Pricing in an equilibrium based model for a large investor. *

David German

Claremont McKenna College
Department of Mathematical Sciences
850 Columbia ave
Claremont, CA 91711, USA
phone: 1(909)607-7261
fax: 1(909)621-8419
dgerman@cmc.edu

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Abstract

We study a financial model with a non-trivial price impact effect. In this model we consider the interaction of a large investor trading in an illiquid security, and a market maker who is quoting prices for this security. We assume that the market maker quotes the prices such that by taking the other side of the investor's demand, the market maker will arrive at maturity with the maximal expected utility of the terminal wealth. Within this model we provide an explicit recursive pricing formula for an exponential utility function, as well as an asymptotic expansion for the price for a "small" simple demand.

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1 Introduction

The study of contingent claim valuation problem accounts for a large number of papers in Finance, Economics and Mathematical Finance in particular. This question was and currently is studied by many authors in various models under different assumptions. A common assumption made by many authors is the basic economic assumption (imposed either implicitly or explicitly) that an economic agent can trade any security in the *desired* quantity at the *same* price. The consequence of this assumption is that the economic agent's actions do not affect the traded security's price, and that there is never a shortage of any security in any quantity.

One way to relax this assumption is to consider a model where agent's actions move prices. We will achieve that by introducing the notion of liquidity into the model. Liquidity is a complex concept standing for *the ease of trading of a security.* (Il)liquidity can have different sources, such as inventory risk – [Sto78], transaction costs – [CK95], uncertain holding horizons – [Hua03], asymmetry of information – [GP04], demand pressure – [GPP09], search friction – [DGP05], stochastic supply curve – [ÇJP04] and demand for immediacy – [GM88], among many others (see [AMP05] for a thorough literature overview).

We will consider the interaction of a large investor trading in an illiquid security, and a market maker who is quoting prices for this security. We will assume that the market maker quotes the prices such that by taking the other side of the investor's demand, she will arrive at maturity with the maximal expected utility of the terminal wealth. This idea was also used in a recent paper [GPP09] by Gârleanu, et. al. In Section 2 we will rigorously define a model for a large investor. Within this model in this paper we will be concerned with the following questions: "Does there exist a price process corresponding to an arbitrary demand of the large investor, and whether this process is unique." An affirmative answer to these questions (under certain conditions) is presented in Section 3.

An equally important problem is the replication of contingent claims in the large investor model with price impact. A companion paper by [Ger10] shows the existence of a unique pricing rule for a broad class of derivative securities and utility functions, as well as the existence of a unique trading strategy that leads to a perfect replication.

Our approach to the model of a large investor follows the traditional framework of Economic Theory. We begin with economic primitives (such as

agent's preferences and market equilibrium) and *then* derive the model. This is different from several papers in Mathematical Finance where the nature of illiquidity is postulated *a priori*, see for example [CM96], [ÇJP04], [BB04] and [Fre98].

2 Large investor market model

We assume that the uncertainty and the flow of information are modeled by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where the filtration $\mathcal{F}$ is generated by a J-dimensional Brownian Motion B, that is,

$$(2.1) \qquad \mathcal{F}_t = \mathcal{F}_t^B, \quad 0 \leq t \leq T.$$

Here T is a finite time horizon, and $\mathcal{F} = \mathcal{F}_T$.

The security market consists of J risky assets and a riskless asset. These assets are traded between the investor and the market maker. We work in discounted terms and (without loss of generality) assume that the return on the riskless asset is zero. We denote by $\mathcal{F}_T$-measurable random variables $f = (f^j)_{1 \leq j \leq J}$ the payoffs of the risky assets at maturity and by $S^H = (S_t^H)_{0 \leq t \leq T}$ the (J-dimensional) price process of the risky assets under the condition that the investor is using the (J-dimensional) trading strategy or *demand process* $H = (H_t)_{0 \leq t \leq T}$. Of course, at maturity the price does not depend on the strategy:

$$(2.2) \qquad S_T^H = f, \text{ for all } H.$$

From here on we will implicitly understand that we have J-dimensional processes, and without loss of generality we will use one-dimensional notation.

The market maker can be viewed then as a *liquidity provider*. She takes the other side of the investor's demand, which can be positive, as well as negative. We assume that the market maker always responds to the investor's demand, that is the market maker always quotes the price (which turns out to be a function of the trade size). Moreover, the market maker quotes the pricies such that she arrives at maturity with the maximized expected utility of her terminal wealth.

It may be tempting to think that the market maker would quote positive infinity price when the investor is buying, and negative infinity when the

investor is selling. However, by the natural economic assumption, the price process S of the contingent claim f is a semimartingale. Moreover, by (2.2) the price process at maturity is equal to f, and therefore the plus/minus infinity price processes are ruled out here.

We will use the standard apparatus of utility functions. We assume that the market maker has a utility function $U : \mathcal{R} \to \mathcal{R}$, which is strictly increasing, strictly concave, continuously differentiable, and satisfies the Inada conditions

$$U'(-\infty) = \lim_{x \to -\infty} U'(x) = \infty,$$
$$U'(\infty) = \lim_{x \to \infty} U'(x) = 0.$$

We shall also require the following two technical assumptions.

Assumption 2.1. *The terminal value of the traded asset $f = (f^j)_{1 \leq j \leq J} \in \mathcal{F}_T$ has finite exponential moments, that is*

$$\mathbb{E}[\exp(\langle q, f \rangle)] < \infty, \quad q \in \mathcal{R}^J.$$

Assumption 2.2. *Utility function $U : \mathcal{R} \longrightarrow \mathcal{R}$ satisfies*

(2.3) $$c_1 < -\frac{U'(x)}{U''(x)} < c_2 \text{ for some } c_1, c_2 > 0.$$

Notice that a linear combination of exponential functions of the form

$$U(x) = \sum_{i=1}^{N} -c_i \frac{e^{-\gamma_i x}}{\gamma_i}, \quad \gamma_i, c_i > 0, \ x \in \mathcal{R}$$

satisfies the assumption above. We also notice that Assumption 2.2 implies the Inada conditions.

We assume that the investor reveals his market orders (his demand process) H to the market maker. The market maker responds to the investor's demand by quoting the price, and by taking the other side of the demand. That is, if H is the investor's strategy, then $-H$ is the market maker's strategy. In other words, the market maker responds to the demand so that the market rests in equilibrium (supply equals demand). The market maker is quoting the price in such a way that she arrives at maturity with the maximal expected utility of the terminal wealth. Formally this can be stated as

Definition 2.1. *Let $x \in \mathcal{R}$ be the initial cash endowment of the market maker. Let $f = (f^j)_{1 \le j \le J}$ be an $\mathcal{F}_T$-measurable contingent claim. Let $H = (H^j)_{1 \le j \le J}$ be a predictable process. The equivalent probability measure $\mathbb{P}^H \sim \mathbb{P}$ is called* the pricing measure of f under demand H, *and the semimartingale S^H is called* the price process of f under demand H *if*

$$(2.4) \qquad \frac{d\mathbb{P}^H}{d\mathbb{P}} \triangleq \frac{U'(x - \int_0^T H_u dS_u^H)}{\mathbb{E}[U'(x - \int_0^T H_u dS_u^H)]},$$

and the price process S^H with the integral $\int H dS^H$ are martingales under $\mathbb{P}^H$. In particular,

$$S_t^H \triangleq \mathbb{E}^H[f | \mathcal{F}_t], \quad 0 \le t \le T.$$

The above definition displays an intimate relationship between the price process and the pricing measure. It may not be clear from the formulation of Definition 2.1 that it reflects the mechanics of the market described in the previous paragraph. However, notice that the density of $\mathbb{P}^H$ is chosen in such a way that the process $-H$ is indeed a solution to the market maker's optimization problem (which will be defined below.) Naturally, the semimartingale S^H is defined in such a way that it is a martingale under the pricing measure. It will become evident from the following lemma, that the numerator of (2.4) is nothing else but the market maker's marginal utility.

Lemma 2.1. *Let $x \in \mathcal{R}$ be the initial cash endowment of the market maker. Suppose f satisfies Assumption 2.1, and U satisfies Assumption 2.2. Let $H = (H^j)_{1 \le j \le J}$ be a predictable process. Suppose that S^H is the price process of f under demand H. Then $-H$ is the unique solution of the optimization problem*

$$(2.5) \qquad u(x) \triangleq \max_{G \in \mathcal{H}(S^H, \mathbb{P}^H)} \mathbb{E}[U(x + \int_0^T G_u dS_u^H)],$$

where $\mathcal{H}(S^H, \mathbb{P}^H)$ is the collection of predictable processes G such that

$$\int_0^\cdot G_u dS_u^H$$

is a $\mathbb{P}^H$-martingale.

The proof of this Lemma is given in the companion paper [Ger10].

In the following section we will be interested in finding the answers to the following questions:

- Does the price process S^H exist for an arbitrary demand H?

- Provided that S^H exists, is it unique?

3 Price process under simple demand and exponential utility

In our model the market maker has to meet the demand H, which forces her strategy to be $-H$. Therefore the value of the market maker's portfolio at time t is $x - \int_0^t H_u dS_u^H$, when S^H is the price that the market maker is quoting depending on the demand H, and x is the market maker's initial wealth. An important question is whether for every predictable demand process H and an $\mathcal{F}_T$-measurable random variable f there is a corresponding price process S^H, that satisfies Definition 2.1.

In the following theorem we show that the price process S^H exists and is unique for an exponential utility U and bounded *simple* demand processes H. Moreover, we provide an explicit recursive algorithm allowing the computation of S^H. We start by recalling the definition of simple strategies.

Definition 3.1. *Let* $(t_i)_{0 \leq i \leq N}$ *be a partition of the time interval* $[0, T]$ *into* N *intervals with* $t_0 = 0$, *and* $t_N = T > 0$. *Let* $\{\theta^i\}_{0 \leq i \leq N-1}$ *be a sequence of* $(\mathcal{F}_{t_i})_{0 \leq i \leq N-1}$-*measurable* J-*dimensional random variables respectively. Then the process*

$$(3.1) \qquad H_t = \theta^0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{N-1} \theta^i \mathbf{1}_{(t_i, t_{i+1}]}(t)$$

is called a simple process.

We can now formulate the main results of this section.

Theorem 3.1. *Assume that the utility function* U *is of exponential form:*

$$U(x) = -\frac{1}{\gamma} e^{-\gamma x}, \gamma > 0,$$

f is bounded, and (2.1) holds true. Then for a bounded simple demand process H given by (3.1) the price process S^H exists and is unique. Moreover, it satisfies the following equations of backward induction:

$$S_T^H = f,$$

$$(3.2) \qquad S_t^H = \frac{\mathbb{E}\left[f \exp\left\{ \gamma \left(\theta^k S_{t_{k+1}}^H + \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H) \right) \right\} \middle| \mathcal{F}_t \right]}{\mathbb{E}\left[\exp\left\{ \gamma \left(\theta^k S_{t_{k+1}}^H + \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H) \right) \right\} \middle| \mathcal{F}_t \right]},$$

for $t \in [t_k, t_{k+1}]$.

The pricing measure $\mathbb{P}^H$ is given by

$$(3.3) \qquad \frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{\exp\{\gamma \sum_{i=0}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H)\}}{\mathbb{E}\left[\exp\{\gamma \sum_{i=0}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H)\} \right]}.$$

Remark 3.1. Theorem 3.1 was formulated in [GPP09] (Theorem 1) in a different form and without a formal proof. For the sake of mathematical completeness we here provide a proof.

Proof. We first observe that since both f and H are bounded, the process S^H is also bounded and well defined. Therefore the density of $\mathbb{P}^H$, which is defined in terms of the process S^H, is well defined.

In this proof we will show that S^H and $\mathbb{P}^H$ defined by (3.2) and (3.3) satisfy Definition 2.1.

For the exponential utility function, (3.3) can be written as

$$\frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{\exp\{\gamma \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H)\}}{\mathbb{E}\left[\exp\{\gamma \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H)\} \right]} = \frac{U'(x - \int_0^T H_u dS_u^H)}{\mathbb{E}[U'(x - \int_0^T H_u dS_u^H)]},$$

for any $x \in \mathcal{R}$.

Next we will verify by backward induction that S^H defined by (3.2) can be also represented as $S_t^H = \mathbb{E}^H[f|\mathcal{F}_t]$. Indeed, let $t \in [t_{N-1}, T]$. Since

$$\sum_{i=0}^{N-2} \theta^i (S_{t_{i+1}}^H - S_{t_i}^H) - \theta^{N-1} S_{t_{N-1}}^H$$

is an $\mathcal{F}_{t_{N-1}}$-measurable random variable,

$$
\begin{aligned}
\mathbb{E}^H[f|\mathcal{F}_t] &= \frac{\mathbb{E}\left[f\exp\left\{\gamma\left(\sum_{i=0}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[\exp\left\{\gamma\left(\sum_{i=0}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]} \\
&= \frac{\exp\left\{\gamma\left(\sum_{i=0}^{N-2}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H) - \theta^{N-1}S_{t_{N-1}}^H\right)\right\}}{\exp\left\{\gamma\left(\sum_{i=0}^{N-2}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H) - \theta^{N-1}S_{t_{N-1}}^H\right)\right\}} \\
&\quad\times \frac{\mathbb{E}\left[fe^{\gamma\theta^{N-1}f}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\gamma\theta^{N-1}f}\Big|\mathcal{F}_t\right]} \\
&= \frac{\mathbb{E}\left[fe^{\gamma\theta^{N-1}f}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\gamma\theta^{N-1}f}\Big|\mathcal{F}_t\right]}.
\end{aligned}
$$

(3.4)

The equality (3.4) shows us that on the time interval $t \in [t_{k-1}, t_k]$ with $k = N-1$, S_t^H given by (3.2) is in fact equal to $\mathbb{E}^H[f|\mathcal{F}_t]$.

Assume now that $S_t^H = \mathbb{E}^H[f|\mathcal{F}_t]$ for $t \in [t_{k+1}, t_{k+2}]$. Let $t \in [t_k, t_{k+1}]$. Since

$$
\sum_{i=0}^{k-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H) - \theta^k S_{t_k}^H
$$

is an $\mathcal{F}_{t_k}$-measurable random variable, by induction

$$
\begin{aligned}
\mathbb{E}^H[f|\mathcal{F}_t] &= \frac{\mathbb{E}\left[f\exp\left\{\gamma\left(\sum_{i=0}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[\exp\left\{\gamma\left(\sum_{i=0}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]} \\
&= \frac{\exp\left\{\gamma\left(\sum_{i=0}^{k-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H) - \theta^k S_{t_k}^H\right)\right\}}{\exp\left\{\gamma\left(\sum_{i=0}^{k-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H) - \theta^k S_{t_k}^H\right)\right\}} \\
&\quad\times \frac{\mathbb{E}\left[f\exp\left\{\gamma\left(\theta^k S_{t_{k+1}}^H + \sum_{i=n+1}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[\exp\left\{\gamma\left(\theta^k S_{t_{k+1}}^H + \sum_{i=n+1}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]} \\
&= \frac{\mathbb{E}\left[f\exp\left\{\gamma\left(\theta^k S_{t_{k+1}}^H + \sum_{i=n+1}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]}{\mathbb{E}\left[\exp\left\{\gamma\left(\theta_k S_{t_{k+1}}^H + \sum_{i=n+1}^{N-1}\theta^i(S_{t_{i+1}}^H - S_{t_i}^H)\right)\right\}\Big|\mathcal{F}_t\right]}.
\end{aligned}
$$

(3.5)

Now we can see that (3.5) is precisely equal to (3.2). Hence we proved that S^H defined by (3.2) is indeed the price process of f under demand H. Uniqueness follows from the construction. □

Example 3.1. Consider a Bachelier model with $f = \zeta B_T$, constant demand $H_t = \theta$, and $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, where $\theta, \gamma, \zeta \in \mathcal{R}$, $0 < \gamma < \infty$. B is a one-dimensional Brownian Motion under $\mathbb{P}$.

Although in this example we would like to illustrate an application of Theorem 3.1, for technical reasons we chose f to be an unbounded random variable (which does not satisfy the conditions of the theorem.) Therefore we cannot directly apply the result of the above theorem. However, instead we will refer to Definition 2.1, and in this simple case we can work out the price process right from the definition.

By the Definition 2.1, the pricing measure $\mathbb{P}^H$ is given by

(3.6)
$$\frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{U'(x - \int_0^T H_u dS_u^H)}{\mathbb{E}[U'(x - \int_0^T H_u dS_u^H)]} = \frac{\exp\{\gamma \int_0^T H_u dS_u^H\}}{\mathbb{E}[\exp\{\gamma \int_0^T H_u dS_u^H\}]}$$
$$= \frac{\exp\{\gamma\theta(S_T^H - S_0^H)\}}{\mathbb{E}[\exp\{\gamma\theta(S_T^H - S_0^H)\}]},$$

where the last equality follows from the fact that $H = \theta$ is a constant. We notice that since $S_T^H = f = \zeta B_T$, and S_0^H is a constant, (3.6) can be written as

$$\frac{d\mathbb{P}^H}{d\mathbb{P}} = \frac{\exp\{\gamma\theta(f - S_0^H)\}}{\mathbb{E}[\exp\{\gamma\theta(f - S_0^H)\}]} = \exp\left\{\gamma\theta\zeta B_T - \frac{1}{2}(\gamma\theta\zeta)^2 T\right\}.$$

Girsanov's Theorem implies that under the probability measure $\mathbb{P}^H$ there exists a Brownian Motion B^H such that

$$B_t^H = B_t - \gamma\theta\zeta t,$$

and therefore since S^H is the price process of ζB_T under demand θ, Definition 2.1 implies that

(3.7)
$$S_t^H = \mathbb{E}^H[f|\mathcal{F}_t] = \mathbb{E}^H[\zeta B_T|\mathcal{F}_t] = \mathbb{E}^H[\zeta(B_T^H + \gamma\theta\zeta T)|\mathcal{F}_t] = \zeta B_t^H + \gamma\theta\zeta^2 Tr$$
$$= \zeta B_t + \gamma\theta\zeta^2(T - t) = S_t^0 + \gamma\int_t^T H_u\zeta^2 du, \ t \in [0, T],$$

where

$$S_t^0 = \zeta B_t = \mathbb{E}[f|\mathcal{F}_t].$$

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Theorem 3.1 provides a unique price process S^H of the contingent claim f under demand H. Unfortunately, S^H is computed in a recursive form, which makes its practical use rather limited. The following theorem gives a convenient asymptotic expansion for S^H in the case of a "small" simple demand ϵH.

Theorem 3.2. *Assume that conditions of Theorem 3.1 hold true. Then for $\epsilon > 0$ we have*

$$(3.8) \qquad S_t^{\epsilon H} = S_t^0 + \epsilon \mathbb{E}\left[\left. \int_t^T H_u d\langle S^0 \rangle_u \right| \mathcal{F}_t\right] + \xi_t(\epsilon), \quad t \in [0, T],$$

where

$$(3.9) \qquad S_t^0 = \mathbb{E}[f | \mathcal{F}_t], \quad t \in [0, T],$$

and

$$(3.10) \qquad \lim_{\epsilon \to 0} \frac{\xi_t(\epsilon)}{\epsilon} = 0 \text{ for any } t \in [0, T],$$

where the convergence is in probability.

Remark 3.2. Theorem 3.2 is qualitatively rather similar to Theorem 3 in [GPP09], but it offers the convenience of not being recursive, unlike Theorem 3 in [GPP09].

Proof. We will proceed by backward induction. According to Theorem 3.1, the price process $S^{\epsilon H}$ under demand ϵH, it is equal to

$$S_t^{\epsilon H} = \frac{\mathbb{E}\left[\left. f \exp\left\{\epsilon\gamma\left(\theta^k S_{t_{k+1}}^{\epsilon H} + \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^{\epsilon H} - S_{t_i}^{\epsilon H})\right)\right\} \right| \mathcal{F}_t\right]}{\mathbb{E}\left[\left. \exp\left\{\epsilon\gamma\left(\theta^k S_{t_{k+1}}^{\epsilon H} + \sum_{i=k+1}^{N-1} \theta^i (S_{t_{i+1}}^{\epsilon H} - S_{t_i}^{\epsilon H})\right)\right\} \right| \mathcal{F}_t\right]},$$

for $t \in [t_k, t_{k+1}]$.

In particular, for $t \in [t_{N-1}, T]$

$$(3.11) \qquad S_t^{\epsilon H} = \frac{\mathbb{E}\left[\left. f e^{\epsilon\gamma\theta^{N-1}f} \right| \mathcal{F}_t\right]}{\mathbb{E}\left[\left. e^{\epsilon\gamma\theta^{N-1}f} \right| \mathcal{F}_t\right]},$$

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and therefore

(3.12)
$$S_t^{\epsilon H}\big|_{\epsilon=0} = \mathbb{E}\left[f\,\middle|\,\mathcal{F}_t\right] = S_t^0.$$

Expression (3.12) gives economical meaning to the process S^0. It is a price process under zero demand, which is a $\mathbb{P}$-martingale.

Notice that since f and θ^{N-1} are bounded, the Dominated Convergence Theorem implies that

$$\frac{\partial}{\partial\epsilon}\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]$$
$$= \lim_{\delta\to 0}\frac{\mathbb{E}\left[e^{(\epsilon+\delta)\gamma\theta^{N-1}f} - e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}{\delta} = \mathbb{E}\left[\gamma\theta^{N-1}e^{\epsilon\gamma\theta f}\,\middle|\,\mathcal{F}_t\right],$$

as well as

$$\frac{\partial}{\partial\epsilon}\mathbb{E}\left[f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[\gamma\theta^{N-1}f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right].$$

Hence

$$\frac{\partial}{\partial\epsilon}S_t^{\epsilon H} = \frac{\partial}{\partial\epsilon}\frac{\mathbb{E}\left[f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}$$

$$= \frac{\mathbb{E}\left[\gamma\theta^{N-1}f^2 e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]^2}$$

(3.13)
$$- \frac{\mathbb{E}\left[\gamma\theta^{N-1}f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]\mathbb{E}\left[f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]^2}, \quad t \in [t_{N-1}, T].$$

We notice that for every ϵ, (3.13) is a finite random variable. It follows from (3.13) that

$$\frac{\partial}{\partial\epsilon}S_t^{\epsilon H}\bigg|_{\epsilon=0} = \gamma\theta^{N-1}(\mathbb{E}\left[f^2\,\middle|\,\mathcal{F}_t\right] - \mathbb{E}\left[f\,\middle|\,\mathcal{F}_t\right]^2) = \mathbb{E}\left[\gamma\int_t^T H_u d\langle S^0\rangle_u\,\middle|\,\mathcal{F}_t\right],$$

for $t \in [t_{N-1}, T]$. Therefore we can compute Taylor's expansion of $S^{\epsilon H}$ around zero on $[t_{N-1}, T]$:

$$S_t^{\epsilon H} = \frac{\mathbb{E}\left[f e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]}{\mathbb{E}\left[e^{\epsilon\gamma\theta^{N-1}f}\,\middle|\,\mathcal{F}_t\right]} = \mathbb{E}\left[f\,\middle|\,\mathcal{F}_t\right] + \epsilon\mathbb{E}\left[\gamma\int_t^T H_u d\langle S^0\rangle_u\,\middle|\,\mathcal{F}_t\right] + \xi_t(\epsilon),$$

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where $\frac{\xi_t(\epsilon)}{\epsilon}$ converges to 0 in probability as $\epsilon \to 0$. This proves the assertion of the Theorem on the interval $[t_{N-1}, T]$.

Now let us assume that for $t \in [t_{m+1}, T]$

(3.14) $$S_t^{\epsilon H} = S_t^0 + \epsilon \mathbb{E}\left[\gamma \int_t^T H_u d\langle S^0 \rangle_u \,\middle|\, \mathcal{F}_t \right] + \xi_t(\epsilon).$$

Then for $t \in [t_m, T]$

$$
\frac{\partial}{\partial \epsilon} S_t^{\epsilon H} = \frac{\partial}{\partial \epsilon} \frac{\mathbb{E}\left[f \exp\left\{ \epsilon\gamma \left(\theta^m S_{t_{m+1}}^{\epsilon H} + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^{\epsilon H} - S_{t_i}^{\epsilon H}) \right) \right\} \,\middle|\, \mathcal{F}_t \right]}{\mathbb{E}\left[\exp\left\{ \epsilon\gamma \left(\theta^m S_{t_{m+1}}^{\epsilon H} + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^{\epsilon H} - S_{t_i}^{\epsilon H}) \right) \right\} \,\middle|\, \mathcal{F}_t \right]}
$$

$$
= \frac{\partial}{\partial \epsilon} \frac{\mathbb{E}\left[f L^m \,\middle|\, \mathcal{F}_t \right]}{\mathbb{E}\left[L^m \,\middle|\, \mathcal{F}_t \right]}
$$

where

$$
L^m = \exp\left\{ \epsilon\gamma \left(\theta^m S_{t_{m+1}}^{\epsilon H} + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^{\epsilon H} - S_{t_i}^{\epsilon H}) \right) \right\}.
$$

We have

$$
L^m\big|_{\epsilon=0} = 1
$$

and

$$
\frac{\partial}{\partial \epsilon} L^m\big|_{\epsilon=0} = \gamma \left(\theta^m S_{t_{m+1}}^0 + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^0 - S_{t_i}^0) \right).
$$

It follows that

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon} S_t^{\epsilon H}\bigg|_{\epsilon=0} &= \gamma \mathbb{E}\left[f\left(\theta^m S_{t_{m+1}}^0 + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^0 - S_{t_i}^0) \right) \bigg| \mathcal{F}_t \right] \\
&\quad - \gamma \mathbb{E}\left[\theta^m S_{t_{m+1}}^0 + \sum_{i=m+1}^{N-1} \theta^i (S_{t_{i+1}}^0 - S_{t_i}^0) \bigg| \mathcal{F}_t \right] \mathbb{E}\left[f | \mathcal{F}_t \right] \\
&= \gamma \mathbb{E}\left[f\left(\int_{t_m}^T H_u dS_u^0 + \theta^m S_{t_m}^0 \right) \bigg| \mathcal{F}_t \right] \\
&\quad - \gamma \mathbb{E}\left[\int_{t_m}^T H_u dS_u^0 + \theta^m S_{t_m}^0 \bigg| \mathcal{F}_t \right] \mathbb{E}\left[f | \mathcal{F}_t \right] \\
&= \gamma \left(\int_0^t dS_u^0 \int_0^t H_u dS_u^0 + \mathbb{E}\left[\int_t^T H_u d\langle S^0 \rangle_u \bigg| \mathcal{F}_t \right] + S_0^0 \int_0^t H_u dS_u^0 \right. \\
&\qquad \left. - S_t^0 \int_0^{t_m} H_u dS_u^0 - S_t^0 \int_0^t H_u dS_u^0 + S_t^0 \int_0^{t_m} H_u dS_u^0 \right) \\
&= \gamma \mathbb{E}\left[\int_t^T H_u d\langle S^0 \rangle_u \bigg| \mathcal{F}_t \right], \quad t \in [t_m, T].
\end{aligned}
$$

Hence, Taylor's expansion on the time interval $[t_m, T]$ is given by

$$
S_t^{\epsilon H} = \mathbb{E}\left[f | \mathcal{F}_t \right] + \epsilon \mathbb{E}\left[\gamma \int_t^T H_u d\langle S^0 \rangle_u \bigg| \mathcal{F}_t \right] + \xi_t(\epsilon), \ t \in [t_m, T]
$$

where $\frac{\xi_t(\epsilon)}{\epsilon}$ converges to 0 in probability as $\epsilon \to 0$. $\qquad \square$

Remark 3.3. Note that in the framework of Example 3.1 the first order expansion (3.8) is exact, that is, in this case the "error" term $\xi_t(\epsilon) = 0$.

4   Conclusion

We studied the problem of pricing of an illiquid asset in the model with price impact. We derived a recursive unique pricing rule for the illiquid asset under the conditions that the market maker's utility function is exponential, the asset is bounded, and the demand is piece-wise constant. The result was proved by construction.

We also derived the asymptotic expansion (3.8), which gives a very convenient expression for the price process under a "small" simple demand. Its

generalization to continuous demand processes presents an interesting future research project.

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