some LOCC operation $\Lambda : AB \mapsto A'B'$.

Definition 4 (One-shot pure-state distillable entanglement)**.** For any given $\varepsilon \geq 0$, the one-shot distillabe entanglement, $E_D(\phi_{AB}; \varepsilon)$, of a pure state ϕ_{AB} is the maximum of all ε-achievable entanglement distillation rates for the state ϕ_{AB}.

Bounds on the one-shot distillable entanglement of a pure state ϕ_{AB} are given by the following theorem.

Theorem 1. *For any bipartite pure state ϕ_{AB} and any $\varepsilon \geq 0$,*

$$S^{\varepsilon}_{\min}(\rho^{\phi}_A) - \Delta \leq E_D(\phi_{AB}; \varepsilon) \leq S^{\varepsilon'}_{\min}(\rho^{\phi}_A) - \log(1 - 2\sqrt{\varepsilon}), \tag{48}$$

where $\rho^{\phi}_A := \mathrm{Tr}_B\, \phi_{AB}$, $\varepsilon' = \sqrt{2\sqrt{\varepsilon}}$, and $0 \leq \Delta \leq 1$ is a number included to ensure that the lower bound in (48) is the logarithm of an integer number.

Remark 1. The above theorem shows that, for any given $\varepsilon \geq 0$, the smoothed min-entropy $S^{\varepsilon}_{\min}(\rho^{\phi}_A)$ essentially characterizes the one-shot distillable entanglement of the bipartite pure state $|\phi_{AB}\rangle$.

[1] For the more general case of mixed states, see [28]

Remark 2. It is interesting to compare the lower bound of Theorem 1 with the one-shot hashing bound proved in Lemma 2 of [28] for an arbitrary (possibly mixed) state. For pure states, using Lemma 6, the latter yields:

$$E_D(\phi_{AB};\varepsilon) \geq S_{\min}^{\varepsilon/8}(\rho_A^\phi) + \log\left(\frac{1}{d} + \frac{\varepsilon^2}{4}\right) - \Delta, \tag{49}$$

where $d = \dim \mathscr{H}_A$. It is evident that the bound in Theorem 1 is tighter than (49), in particular because there is no explicit logarithmic dependence on the smoothing parameter ε. From the technical point of view, this arises because, for the case of pure states, we can directly employ Nielsen's majorization criterion and hence do not need to use random coding arguments, which are necessary in the general case.

The proof of Theorem 1 can be divided into the following two lemmas.

Lemma 7. *For any bipartite pure state ϕ_{AB} and any $\varepsilon \geq 0$,*

$$E_D(\phi_{AB};\varepsilon) \geq S_{\min}^{\varepsilon}(\rho_A^\phi) - \Delta, \tag{50}$$

where $\Delta \geq 0$ is the least number such that the left hand side is equal to the logarithm of a positive integer.

Proof. Let us begin by considering the case $\varepsilon = 0$. In this case, Nielsen's majorization theorem [16] implies that, using LOCC, it is possible to exactly convert any pure state $|\phi_{AB}\rangle$ to a maximally entangled state of rank equal to $\left\lfloor \frac{1}{\lambda_{\max}} \right\rfloor$, where $\lambda_{\max}$ denotes the maximum eigenvalue of the reduced density matrix ρ_A^ϕ. Using the definition (27) of the min-entropy we then infer that

$$E_D(\phi_{AB};0) \geq \log\left\lfloor 2^{S_{\min}(\rho_A^\phi)} \right\rfloor. \tag{51}$$

If we allow a finite accuracy in the conversion, a lower bound to the distillable entanglement can be given as follows.

For any $|\bar{\phi}_{AB}\rangle \in \mathfrak{b}_*(\phi_{AB};\varepsilon)$, by Nielsen's theorem, there exists an LOCC map $\bar{\Lambda}$ such that

$$F^2\left(\bar{\Lambda}\left(\bar{\phi}_{AB}\right), \Psi_{A'B'}^{\bar{M}}\right) = 1, \tag{52}$$

where $\log \bar{M} := S_{\min}\left(\rho_A^{\bar{\phi}}\right)$.

On the other hand, due to the monotonicity of fidelity under the action of a completely positive trace-preserving map,

$$\begin{aligned} 1 - \varepsilon \leq 1 - \varepsilon^2 &\leq F^2(\bar{\phi}_{AB}, \phi_{AB}) \\ &\leq F^2\left(\bar{\Lambda}\left(\bar{\phi}_{AB}\right), \bar{\Lambda}(\phi_{AB})\right) \\ &= F^2\left(\Psi_{A'B'}^{\bar{M}}, \bar{\Lambda}(\phi_{AB})\right). \end{aligned} \tag{53}$$

This yields the bound $E_D(\phi_{AB};\varepsilon) \geq \log \bar{M}$, for any $|\bar{\phi}_{AB}\rangle \in \mathfrak{b}_*(\phi_{AB};\varepsilon)$. In particular, we have that

$$E_D(\phi_{AB};\varepsilon) \geq \max_{\bar{\phi}_{AB}\in\mathfrak{b}_*(\phi_{AB};\varepsilon)} \log \left\lfloor 2^{S_{\min}(\rho^{\bar{\phi}}_A)} \right\rfloor. \tag{54}$$

Since the two sets $\{\mathrm{Tr}_B[\bar{\phi}_{AB}] : \bar{\phi}_{AB} \in \mathfrak{b}_*(\phi_{AB})\}$ and $\mathfrak{b}(\rho^{\phi}_A;\varepsilon)$ coincide [25], we finally arrive at

$$E_D(\phi_{AB};\varepsilon) \geq \log \left\lfloor 2^{S^{\varepsilon}_{\min}(\rho^{\phi}_A)} \right\rfloor. \tag{55}$$

∎

Lemma 8. *For any bipartite pure state ϕ_{AB} and any $\varepsilon \geq 0$,*

$$E_D(\phi_{AB};\varepsilon) \leq S^{\varepsilon'}_{\min}(\rho^{\phi}_A), \tag{56}$$

for $\varepsilon' = \sqrt{2\sqrt{\varepsilon}}$.

Proof. Let r be the maximum of all achievable rates of entanglement distillation for the pure state ϕ_{AB}, i.e. $\log r = E_D(\phi_{AB};\varepsilon)$. This means that there exists an LOCC transformation Λ that maps $|\phi_{AB}\rangle$ into a state $\omega_{A'B'} = \Lambda(\phi_{AB})$ which is ε-close to a maximally entangled state $|\Psi^r_{A'B'}\rangle$ of rank r, i.e., $F^2(\Lambda(\phi_{AB}, \Psi^r_{A'B'}) \geq 1-\varepsilon$. Then,

$$\begin{aligned} E_D(\phi_{AB};\varepsilon) &= \log r \\ &= I_0^{A'\to B'}(\Psi^r_{A'B'}) \\ &\leq \widetilde{I}^{A'\to B'}_{0,\varepsilon}(\Lambda(\phi_{AB})) \\ &\leq \widetilde{I}^{A\to B}_{0,2\sqrt{\varepsilon}}(\phi_{AB}) \\ &\leq S^{\varepsilon'}_{\min}(\rho^{\phi}_A) - \log(1-2\sqrt{\varepsilon}), \end{aligned} \tag{57}$$

for $\varepsilon' = \sqrt{2\sqrt{\varepsilon}}$, where the first, second and third inequalities follow from Lemma 4, Lemma 5 and Lemma 6, respectively. ∎

4 One-shot entanglement of assistance

As stated in the introduction, the definition of the EoA arises naturally when considering the task in which Alice and Bob distill entanglement from an initial mixed bipartite state ρ_{AB} which they share, when a third party (say Charlie), who holds the purification of the state, assists them, by doing local operations on his share and communicating classical bits to Alice and Bob.

In order to express these ideas in a mathematically sound form, we start by noticing that any strategy that Charlie may employ can be described

as the measurement of a positive operator-valued measure (POVM) $\{P_C^i\}_i$, followed by the communication, to both Alice and Bob, of the resulting classical outcome i. Since the state shared between Alice, Bob, and Charlie is pure, say $|\Psi_{ABC}^{\rho}\rangle$, Charlie's POVM's are in one-to-one correspondence with decompositions of ρ_{AB} into ensembles $\{p_i, \rho_{AB}^i\}_i$, via the relation $p_i\rho_{AB}^i := \mathrm{Tr}_C[\Psi_{ABC}^{\rho}\ (\mathbb{1}_{AB} \otimes P_C^i)]$. The fact that Charlie announces which outcome he got, means that Alice and Bob can apply a different LOCC map for each value of i.

An important point to stress now is that, in general, the distillation process is allowed to be approximate. This is needed, in particular, if one later wants to recover, from the one-shot setting, the usual asymptotic scenario, where errors are required to vanish asymptotically but are finite otherwise. In the classically-assisted case we are studying here, since the index i is visible to Alice and Bob, they can apply a different LOCC map Λ_i for each state ρ_{AB}^i. We can hence choose to evaluate the distillation accuracy according to a worst-case or an average criterion. Here we choose the average fidelity as a measure of the "expected" accuracy. This leads us to define the maximum amount of entanglement that can be distilled in the assisted case, namely, the *one-shot entanglement of assistance*, as,

$$\begin{aligned} &E_A(\rho_{AB};\varepsilon) \\ &:= \max_{\{P_C^i\}_i} \max_{M\in\mathbb{N}} \left\{ \log M : \max_{\{\Lambda_{AB}^i\}_i} F^2\left(\sum_i p_i \Lambda^i(\rho_{AB}^i), \Psi_{A'B'}^M\right) \geq 1-\varepsilon \right\}, \end{aligned} \tag{58}$$

where each Λ^i is an LOCC map from AB to $A'B'$.

As proved in Appendix A, the maximization over Charlie's measurement in the above definition can always be restricted, without loss of generality, to rank-one POVM's. Since rank-one POVM's at Charlie's side are in one-to-one correspondence with pure state ensemble decompositions of ρ_{AB}, we can equivalently write

$$\begin{aligned} &E_A(\rho_{AB};\varepsilon) \\ &= \max_{\substack{\{p_i,\phi_{AB}^i\}_i \\ \sum_i p_i\phi_{AB}^i = \rho_{AB}}} \max_{M\in\mathbb{N}} \left\{ \log M : \max_{\{\Lambda_{AB}^i\}_i} F^2\left(\sum_i p_i \Lambda^i(\phi_{AB}^i), \Psi_{A'B'}^M\right) \geq 1-\varepsilon \right\}. \end{aligned} \tag{59}$$

In order to quantify $E_A(\rho_{AB};\varepsilon)$ then, it is sufficient to quantify the maximum expected amount of entanglement that can be distilled, in average, from any given ensemble of pure bipartite states. This is the aim of the following section.

5 Distillable entanglement of an ensemble of pure states

Given an ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ of pure states, we define, for any given $\varepsilon \geq 0$ the one-shot distillable entanglement of $\mathfrak{E}$ as

$$E_D(\mathfrak{E};\varepsilon) := \max_{M\in\mathbb{N}} \left\{ \log M : \max_{\{\Lambda^i_{AB}\}_i} F^2\left(\sum_i p_i \Lambda^i(\phi^i_{AB}), \Psi^M_{A'B'}\right) \geq 1-\varepsilon \right\}, \tag{60}$$

where each Λ^i is an LOCC map from AB to $A'B'$. According with equation (59), the one-shot entanglement of assistance E_A of a given mixed state ρ_{AB} is given by

$$E_A(\rho_{AB};\varepsilon) = \max_{\mathfrak{E}} E_D(\mathfrak{E};\varepsilon), \tag{61}$$

where the maximum is over all possible pure state ensemble decompositions $\mathfrak{E}$ of ρ_{AB}.

For any given ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ of pure states, we define the quantity

$$F_{\min}(\mathfrak{E}) := \min_i S_{\min}(\rho^{\phi^i}_A), \tag{62}$$

where $\rho^{\phi^i}_A := \mathrm{Tr}_B\, \phi^i_{AB}$. This quantity can be intuitively interpreted as a conservative estimate of the amount of entanglement present in the ensemble $\mathfrak{E}$. Further, for any such ensemble, and any given $\varepsilon \geq 0$, let us define the set

$$\mathcal{S}_{\leqslant}(\mathfrak{E};\varepsilon) := \left\{ \bar{\mathfrak{E}} = \{\bar{\varphi}^i_{AB}\}_i : \mathrm{Tr}\,\bar{\varphi}^i_{AB} \leq 1, \sum_i p_i F(\bar{\varphi}^i_{AB}, \phi^i_{AB}) \geq 1-\varepsilon \right\}, \tag{63}$$

and let $\mathcal{S}_{=}(\mathfrak{E};\varepsilon)$ denote the set obtained from $\mathcal{S}_{\leqslant}(\mathfrak{E};\varepsilon)$ by restricting the pure states $\bar{\varphi}^i_{AB}$ to be normalized.

Theorem 2. *For any given ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ of pure states, and any $\varepsilon \geq 0$,*

$$\max_{\bar{\mathfrak{E}}\in\mathcal{S}_{=}(\mathfrak{E};\varepsilon')} F_{\min}(\bar{\mathfrak{E}}) - \Delta \;\leq\; E_D(\mathfrak{E};\varepsilon) \;\leq\; \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\varepsilon'')} F_{\min}(\bar{\mathfrak{E}}), \tag{64}$$

where $\varepsilon' = \varepsilon/2$, $\varepsilon'' := \sqrt{2\sqrt{\varepsilon}}$, and $0 \leq \Delta \leq 1$ is a number which is included to ensure that the lower bound in (64) is the logarithm of an integer number.

The proof of this theorem is divided into the following two lemmas.

Lemma 9 (Direct part)**.** *For any pure state ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ and any $\varepsilon \geq 0$,*

$$E_D(\mathfrak{E};\varepsilon) \geq \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{=}(\mathfrak{E};\varepsilon')} F_{\min}(\bar{\mathfrak{E}}) - \Delta, \tag{65}$$

where Δ is the minimum number in $[0,1]$ such that the right hand side is equal to the logarithm of an integer number $M \geq 1$.

Proof. From Theorem 1, we know that, given the pure bipartite state ϕ^i_{AB}, Alice and Bob can distill $\log \left\lfloor 2^{S_{\min}\left(\rho^{\phi^i}_A\right)} \right\rfloor$ ebits with zero error. Hence, given the ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$, Alice and Bob can distill, *without error*, at least $\min_i \log \left\lfloor 2^{S_{\min}\left(\rho^{\phi^i}_A\right)} \right\rfloor$ ebits. For any pure state ensemble $\mathfrak{E}$, let us then introduce the quantity $M(\mathfrak{E}) := \min_i \left\lfloor 2^{S_{\min}\left(\rho^{\phi^i}_A\right)} \right\rfloor$.

If a finite accuracy $\varepsilon > 0$ is allowed, then it is possible to give a lower bound on the one-shot distillable entanglement $E_D(\mathfrak{E}; \varepsilon)$ as follows. Let us consider the set of ensembles of normalized pure states of the form $\bar{\mathfrak{E}} = \{p_i, \bar{\varphi}^i_{AB}\}$, such that $\sum_i p_i F(\phi^i_{AB}, \bar{\varphi}^i_{AB}) \geq 1 - \varepsilon$. Then, for any ensemble $\bar{\mathfrak{E}}$ in such a set, there exist LOCC maps $\Lambda^i : AB \to A'B'$ such that

$$F\left(\sum_i p_i \Lambda^i(\bar{\varphi}^i_{AB}), \Psi^{M(\bar{\mathfrak{E}})}_{AB}\right) = 1, \tag{66}$$

where $\Psi^{M(\bar{\mathfrak{E}})}_{A'B'}$ denotes a maximally entangled state of rank $M(\bar{\mathfrak{E}})$. Equivalently, $\Lambda^i(\bar{\varphi}^i_{AB}) = \Psi^{M(\bar{\mathfrak{E}})}_{AB}$, for all i. Then,

$$\begin{aligned} 1 - \varepsilon &\leq \sum_i p_i F(\phi^i_{AB}, \bar{\varphi}^i_{AB}) \\ &\leq \sum_i p_i F\left(\Lambda^i(\phi^i_{AB}), \Lambda^i(\bar{\varphi}^i_{AB})\right) \\ &\leq F\left(\sum_i p_i \Lambda^i(\phi^i_{AB}), \sum_i p_i \Lambda^i(\bar{\varphi}^i_{AB})\right) \\ &= F\left(\sum_i p_i \Lambda^i(\phi^i_{AB}), \Psi^{M(\bar{\mathfrak{E}})}_{AB}\right), \end{aligned} \tag{67}$$

where the second line follows from the monotonicity of fidelity under completely positive trace-preserving (CPTP) maps, the third line follows from the concavity of the fidelity, and the last identity follows from (66). Hence, we conclude that there exist LOCC maps Λ^i for which

$$F^2\left(\sum_i p_i \Lambda^i(\phi^i_{AB}), \Psi^{M(\bar{\mathfrak{E}})}_{A'B'}\right) \geq 1 - 2\varepsilon, \tag{68}$$

that is,

$$E_D(\mathfrak{E}; 2\varepsilon) \geq \log M(\bar{\mathfrak{E}}), \tag{69}$$

for any $\bar{\mathfrak{E}}$ in the set introduced above. By maximizing $M(\bar{\mathfrak{E}})$ over all such ensembles and comparing the result with the definition in (62), we obtain the statement of the lemma. ∎

Lemma 10 (Converse part)**.** *For any pure state ensemble* $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ *and any* $\varepsilon \geq 0$*,*

$$E_D(\mathfrak{E};\varepsilon) \leq \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\varepsilon')} F_{\min}(\bar{\mathfrak{E}}), \tag{70}$$

where $\varepsilon' = \sqrt{2\sqrt{\varepsilon}}$*.*

Proof. Let r be a positive integer such that $E_D(\mathfrak{E};\varepsilon) = \log r$. According to (60), this means that there exist LOCC maps $\Lambda^i : AB \to A'B'$ such that

$$\mathrm{Tr}\left[\sum_i p_i \Lambda^i(\phi^i_{AB})\ \Psi^r_{A'B'}\right] \geq 1-\varepsilon. \tag{71}$$

Since the maps Λ^i act on pure states, without loss of generality we can assume them to be of the Lo-Popescu form (12).

Further, equation (71) above, in particular, informs us that

$$\Psi^r_{A'B'} \in \mathfrak{p}\left(\sum_i p_i \Lambda^i(\phi^i_{AB});\varepsilon\right). \tag{72}$$

This fact in turns implies that

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &= \log r \\ &= I_0^{A'\to B'}\left(\Psi^r_{A'B'}\right) \\ &\equiv \min_{\sigma_{B'}}\left\{-\log \mathrm{Tr}\left[\Psi^r_{A'B'}\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\} \\ &\leq -\log \mathrm{Tr}\left[\Psi^r_{A'B'}\ (\mathbb{1}_{A'}\otimes\tilde{\sigma}_{B'})\right] \\ &\leq -\log \mathrm{Tr}\left[\left(\Psi^r_{A'B'}\ \Pi_{\sum_i p_i\Lambda^i(\phi^i_{AB})}\ \Psi^r_{A'B'}\right)\ (\mathbb{1}_{A'}\otimes\tilde{\sigma}_{B'})\right], \end{aligned} \tag{73}$$

for any state $\tilde{\sigma}_{B'}$. To obtain the last inequality, we simply used the fact that $\Psi^r_{A'B'} \geq \Psi^r_{A'B'}\Pi\Psi^r_{A'B'}$, for any $0 \leq \Pi \leq \mathbb{1}$. We then choose $\tilde{\sigma}_{B'}$ so that

$$\begin{aligned} &-\log \mathrm{Tr}\left[\left(\Psi^r_{A'B'}\ \Pi_{\sum_i p_i\Lambda^i(\phi^i_{AB})}\ \Psi^r_{A'B'}\right)\ (\mathbb{1}_{A'}\otimes\tilde{\sigma}_{B'})\right] \\ &= \min_{\sigma_{B'}}\left\{-\log \mathrm{Tr}\left[\left(\Psi^r_{A'B'}\ \Pi_{\sum_i p_i\Lambda^i(\phi^i_{AB})}\ \Psi^r_{A'B'}\right)\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\}. \end{aligned} \tag{74}$$

From (72), (73) and (74) we infer that

$$E_D(\mathfrak{E};\varepsilon) \leq \widetilde{I}^{A'\to B'}_{0,\varepsilon}\left(\sum_i p_i \Lambda_i(\phi^i_{AB})\right). \tag{75}$$

Let us now introduce an auxiliary system Z and an orthonormal basis for it $\{|i_Z\rangle\}$ that keeps track of the classical outcome i labeling the states in $\mathfrak{E}$. Let us denote by π^i_Z the projector $|i\rangle\langle i|_Z$. By further introducing the states $\omega_{A'B'} := \sum_i p_i \Lambda_i(\phi^i_{AB})$ and $\omega_{A'B'Z} := \sum_i p_i \Lambda_i(\phi^i_{AB}) \otimes \pi^i_Z$, so that $\omega_{A'B'} = \mathrm{Tr}_Z\, \omega_{A'B'Z}$, we have

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &\leq \widetilde{I}^{A'\to B'}_{0,\varepsilon}(\omega_{A'B'}) \\ &\equiv \max_{P\in\mathfrak{p}(\omega_{A'B'};\varepsilon)} \min_{\sigma_{B'}} \left\{ -\log \mathrm{Tr}\left[\sqrt{P}\Pi_{\omega_{A'B'}}\sqrt{P}\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\} \\ &= \min_{\sigma_{B'}} \left\{ -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ (\mathbb{1}_{A'}\otimes\sigma_{B'})\right]\right\} \\ &\leq -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ (\mathbb{1}_{A'}\otimes\bar{\nu}_{B'})\right], \end{aligned} \tag{76}$$

where the operator P_0 in the third line is the one achieving the maximum, and $\bar{\nu}_{B'}$ in the fourth line is any state in $\mathfrak{S}(\mathscr{H}_{B'})$. In particular, since $\Pi_{\omega_{A'B'}}\otimes\mathbb{1}_Z \geq \Pi_{\omega_{A'B'Z}}$, we have that

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &\leq -\log \mathrm{Tr}\left[\sqrt{P_0}\Pi_{\omega_{A'B'}}\sqrt{P_0}\ (\mathbb{1}_{A'}\otimes\bar{\nu}_{B'})\right] \\ &= -\log \mathrm{Tr}\left[\sqrt{P_0\otimes\mathbb{1}_Z}(\Pi_{\omega_{A'B'}}\otimes\mathbb{1}_Z)\sqrt{P_0\otimes\mathbb{1}_Z}\ (\mathbb{1}_{A'}\otimes\bar{\nu}_{B'Z})\right] \\ &\leq -\log \mathrm{Tr}\left[\sqrt{P_0\otimes\mathbb{1}_Z}\Pi_{\omega_{A'B'Z}}\sqrt{P_0\otimes\mathbb{1}_Z}\ (\mathbb{1}_{A'}\otimes\bar{\nu}_{B'Z})\right], \end{aligned} \tag{77}$$

for any state $\bar{\nu}_{B'Z}$.

Let us then choose $\bar{\nu}_{B'Z}$ to be the state such that

$$\begin{aligned} &-\log \mathrm{Tr}\left[\sqrt{P_0\otimes\mathbb{1}_Z}\Pi_{\omega_{A'B'Z}}\sqrt{P_0\otimes\mathbb{1}_Z}\ (\mathbb{1}_{A'}\otimes\bar{\nu}_{B'Z})\right] \\ &= \min_{\nu_{B'Z}} \left\{ -\log \mathrm{Tr}\left[\sqrt{P_0\otimes\mathbb{1}_Z}\Pi_{\omega_{A'B'Z}}\sqrt{P_0\otimes\mathbb{1}_Z}\ (\mathbb{1}_{A'}\otimes\nu_{B'Z})\right]\right\}. \end{aligned} \tag{78}$$

Moreover, note that $(P_0\otimes\mathbb{1}_Z)\in\mathfrak{p}(\omega_{A'B'Z};\varepsilon)$, since $P_0\in\mathfrak{p}(\omega_{A'B'};\varepsilon)$. In fact, the operator $(P_0\otimes\mathbb{1}_Z)$ also belongs to the following set of quantum-classical (q-c) operators:

$$\begin{aligned} &\mathfrak{p}_{\mathrm{qc}}(\sigma_{ABZ};\varepsilon) := \\ &\left\{ P_{ABZ} = \sum_i P^i_{AB}\otimes\pi^i_Z \,\middle|\, 0\leq P^i_{AB}\leq\mathbb{1}_{AB},\ \mathrm{Tr}\big(P_{ABZ}\sigma_{ABZ}\big)\geq 1-\varepsilon \right\}. \end{aligned} \tag{79}$$

Hence, we can write

$$E_D(\mathfrak{E};\varepsilon) \leq \max_{Q\in\mathfrak{p}_{\mathrm{qc}}(\omega_{A'B'Z};\varepsilon)} \min_{\nu_{B'Z}} \left\{ -\log \mathrm{Tr}\left[\sqrt{Q}\Pi_{\omega_{A'B'Z}}\sqrt{Q}\ (\mathbb{1}_{A'}\otimes\nu_{B'Z})\right]\right\} \tag{80}$$

Let the Kraus representations of the CPTP maps $\Lambda_i : AB \mapsto A'B'$ satisfying (71) be written as $\Lambda_i(\rho) = \sum_{\mu_i} V_{\mu_i} \rho V_{\mu_i}^\dagger$, so that $\sum_{\mu_i} V_{\mu_i}^\dagger V_{\mu_i} = \mathbb{1}_{AB}$ for all i. Using these, we construct a CPTP map $\mathscr{M} : ABZ \to A'B'Z$ as

$$\mathscr{M}(\rho_{ABZ}) := \sum_i \sum_{\mu_i} \left(V_{\mu_i} \otimes \pi_Z^i\right) \rho_{ABZ} \left(V_{\mu_i} \otimes \pi_Z^i\right)^\dagger . \tag{81}$$

In terms of the map $\mathscr{M}$ so constructed,

$$\omega_{A'B'Z} = \mathscr{M}\left(\sum_i p_i \phi_{AB}^i \otimes \pi_Z^i\right). \tag{82}$$

Defining the quantum-classical (q-c) state $\sigma_{ABZ} := \sum_i p_i \phi_{AB}^i \otimes \pi_Z^i$, we have, continuing from (80),

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &\leq \max_{Q \in \mathfrak{p}_{\mathrm{qc}}(\mathscr{M}(\sigma_{ABZ});\varepsilon)} \min_{\nu_{B'Z}} \left\{ -\log \mathrm{Tr}\left[\sqrt{Q}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q}\ \ (\mathbb{1}_{A'} \otimes \nu_{B'Z})\right]\right\} \\ &\equiv \min_{\nu_{B'Z}} \left\{ -\log \mathrm{Tr}\left[\sqrt{Q_0}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q_0}\ \ (\mathbb{1}_{A'} \otimes \nu_{B'Z})\right]\right\}, \end{aligned} \tag{83}$$

where $Q_0 \in \mathfrak{p}_{\mathrm{qc}}(\mathscr{M}(\sigma_{ABZ});\varepsilon)$ is the q-c operator achieving the maximum in the second line. This implies that

$$E_D(\mathfrak{E};\varepsilon) \leq -\log \mathrm{Tr}\left[\sqrt{Q_0}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q_0}\ \ (\mathbb{1}_{A'} \otimes \nu_{B'Z})\right], \tag{84}$$

for any state $\nu_{B'Z}$.

Due to the fact that the maps Λ_i are in the Lo-Popescu form (12), it follows that the map $\mathscr{M}$ (obtained from the Λ_i's) is also in the Lo-Popescu form. The identity (13) then implies that

$$E_D(\mathfrak{E};\varepsilon) \leq -\log \mathrm{Tr}\left[\sqrt{Q_0}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q_0}\ \ \mathscr{M}(\mathbb{1}_A \otimes \tilde{\nu}_{BZ})\right], \tag{85}$$

for any state $\tilde{\nu}_{BZ}$. By using the dual map $\mathscr{M}^*$,

$$E_D(\mathfrak{E};\varepsilon) \leq -\log \mathrm{Tr}\left[\mathscr{M}^*\left(\sqrt{Q_0}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q_0}\right)\ \ (\mathbb{1}_A \otimes \tilde{\nu}_{BZ})\right], \tag{86}$$

for any state $\tilde{\nu}_{BZ}$. By denoting the operator $\mathscr{M}^*\left(\sqrt{Q_0}\Pi_{\mathscr{M}(\sigma_{ABZ})}\sqrt{Q_0}\right)$ as $\tilde{Q}_{ABZ}$, we have, for any state $\tilde{\nu}_{BZ}$,

$$E_D(\mathfrak{E};\varepsilon) \leq -\log \mathrm{Tr}\left[\sqrt{\tilde{Q}_{ABZ}}\Pi_{\sigma_{ABZ}}\sqrt{\tilde{Q}_{ABZ}}\ (\mathbb{1}_A \otimes \tilde{\nu}_{BZ})\right], \tag{87}$$

since $\tilde{Q}_{ABZ} \geq \sqrt{\tilde{Q}_{ABZ}}\Pi_{\sigma_{ABZ}}\sqrt{\tilde{Q}_{ABZ}}$. Let us also choose $\tilde{\nu}_{BZ}$ so that

$$\begin{aligned} &-\log \mathrm{Tr}\left[\sqrt{\tilde{Q}_{ABZ}}\Pi_{\sigma_{ABZ}}\sqrt{\tilde{Q}_{ABZ}}\ (\mathbb{1}_A \otimes \tilde{\nu}_{BZ})\right] \\ &= \min_{\nu_{BZ}} \left\{ -\log \mathrm{Tr}\left[\sqrt{\tilde{Q}_{ABZ}}\Pi_{\sigma_{ABZ}}\sqrt{\tilde{Q}_{ABZ}}\ (\mathbb{1}_A \otimes \nu_{BZ})\right]\right\}. \end{aligned} \tag{88}$$

Using the particular form (81) of $\mathscr{M}$, and the facts that σ_{ABZ} is a q-c state and $Q_0 \in \mathfrak{p}_{\text{qc}}(\sigma_{ABZ};\varepsilon)$, we can prove that the operator $\tilde{Q}_{ABZ} \in \mathfrak{p}_{\text{qc}}(\sigma_{ABZ};2\sqrt{\varepsilon})$, using arguments similar to those leading to (39).

Hence, continuing from equation (87), we can write

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &\leq \min_{\nu_{BZ}} \left\{ -\log \text{Tr}\left[\sqrt{\tilde{Q}_{ABZ}}\Pi_{\sigma_{ABZ}}\sqrt{\tilde{Q}_{ABZ}}\ (\mathbb{1}_A \otimes \nu_{BZ})\right]\right\} \\ &\leq \max_{P\in\mathfrak{p}_{\text{qc}}(\sigma_{ABZ};2\sqrt{\varepsilon})} \min_{\nu_{BZ}} \left\{ -\log \text{Tr}\left[\sqrt{P}\Pi_{\sigma_{ABZ}}\sqrt{P}\ \ (\mathbb{1}_A \otimes \nu_{BZ})\right]\right\}. \end{aligned} \tag{89}$$

Let $\varepsilon' := 2\sqrt{\varepsilon}$. Then, for any $P = \sum_i P^i_{AB} \otimes \pi^i_Z$ in $\mathfrak{p}_{\text{qc}}(\sigma_{ABZ};\varepsilon')$, let us define $|\varphi^i_{AB}\rangle := \sqrt{P^i_{AB}}|\phi^i_{AB}\rangle$. As a consequence of Lemma 2, we have that $\sum_i p_i F(\varphi^i_{AB},\phi^i_{AB}) \geq 1-\sqrt{\varepsilon'}$, so that

$$\begin{aligned} E_D(\mathfrak{E};\varepsilon) &\leq \max_{P\in\mathfrak{p}_{\text{qc}}(\sigma_{ABZ};\varepsilon')} \min_{\nu_{BZ}} \left\{ -\log \text{Tr}\left[\sqrt{P}\Pi_{\sigma_{ABZ}}\sqrt{P}\ \ (\mathbb{1}_A \otimes \nu_{BZ})\right]\right\} \\ &\leq \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\sqrt{\varepsilon'})} \min_{\nu_{BZ}} \left\{ -\log \text{Tr}\left[\left(\bar{\varphi}^i_{AB} \otimes \pi^i_Z\right)\ \ (\mathbb{1}_A \otimes \nu_{BZ})\right]\right\} \\ &= \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\sqrt{\varepsilon'})} \min_{i} \min_{\nu_B} \left\{ -\log \text{Tr}\left[\rho^{\bar{\varphi}^i}_B \nu_B)\right]\right\} \\ &= \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\sqrt{\varepsilon'})} \min_{i} \left[-\log \lambda_{\max}(\rho^{\bar{\varphi}^i}_B)\right], \\ &= \max_{\bar{\mathfrak{E}}\in\mathcal{S}_{\leqslant}(\mathfrak{E};\sqrt{\varepsilon'})} \min_{i} S_{\min}(\rho^{\bar{\varphi}^i}_A), \end{aligned} \tag{90}$$

where we used the fact that $\lambda_{\max}(\rho^{\bar{\varphi}^i}_B) = \lambda_{\max}(\rho^{\bar{\varphi}^i}_A) = S_{\min}(\rho^{\bar{\varphi}^i}_A)$, since $\bar{\varphi}^i_{AB}$ is a pure state. ∎

6 Asymptotic entanglement of assistance

Consider the situation in which three parties, Alice, Bob and Charlie jointly possess multiple (say n) copies of a tripartite pure state $|\Psi_{ABC}\rangle$. Alice and Bob, considered in isolation, therefore possess n copies of the state $\rho_{AB} := \text{Tr}_C\, \Psi_{ABC}$, i.e., they share the state $\rho^{\otimes n}_{AB}$. We refer to this situation as the "i.i.d. scenario", in analogy with the classical case of independent and identically distributed (i.i.d.) random variables. We define the asymptotic entanglement of assistance of a state ρ_{AB} as

$$E^{\infty}_A(\rho_{AB}) := \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} E_A(\rho^{\otimes n}_{AB};\varepsilon), \tag{91}$$

where for any $\varepsilon \geq 0$, $E_A(\rho^{\otimes n}_{AB};\varepsilon)$ denotes the one-shot entanglement of assistance of the state $\rho^{\otimes n}_{AB}$, defined in (58) and quantified in (61) and (64).

The same notation $E_A^\infty(\rho_{AB})$ was used in Ref. [7] to denote the *regularized* EoA, formally defined as $\lim_{n\to\infty}\frac{1}{n}E_A(\rho_{AB}^{\otimes n})$ from (2). The aim of this section is to show that the two quantities coincide, so that, in fact, there is no notational inconsistency. At the same time, this provides an alternative proof of the operational interpretation of the regularized EoA given in [7].

The main result of this section is the following theorem:

Theorem 3. *For any bipartite state ρ_{AB}*

$$E_A^\infty(\rho_{AB}) := \lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}E_A(\rho_{AB}^{\otimes n};\varepsilon) = \lim_{n\to\infty}\frac{1}{n}E_A(\rho_{AB}^{\otimes n}), \tag{92}$$

where for any state ω_{AB},

$$E_A(\omega_{AB}) := \max_{\substack{\{p_i,|\varphi_{AB}^i\rangle\}\\ \omega_{AB}=\sum_i p_i\varphi_{AB}^i}} \sum_i p_i S(\rho_A^{\varphi^i}), \tag{93}$$

denotes its entanglement of assistance, with $\rho_A^{\varphi^i} = \mathrm{Tr}_B[\varphi_{AB}^i]$.

In order to prove this, we first need to introduce a few more definitions. Let σ_{ABZ} be a quantum-classical (qc) state, i.e.

$$\sigma_{ABZ} = \sum_i p_i \sigma_{AB}^i \otimes \pi_Z^i, \tag{94}$$

for some probabilities $p_i \geq 0$, $\sum_i p_i = 1$, some normalized states $\sigma_{AB}^i \in \mathfrak{S}(\mathscr{H}_A\otimes\mathscr{H}_B)$, and some orthogonal rank-one projectors $\pi_Z^i = |i\rangle\langle i|_Z$ (that we fix here once and for all). As it has been done already in (79), along the proof of Lemma 10, we define the sets

$$\mathfrak{p}_{\mathrm{qc}}(\sigma_{ABZ};\varepsilon) := \left\{ P_{ABZ} = \sum_i P_{AB}^i \otimes \pi_Z^i \;\middle|\; \begin{array}{l} 0 \leq P_{AB}^i \leq \mathbb{1}_{AB}, \\ \mathrm{Tr}[P\sigma] \geq 1-\varepsilon \end{array} \right\}, \tag{95}$$

and

$$\mathfrak{b}_{\mathrm{qc}}(\sigma_{ABZ};\varepsilon) := \left\{ \bar{\omega}_{ABZ} = \sum_i p_i \bar{\varphi}_{AB}^i \otimes \pi_Z^i \;\middle|\; \begin{array}{l} \left\|\bar{\varphi}_{AB}^i\right\|_1 = \left\|\bar{\varphi}_{AB}^i\right\|_\infty = 1, \\ F(\bar{\omega},\sigma) = \sum_i p_i F(\bar{\varphi}^i,\sigma^i) \geq 1-\varepsilon \end{array} \right\}. \tag{96}$$

The sets defined above are analogous to those introduced in (23) and (24), with the difference that the quantum-classical structure of the argument σ_{ABZ} is here maintained.

For technical reasons that will be apparent in the proofs, we also need to introduce an additional smoothed zero-coherent information, besides those

in (28) and (29), defined as, for any qc state σ_{ABZ} and any $\varepsilon \geq 0$,

$$I_{0,\varepsilon}^{A\rightsquigarrow BZ}(\sigma_{ABZ}) := \max_{\bar{\sigma}_{ABZ}\in\mathfrak{b}_{\mathrm{qc}}(\sigma_{ABZ};\varepsilon)} \min_{\nu_{BZ}\in\mathfrak{S}(\mathscr{H}_B\otimes\mathscr{H}_Z)} S_0(\bar{\sigma}_{ABZ}\|\mathbb{1}_A\otimes\nu_{BZ}). \quad (97)$$

We then proceed by proving the following lemma, which is nothing but a convenient reformulation of Theorem 2:

Lemma 11. *For any bipartite state ρ_{AB} and any $\varepsilon \geq 0$,*

$$\max_{\mathfrak{E}} I_{0,\varepsilon/2}^{A\rightsquigarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}}) - \Delta \leq E_A(\rho_{AB};\varepsilon) \leq \max_{\mathfrak{E}} \widetilde{I}_{0,2\sqrt{\varepsilon}}^{A\rightarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}}), \quad (98)$$

where the maxima are taken over all possible pure state ensembles $\mathfrak{E} = \{p_i, \phi_{AB}^i\}$ such that $\rho_{AB} = \sum_i p_i\phi_{AB}^i$, and for a given ensemble $\mathfrak{E} = \{p_i, \phi_{AB}^i\}$, $\sigma_{ABZ}^{\mathfrak{E}} = \sum_i p_i\phi_{AB}^i\otimes\pi_Z^i$. In the above, the real number $0 \leq \Delta \leq 1$ is included to ensure that the lower bound is equal to the logarithm of a positive integer.

For the sake of clarity, we divide the proof of the Lemma above into two separate lemmas. The first is the following:

Lemma 12. *For any given ensemble $\mathfrak{E} = \{p_i, \phi_{AB}^i\}$ of pure states, and any $\varepsilon \geq 0$,*

$$E_D(\mathfrak{E};\varepsilon) \leq \widetilde{I}_{0,2\sqrt{\varepsilon}}^{A\rightarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}}), \quad (99)$$

where $\sigma_{ABZ}^{\mathfrak{E}} := \sum_i p_i\phi_{AB}^i\otimes\pi_Z^i$, and $\widetilde{I}_{0,2\sqrt{\varepsilon}}^{A\rightarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}})$ is defined in (29).

Proof. The equation number (89) in the proof of Theorem 2, that is,

$$E_D(\mathfrak{E};\varepsilon) \leq \max_{P\in\mathfrak{p}_{\mathrm{qc}}(\sigma_{ABZ};2\sqrt{\varepsilon})} \min_{\nu_{BZ}} \left\{ -\log \mathrm{Tr}\left[\sqrt{P}\Pi_{\sigma_{ABZ}}\sqrt{P}\ \ (\mathbb{1}_A\otimes\nu_{BZ})\right]\right\} \quad (100)$$

already proves the statement, since $\mathfrak{p}_{\mathrm{qc}}(\sigma_{ABZ};2\sqrt{\varepsilon}) \subset \mathfrak{p}(\sigma_{ABZ};2\sqrt{\varepsilon})$. ∎

Lemma 13. *For any given ensemble $\mathfrak{E} = \{p_i, \phi_{AB}^i\}$ of pure states, and any $\varepsilon \geq 0$,*

$$E_D(\mathfrak{E};\varepsilon) \geq I_{0,\varepsilon/2}^{A\rightsquigarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}}). \quad (101)$$

where $\sigma_{ABZ}^{\mathfrak{E}} := \sum_i p_i\phi_{AB}^i\otimes\pi_Z^i$ and $I_{0,\varepsilon/2}^{A\rightsquigarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}})$ is defined in (97).

Proof. The statement is a direct consequence of the lower bound in Theorem 2. This can be shown as follows:

$$\begin{aligned} I_{0,\varepsilon/2}^{A\rightsquigarrow BZ}(\sigma_{ABZ}^{\mathfrak{E}}) :&= \max_{\bar{\sigma}_{ABZ}\in\mathfrak{b}_{\mathrm{qc}}(\sigma_{ABZ};\varepsilon/2)} \min_{\nu_{BZ}} \left\{-\log \mathrm{Tr}\left[\Pi_{\bar{\sigma}_{ABZ}}\ (\mathbb{1}_A\otimes\nu_{BZ})\right]\right\} \\ &= \max_{\substack{\{\bar{\varphi}_{AB}^i\}_i:\mathrm{Tr}\,\bar{\varphi}_{AB}^i=1 \\ \sum_i p_i F(\bar{\varphi}_{AB}^i,\phi_{AB}^i)\geq 1-\varepsilon/2}} \min_i \min_{\nu_B} \left\{-\log \mathrm{Tr}\left[\rho_B^{\bar{\varphi}^i}\ \nu_B\right]\right\} \\ &= \max_{\substack{\{\bar{\varphi}_{AB}^i\}_i:\mathrm{Tr}\,\bar{\varphi}_{AB}^i=1 \\ \sum_i p_i F(\bar{\varphi}_{AB}^i,\phi_{AB}^i)\geq 1-\varepsilon/2}} \min_i \left\{-\log\lambda_{\max}\left(\rho_B^{\bar{\varphi}^i}\right)\right\} \\ &= \max_{\substack{\{\bar{\varphi}_{AB}^i\}_i:\mathrm{Tr}\,\bar{\varphi}_{AB}^i=1 \\ \sum_i p_i F(\bar{\varphi}_{AB}^i,\phi_{AB}^i)\geq 1-\varepsilon/2}} \min_i S_{\min}(\rho_A^{\bar{\varphi}^i}), \end{aligned} \quad (102)$$

since $\lambda_{\max}(\rho_B^{\bar{\varphi}^i}) = \lambda_{\max}(\rho_A^{\bar{\varphi}^i}) = S_{\min}(\rho_A^{\bar{\varphi}^i})$, with $\rho_B^{\bar{\varphi}^i} := \mathrm{Tr}_A(\bar{\varphi}^i)$ and $\rho_A^{\bar{\varphi}^i} := \mathrm{Tr}_B(\bar{\varphi}^i)$, because $\bar{\varphi}^i_{AB}$ is a pure state. To obtain the identity on the third line, we made use of the fact that $\Pi_{\bar{\sigma}_{ABZ}} = \sum_i \bar{\varphi}^i_{AB} \otimes \pi^i_Z$. ∎

The proof of Theorem 3 can be divided into the following two lemmas.

Lemma 14. *For any bipartite state ρ_{AB},*

$$E_A^\infty(\rho_{AB}) \geq \lim_{n\to\infty} \frac{1}{n} E_A(\rho_{AB}^{\otimes n}), \tag{103}$$

Proof. Let $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$ be an ensemble of pure states for ρ_{AB} and $\mathfrak{E}_n = \{p_i^n, \phi^i_{A_nB_n}\}$ be an ensemble of pure states for $\rho_{AB}^{\otimes n}$. First of all, note that the pure states $\phi^i_{A_nB_n}$ need not be factorized. For this ensemble, define the tripartite state

$$\sigma^{\mathfrak{E}_n}_{ABZ} = \sum_i p_i^n \phi^i_{A_nB_n} \otimes \pi_Z^{n,i} \in \mathcal{B}\big(\mathscr{H}_A^{\otimes n} \otimes \mathscr{H}_B^{\otimes n} \otimes \mathscr{H}_Z^{\otimes n}\big), \tag{104}$$

where $\pi_Z^{n,i} = |i_n\rangle\langle i_n| \in \mathfrak{S}(\mathscr{H}_Z^{\otimes n})$, with $\{|i_n\rangle\}_i$ being an orthonormal basis of $\mathscr{H}_Z^{\otimes n}$.

From (98) of Lemma 11 we have, for any given $\varepsilon \geq 0$,

$$E_A(\rho_{AB}^{\otimes n}; \varepsilon) \geq \max_{\mathfrak{E}_n} I^{A_n \rightsquigarrow B_nZ_n}_{0,\varepsilon/2}(\sigma^{\mathfrak{E}_n}_{ABZ}) - \Delta_n \tag{105}$$

with $0 \leq \Delta_n \leq 1$. We then have:

$$\begin{aligned}
E_A^\infty(\rho_{AB}) :=& \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} E_A(\rho_{AB}^{\otimes n}; \varepsilon), \\
\geq& \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathfrak{E}_n} I^{A_n \rightsquigarrow B_nZ_n}_{0,\varepsilon/2}(\sigma^{\mathfrak{E}_n}_{ABZ}) \\
\geq& \lim_{\varepsilon\to 0}\lim_{n\to\infty} \frac{1}{n} \max_{\mathfrak{E}} I^{A_n \rightsquigarrow B_nZ_n}_{0,\varepsilon/2}\left((\sigma^{\mathfrak{E}}_{ABZ})^{\otimes n}\right) \\
=& \max_{\mathfrak{E}} \Big[I^{A\to BZ}(\sigma^{\mathfrak{E}}_{ABZ}) \Big].
\end{aligned} \tag{106}$$

The proof of (106) can be found in Appendix B

From the definition of the state $\sigma^{\mathfrak{E}}_{ABZ}$ it follows that for the ensemble $\mathfrak{E} = \{p_i, \phi^i_{AB}\}$,

$$I^{A\to BZ}(\sigma^{\mathfrak{E}}_{ABZ}) = \sum_i p_i S(\rho_B^{\phi^i}), \tag{107}$$

where $\rho_B^{\phi^i} = \mathrm{Tr}_{AZ}\big(\sigma^{\mathfrak{E}}_{ABZ}\big)$. From (106) and (107) we hence obtain

$$\begin{aligned}
E_A^\infty(\rho_{AB}) \geq& \max_{\mathfrak{E}} \sum_i p_i S(\rho_B^{\phi^i}) \\
=& E_A(\rho_{AB}).
\end{aligned} \tag{108}$$

The statement of the lemma can then be obtained by the usual blocking argument. ∎

Lemma 15. *For any bipartite state ρ_{AB},*

$$E_A^{\infty}(\rho_{AB}) \leq \lim_{n\to\infty} \frac{1}{n} E_A(\rho_{AB}^{\otimes n}), \tag{109}$$

Proof. From (98) of Lemma 11 we have, for any given $\varepsilon \geq 0$,

$$E_A(\rho_{AB}^{\otimes n}; \varepsilon) \leq \max_{\mathfrak{E}_n} I_{0,2\sqrt{\varepsilon}}^{A_n \to B_n Z_n}(\sigma_{ABZ}^{\mathfrak{E}_n}), \tag{110}$$

where the maximisation is over all possible pure state decompositions of the satte $\rho_{AB}^{\otimes n}$.

From Lemma 14 of [22] we have the following inequality relating the smoothed zero-coherent information to the ordinary coherent information:

$$\begin{aligned} I_{0,2\sqrt{\varepsilon}}^{A_n \to B_n Z_n}(\sigma_{ABZ}^{\mathfrak{E}_n}) \leq & \frac{I^{A_n \to B_n Z_n}(\sigma_{ABZ}^{\mathfrak{E}_n})}{1-\varepsilon''} \\ & + \frac{4\big(\varepsilon'' \log\big(d_A^n d_{BZ}^n\big) + 1\big)}{1-\varepsilon''}, \end{aligned} \tag{111}$$

where $\varepsilon' = 2\sqrt{\varepsilon}$, $\varepsilon'' = 2\sqrt{\varepsilon'}$, $d_A^n = \dim \mathscr{H}_A^{\otimes n}$ and $d_{BZ}^n = \dim\big(\mathscr{H}_B^{\otimes n} \otimes \mathscr{H}_Z^{\otimes n}\big)$. Moreover, analogous to (107) we have

$$I^{A_n \to B_n Z_n}(\sigma_{ABZ}^{\mathfrak{E}_n}) = \sum_i p_i^n S(\rho_{\phi^i}^{B_n}). \tag{112}$$

Hence,

$$\begin{aligned} E_A^{\infty}(\rho_{AB}) &\leq \lim_{n\to\infty} \frac{1}{n} \max_{\mathfrak{E}_n} I^{A_n \to B_n Z_n}(\sigma_{ABZ}^{\mathfrak{E}_n}) \\ &= \lim_{n\to\infty} \frac{1}{n} \max_{\mathfrak{E}_n} \sum_i p_i^n S(\rho_{\phi^i}^{B_n}) \\ &= \lim_{n\to\infty} \frac{1}{n} E_A\big(\rho_{AB}^{\otimes n}\big) \end{aligned} \tag{113}$$

∎

7 Discussion

In this paper we evaluated the one-shot entanglement of assistance for an arbitrary bipartite state ρ_{AB}. In doing this, we proved a result, which is of interest on its own, namely a characterization of the one-shot distillable entanglement of a bipartite pure state. This result turned out to be stronger than what one obtains by simply specializing the one-shot hashing bound, obtained in [28], to pure states.

Further, we showed how our one-shot result yields the operational interpretation of the asymptotic entanglement of assistance in the asymptotic i.i.d. scenario. In this context, an interesting open question is to find a one-shot analogue of the result $E_A^{\infty}(\rho_{AB}) = \min\{S(\rho_A), S(\rho_B)\}$ proved in [7].

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A Appendix A: optimality of rank-one measurements in (58)

Suppose in fact that the optimal assisting measurement at Charlie's is given by the POVM $\{P_C^i\}_i$ (not necessarily rank-one). Then the resulting shared

state will be $\sum_i p(i)\rho^i_{AB}\otimes\pi^i_X\otimes\pi^i_Y$, where $p(i)\rho^i_{AB} = \mathrm{Tr}_C\left[(\mathbb{1}_{AB}\otimes P^i_C)\ \Psi_{ABC}\right]$, and π^i is the shorthand notation for the projector $|i\rangle\langle i|$. In this form, the systems X and Y, at Alice's and Bob's side respectively, are classical registers carrying the information about the outcome of Charlie's measurement.

Now, consider the situation where Charlie actually performs the rank-one POVM $\{|\mu_i\rangle\langle\mu_i|_C\}_{(i,\mu_i)}$, with $\sum_{\mu_i}|\mu_i\rangle\langle\mu_i|_C = P^i_C$, and communicates the double index outcome (i,μ_i) to Alice and Bob. In this case, the shared state between Alice and Bob can be written as $\sum_{i,\mu_i} p(i,\mu_i)|\varphi^{(i,\mu_i)}\rangle\langle\varphi^{(i,\mu_i)}|_{AB}\otimes \pi^i_X\otimes\pi^{\mu_i}_{X'}\otimes\pi^i_Y\otimes\pi^{\mu_i}_{Y'}$, where

$$p(i,\mu_i)|\varphi^{(i,\mu_i)}\rangle\langle\varphi^{(i,\mu_i)}|_{AB} = \mathrm{Tr}_C\left[(\mathbb{1}_{AB}\otimes|\mu_i\rangle\langle\mu_i|_C)\ \ \Psi_{ABC}\right].$$

It is easy to verify that $\sum_{\mu_i} p(i,\mu_i)|\varphi^{(i,\mu_i)}\rangle\langle\varphi^{(i,\mu_i)}|_{AB} = p(i)\rho^i_{AB}$, so that, in order to retrieve the optimal case, Alice and Bob simply have to first perform a partial trace over the registers X' and Y', respectively, and then proceed with the required LOCC transformation. The partial trace can be effectively seen as a coarse-graining of Charlie's measurement.

B Appendix B: proof of equation (106)

Equation (106) is proved by using Lemma 16 and Lemma 17, given below. However, before stating and proving these lemmas, we need to recall some definitions and notations extensively used in the Quantum Information Spectrum Approach [32, 33]. A fundamental quantity used in this approach is the *quantum spectral inf-divergence rate*, defined as follows [33]:

Definition 5 (Spectral inf-divergence rate)**.** Given a sequence of states $\hat\rho = \{\rho_n\}_{n=1}^\infty$, $\rho_n\in\mathfrak{S}(\mathscr{H}^{\otimes n})$, and a sequence of positive operators $\hat\sigma = \{\sigma_n\}_{n=1}^\infty$, with $\sigma_n\in\mathcal{B}(\mathscr{H}^{\otimes n})$, the *quantum spectral inf-divergence rate* is defined in terms of the difference operators $\Delta_n(\gamma) := \rho_n - 2^{n\gamma}\sigma_n$ as follows:

$$\underline{D}(\hat\rho\|\hat\sigma) := \sup\left\{\gamma : \liminf_{n\to\infty}\mathrm{Tr}\left[\{\Delta_n(\gamma)\geq 0\}\Delta_n(\gamma)\right] = 1\right\}, \tag{114}$$

where the notation $\{X\geq 0\}$, for a self-adjoint operator X, is used to indicate the projector onto the non-negative eigenspace of X.

Lemma 16. *For any given bipartite state ρ_{AB}, let $\mathfrak{E}$ denote a pure-state ensemble decomposition, and let $\mathfrak{E}_n$ denote a pure-state ensemble decomposition of the state $\rho^{\otimes n}_{AB}$. Then, using the notation of (104), we have*

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\max_{\mathfrak{E}_n} I^{A_n\rightsquigarrow B_nZ_n}_{0,\varepsilon}(\sigma^{\mathfrak{E}_n}_{ABZ}) \geq \max_{\mathfrak{E}}\min_{\hat\nu_{BZ}}\underline{D}(\hat\sigma^{\mathfrak{E}}_{ABZ}\|\hat{\mathbb{1}}_A\otimes\hat\nu_{BZ}), \tag{115}$$

where $\hat\sigma^{\mathfrak{E}}_{ABZ} := \left\{(\sigma^{\mathfrak{E}}_{ABZ})^{\otimes n}\right\}_{n\geq 1}$, $\hat{\mathbb{1}}_A := \{\mathbb{1}_A^{\otimes n}\}_{n\geq 1}$, and $\hat\nu_{BZ} := \{\nu^n_{BZ}\in\mathfrak{S}(\mathscr{H}_B^{\otimes n}\otimes\mathscr{H}_Z^{\otimes n})\}_{n\geq 1}$.

Proof. Let $\bar{\mathfrak{E}}$ be the pure state ensemble decomposition of ρ_{AB} for which the maximum on the r.h.s. of eq. (115) is achieved. Since $\bar{\mathfrak{E}}$ is fixed, in the following, we drop the superscript $\bar{\mathfrak{E}}$ whenever no confusion arises, denoting $\sigma^{\bar{\mathfrak{E}}}_{ABZ}$ simply as σ_{ABZ}.

From the definition (97) it follows that, for any fixed $\varepsilon > 0$,

$$\begin{aligned}
&\max_{\mathfrak{E}_n} I^{A_n \rightsquigarrow B_n Z_n}_{0,\varepsilon}(\sigma^{\mathfrak{E}_n}_{ABZ}) \\
=&\max_{\mathfrak{E}_n} \max_{\bar{\sigma}^n_{A_nB_nZ_n} \in \mathfrak{b}_{\mathrm{qc}}(\sigma^{\mathfrak{E}_n}_{ABZ};\varepsilon)} \min_{\nu^n_{B_nZ_n}} S_0(\bar{\sigma}^{\mathfrak{E}_n}_{A_nB_nZ_n} \| \mathbb{1}^{\otimes n}_A \otimes \nu^n_{B_nZ_n}) \\
\geq&\max_{\mathfrak{E}} \max_{\bar{\sigma}^n_{A_nB_nZ_n} \in \mathfrak{b}_{\mathrm{qc}}((\sigma^{\mathfrak{E}}_{ABZ})^{\otimes n};\varepsilon)} \min_{\nu^n_{B_nZ_n}} S_0(\bar{\sigma}^n_{A_nB_nZ_n} \| \mathbb{1}^{\otimes n}_A \otimes \nu^n_{B_nZ_n}) \\
\geq&\max_{\bar{\sigma}^n_{A_nB_nZ_n} \in \mathfrak{b}_{\mathrm{qc}}(\sigma^{\otimes n}_{ABZ};\varepsilon)} \min_{\nu^n_{B_nZ_n}} S_0(\bar{\sigma}^n_{A_nB_nZ_n} \| \mathbb{1}^{\otimes n}_A \otimes \nu^n_{B_nZ_n}). \qquad (116)
\end{aligned}$$

For each $\nu^n_{B_nZ_n}$ and any $\gamma \in \mathbb{R}$, define the projector

$$P^\gamma_n \equiv P^\gamma_n(\nu^n_{B_nZ_n}) := \{\sigma^{\otimes n}_{ABZ} - 2^{n\gamma}(\mathbb{1}^{\otimes n}_A \otimes \nu^n_{B_nZ_n}) \geq 0\}. \qquad (117)$$

Since the operator $\bar{\sigma}^n_{A_nB_nZ_n}$ in (116) is a qc operator, it is clear that the minimization over $\nu^n_{B_nZ_n}$ in (116) can be restricted to states diagonal in the basis chosen in representing qc operators. Consequently, also P^γ_n has the same qc structure.

Next, let us denote by $\hat{\sigma}_{ABZ}$ the i.i.d. sequence of states $\{\sigma^{\otimes n}_{ABZ}\}_{n\geq 1}$. For any sequence $\hat{\nu}_{BZ} := \{\nu^n_{B_nZ_n}\}_{n\geq 1}$, fix $\delta > 0$ and choose $\gamma \equiv \gamma(\hat{\nu}_{BZ}) := \underline{D}(\hat{\sigma}_{ABZ} \| \hat{\mathbb{1}}_A \otimes \hat{\nu}_{BZ}) - \delta$. Then it follows from the definition (114) that, for n large enough,

$$\mathrm{Tr}\left[P^\gamma_n \ \sigma^{\otimes n}_{ABZ}\right] \geq 1 - \frac{\varepsilon^2}{4}, \qquad (118)$$

for any $\varepsilon > 0$. Further, define

$$\omega^{n,\gamma}_{A_nB_nZ_n} \equiv \omega^{n,\gamma}_{A_nB_nZ_n}(\nu^n_{B_nZ_n}) := \frac{\sqrt{P^\gamma_n}\sigma^{\otimes n}_{ABZ}\sqrt{P^\gamma_n}}{\mathrm{Tr}\left[P^\gamma_n \sigma^{\otimes n}_{ABZ}\right]}, \qquad (119)$$

which, by Lemma 3, is clearly in $\mathfrak{b}_{\mathrm{qc}}(\sigma^{\otimes n}_{ABZ};\varepsilon)$, the qc-ball around the state $\sigma^{\otimes n}_{ABZ}$,defined by (96).

Then, using the fact that $\Pi_{\omega^{n,\gamma}_{A_nB_nZ_n}} \leq P^\gamma_n$, and Lemma 2 of [34], we have,

for any fixed $\varepsilon > 0$,

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n} \{\text{r.h.s. of (116)}\} \\
\geq\ & \lim_{n\to\infty} \frac{1}{n} \min_{\nu^n_{B_nZ_n}} S_0(\omega^{n,\gamma}_{R_nA_n} \| \mathbb{1}_A^{\otimes n} \otimes \nu^n_{B_nZ_n}) \\
=\ & \lim_{n\to\infty} \frac{1}{n} \min_{\nu^n_{B_nZ_n}} \left\{ -\log \mathrm{Tr} \left[\Pi_{\omega^{n,\gamma}_{A_nB_nZ_n}} (\mathbb{1}_A^{\otimes n} \otimes \nu^n_{B_nZ_n}) \right] \right\} \\
\geq\ & \lim_{n\to\infty} \frac{1}{n} \min_{\nu^n_{B_nZ_n}} \left\{ -\log \mathrm{Tr} \left[P_n^{\gamma} (\mathbb{1}_A^{\otimes n} \otimes \nu^n_{B_nZ_n}) \right] \right\} \\
\geq\ & \min_{\hat{\nu}_{BZ}} \gamma(\hat{\nu}_{BZ}) \\
=\ & \min_{\hat{\nu}_{BZ}} \underline{D}(\hat{\sigma}_{ABZ} \| \hat{\mathbb{1}}_A \otimes \hat{\nu}_{BZ}) - \delta \\
=\ & \max_{\mathfrak{E}} \min_{\hat{\nu}_{BZ}} \underline{D}(\hat{\sigma}^{\mathfrak{E}}_{ABZ} \| \hat{\mathbb{1}}_A \otimes \hat{\nu}_{BZ}) - \delta
\end{aligned} \tag{120}$$

Since this holds for any arbitrary $\delta > 0$, it yields the required inequality (115) in the limit $\varepsilon \to 0$. ∎

We also use the following lemma [4], which employs the Generalized Stein's Lemma [35] and Lemma 4 of [22]. We include its proof for the sake of completeness.

Lemma 17. *For any given bipartite state ρ_{AR},*

$$\min_{\hat{\sigma}_R} \underline{D}(\hat{\rho}_{AR} \| \hat{\mathbb{1}}_A \otimes \hat{\sigma}_R) = S(\rho_{AR} \| \mathbb{1}_A \otimes \rho_R), \tag{121}$$

where $\hat{\rho}_{AR} = \{\rho_{AR}^{\otimes n}\}_{n\geq 1}$, $\rho_R = \mathrm{Tr}_A \rho_{AR}$, $\hat{\sigma}_R := \{\sigma^n_{R_n} \in \mathfrak{S}(\mathscr{H}_R^{\otimes n})\}_{n\geq 1}$, and $\hat{\mathbb{1}}_A := \{\mathbb{1}_A^{\otimes n}\}_{n\geq 1}$.

Proof. Consider the family of sets $\mathcal{M} := \{\mathcal{M}_n\}_{n\geq 1}$

$$\mathcal{M}_n := \left\{ \tau^n_{A_n} \otimes \sigma^n_{R_n} \in \mathfrak{S}(\mathscr{H}_A^{\otimes n} \otimes \mathscr{H}_R^{\otimes n}) \right\}, \tag{122}$$

such that $\tau^n_{A_n} := (\mathbb{1}_A/d_A)^{\otimes n}$. For this family, the Generalized Stein's Lemma (Proposition III.1 of [35]) holds.

More precisely, for a given bipartite state ρ_{AR}, let us define

$$S^{\infty}_{\mathcal{M}}(\rho_{AR}) := \lim_{n\to\infty} \frac{1}{n} S_{\mathcal{M}_n}(\rho_{AR}^{\otimes n}), \tag{123}$$

with $S_{\mathcal{M}_n}(\rho_{AR}^{\otimes n}) := \min_{\omega^n_{A_nR_n} \in \mathcal{M}_n} S(\rho_{AR}^{\otimes n} \| \omega^n_{A_nR_n})$, and $\Delta_n(\gamma) = \rho_{AR}^{\otimes n} - 2^{n\gamma} \omega^n_{A_nR_n}$. From the Generalized Stein's Lemma [35] it follows that, for $\gamma > S^{\infty}_{\mathcal{M}}(\rho_{AR})$,

$$\lim_{n\to\infty} \min_{\omega^n_{A_nR_n} \in \mathcal{M}_n} \mathrm{Tr} \left[\{\Delta_n(\gamma) \geq 0\} \Delta_n(\gamma) \right] = 0, \tag{124}$$

implying that $\min_{\hat{\omega}_{AR}\in\mathcal{M}} \underline{D}(\hat{\rho}_{AR}\|\hat{\omega}_{AR}) \leq S^{\infty}_{\mathcal{M}}(\rho_{AR})$. On the other hand, for $\gamma < S^{\infty}_{\mathcal{M}}(\rho_{AR})$,

$$\lim_{n\to\infty} \omega^n_{A_nR_n}\in\min_{\mathcal{M}_n} \mathrm{Tr}\left[\{\Delta_n(\gamma)\geq 0\}\Delta_n(\gamma)\right] = 1, \tag{125}$$

implying that $\min_{\hat{\omega}_{AR}\in\mathcal{M}} \underline{D}(\hat{\rho}_{AR}\|\hat{\omega}_{AR}) \geq S^{\infty}_{\mathcal{M}}(\rho_{AR})$. Hence

$$\min_{\hat{\omega}_{AR}\in\mathcal{M}} \underline{D}(\hat{\rho}_{AR}\|\hat{\omega}_{AR}) = S^{\infty}_{\mathcal{M}}(\rho_{AR}).$$

Finally, by noticing that, due to the definition (122) of $\mathcal{M}$,

$$\begin{aligned}&\min_{\hat{\omega}_{AR}\in\mathcal{M}} \underline{D}(\hat{\rho}_{AR}\|\hat{\omega}_{AR})\\ &= \min_{\hat{\sigma}_R} \underline{D}(\hat{\rho}_{AR}\|\hat{\mathbb{1}}_A\otimes\hat{\sigma}_R) + \log d_A,\end{aligned} \tag{126}$$

and that, due to Lemma 4 in [22],

$$S^{\infty}_{\mathcal{M}}(\rho_{AR}) = S(\rho_{AR}\|\mathbb{1}_A\otimes\rho_R) + \log d_A, \tag{127}$$

we obtain the statement of the lemma. ∎

From Lemma 16 and Lemma 17 we conclude that

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{n\to\infty}\frac{1}{n}\max_{\mathfrak{E}_n} I^{A_n\rightsquigarrow B_nZ_n}_{0,\varepsilon}(\sigma^{\mathfrak{E}_n}_{ABZ}) &\geq \max_{\mathfrak{E}}\min_{\hat{\nu}_{BZ}} \underline{D}(\hat{\sigma}^{\mathfrak{E}}_{ABZ}\|\hat{\mathbb{1}}_A\otimes\hat{\nu}_{BZ})\\ &= \max_{\mathfrak{E}} S(\sigma^{\mathfrak{E}}_{ABZ}\|\mathbb{1}_A\otimes\sigma^{\mathfrak{E}}_{BZ})\\ &= \max_{\mathfrak{E}}\Big[I^{A\to BZ}(\sigma^{\mathfrak{E}}_{ABZ})\Big],\end{aligned} \tag{128}$$

where $\sigma^{\mathfrak{E}}_{BZ} = \mathrm{Tr}_A\,\sigma^{\mathfrak{E}}_{ABZ}$. Thus (106) is proved.