Detection efficiency for loophole-free Bell experiments with postselection

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It is generally assumed that the violation of a Bell inequality by postselected events cannot be used for loophole-free tests of quantum nonlocality. We show that this is not the case if the postselection is performed locally and without communication between the observers. This permits the adoption of certain setups of locally selected entanglement, which are simpler and more efficient, for long-distance free-space Bell tests. For these setups, we investigate which is the minimum detection efficiency for loophole-free experiments.

Introduction.—An experimental loophole-free violation of a Bell inequality is of fundamental importance not only for ruling out the possibility of describing nature with local hidden variable theories [1], but also for proving entanglement-assisted reduction of classical communication complexity [2], device-independent eternally secure communication [3], and random number generation with randomness certified by fundamental physical principles [4]. This explains the interest in loophole-free Bell test over long distances.

There are three types of loopholes. The locality loophole [5] occurs when the distance between the local measurements is too small to prevent causal influences between one observer's measurement choice and the other observer's result. To avoid this possibility these two events must be spacelike separated.

The detection loophole [6] occurs when the overall detection efficiency is below a minimum value, so although the events in which both observer's have results might violate the Bell inequality, there is still the possibility to make a local hidden variable model which reproduces all the experimental results. To avoid this possibility the overall detection efficiency must be larger than a threshold value.

Finally, the postselection loophole [7–9] occurs when the setup does not always prepare the desired state, so the experimenter postselects those events with the required properties. It has been shown that, in certain configurations, the rejection of "undesired" events can be exploited by a *local* model to imitate the predictions of quantum mechanics [7]. However, it has been recently pointed out that the postselection loophole is not due to the rejection of undesired events itself, but rather to the geometry of the setup. The loophole can be fixed with a suitable geometry, without renouncing to the postselection [8, 10]. This leads to the question of when Bell experiments with postselection are legitimate. The answer is interesting since sources with postselection can be simpler and more efficient. On the other hand, most n-photon entangled states with $n > 2$ are generated by postselection [11–13].

In this paper we focus on sources emitting exactly two

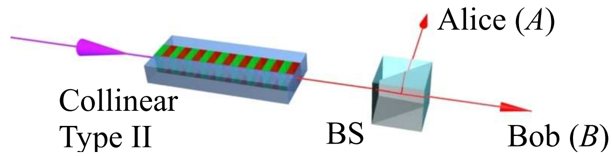

FIG. 1: Collinear SPDC source of two photon states. The two photons, produced by type II phase matching, are divided in the two modes A and B by postselecting the events in which one photon per mode is detected.

photons, such that sometimes one photon ends in Alice's detectors and the other in Bob's, but sometimes both end in the same party's detectors. Hereafter we will consider as benchmark the source shown in Fig. 1 [14]: Two photons with horizontal and vertical polarization are generated via collinear spontaneous parametric down conversion (SPDC) and a beamsplitter (BS) splits the photons over the modes A and B. Three different cases may occur: Both photons emerge on mode A ($|HV\rangle_A$), both on mode B ($|HV\rangle_B$), or the two photons are divided into different modes $[(|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B)/\sqrt{2}]$. A maximally entangled state is obtained by postselecting the events in which there is one photon per mode. Such a setup is a natural candidate for long-distance free-space experiments requiring quantum entanglement, and specifically for satellite-based quantum communications [15, 16], since it satisfies the requirements of high efficiency (due to the adoption of periodically poled crystals), stability, compactness (the beam splitter could even be manufactured onto a single chip with the SPDC source), and emission over a single spatial mode.

It is therefore important to know when these setups of locally selected entanglement can be used for loophole-free Bell experiments, and which is the minimum detection efficiency required to avoid the detection loophole. This paper addresses both questions.

Perfect detectors.—The starting point is the following: Any Bell experiment with local postselection, in the sense that it does not require Alice and Bob to communicate, is free of the postselection loophole. Local postselection is

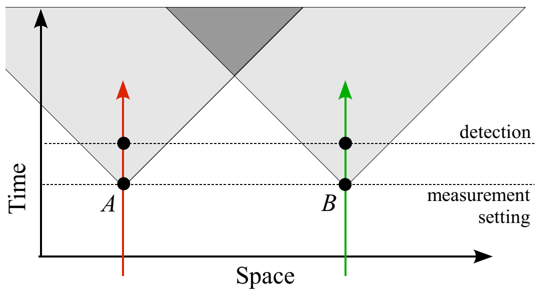

FIG. 2: Space-time diagram for loophole-free Bell experiments.

	$\eta = 1$			$\eta \neq 1$				
	A	B		A	B	fs	CH (no-fs)	CHSH (no-fs)
Events I	1	1	✓	1	1	✓	✓	✓
				1	0	NO	✓	✓
				0	1	NO	✓	✓
				0	0	NO	×	✓
Events II	2	0	NO	2	0	NO	×	✓
				1	0	NO	✓	✓
				0	0	NO	×	✓
Events III	0	2	NO	0	2	NO	×	✓
				0	1	NO	✓	✓
				0	0	NO	×	✓

TABLE I: Types of events. NO: Discarded events, ×: Events which do not contribute to the inequality, ✓: Events which contribute to the inequality. "fs" stands for fair-sampling. The numbers are the number of photons detected by each observer.

not a *necessary requirement* to be free of the postselection loophole (see [7] for a counterexample), but is a *sufficient property* to be free of this loophole.

We will consider two scenarios. In the first one, the detection efficiency is equal to 1 both for Alice and Bob's detectors, i.e., $\eta_A = 1$ and $\eta_B = 1$, respectively. The source of entangled pairs generates with probability p an entangled pair of photons shared by Alice and Bob, while two photons are sent to Alice and zero to Bob, or viceversa, with probability $1 - p$. Hence, Alice and Bob, depending on the measured number of photons, locally decide to keep only those events in which entanglement has been successfully distributed. See column $\eta = 1$ of Table I. The probability p will depend on the particular configuration. For instance, for the source of Fig. 1, $p = 2T(1 - T)$, where T is the transmittance of the beam splitter.

Is such local discarding of events introducing any loophole? The answer is no, if the selection and rejection of events is independent of the local measurement settings; otherwise a local hidden variables model could exploit the selection to violate the inequality. This is a crucial point which deserves a detailed examination. First consider a *selected* event: The two photons have been detected at different locations, corresponding one to Alice's detector D_A and the other to the Bob's detector D_B. If the detection in B is outside the forward light-cone of the measurement setting in A (this is precisely the locality condition), no mechanism could turn a rejected event into a selected one (see Fig. 2).

Let us consider a *rejected* event, for example when two photons have been detected at D_A. Both Alice (since she registers a double detection) and Bob (since he does not register any detection) locally discard the event. Again, due to the locality condition, the double detection at Alice's side cannot be caused by Bob's measurement setting, and the absence of Bob detection cannot be influenced by Alice's measurement setting. The same happens when two photons go towards Bob's side.

Hence, when Alice or Bob locally discard the events, there is no physical mechanism preserving locality which can turn a selected (rejected) event into a rejected (selected) event. The selected events are *independent* of the local settings. For the selected events only the result can depend on the local settings. This is exactly the condition under which the Bell's inequalities are valid. Therefore, an experimental violation of them based on a postselected source of entangled photons provides a conclusive test of local realism when perfect detectors are assumed.

Fair sampling.—In the second scenario, the detection efficiency of the measurement apparatus is not perfect, hence some of the events are lost, and Alice and Bob only keep coincidences and assume fair sampling (i.e., that the coincidences are a statistically fair sample of the pairs in which one photon has gone to Alice and the other to Bob). Under the fair sampling assumption, Bell experiments based on postselection are able to show violations of Bell's inequalities. Indeed, in the case of $p = 1$, the fair sampling assumption allows Alice and Bob to discard the contribution where just one particle is detected. When $p \neq 1$, we have already shown that the contribution due to double particle detection on the same observer can also be discarded. Hence, when fair sampling is assumed, there is no difference between Bell experiments with or without postselection.

Loophole-free test.—What happens when the fair sampling assumption is not considered? In this case we have to determine the threshold detection efficiency needed for a loophole-free violation. For the Clauser-Horne (CH) [17] and the Clauser-Horne-Shimony-Holt (CHSH) Bell inequalities [18], this threshold can be obtained as follows.

Let us consider an experiment with postselection producing with probability p two photons in different locations, two photons in Alice's side with probability $\frac{1-p}{2}$, and two photons on Bob's side with probability $\frac{1-p}{2}$. Detection efficiencies are η_A and η_B. Alice and Bob also have photon number discriminators. In the perfect detection scenario, all the events in which two particles are sent to the same observer

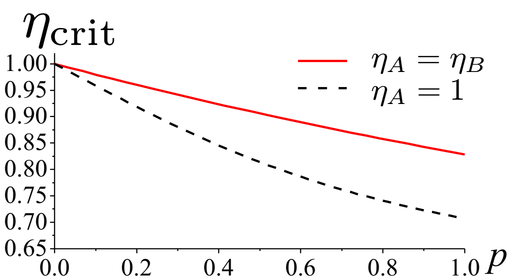

FIG. 3: Critical efficiency for a loophole-free Bell test in the symmetric ($\eta_A = \eta_B > \eta_{\rm crit}$) or in the asymmetric ($\eta_A = 1$, $\eta_B > \eta_{\rm crit}$) case.

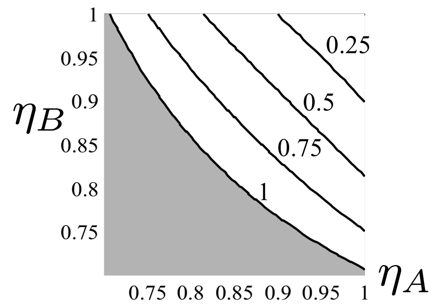

FIG. 4: Allowed values of η_A and η_B for a loophole-free Bell test for different values of p (1, 0.75, 0.5, 0.25). For each value of p the allowed zone is in the upper-right part of the corresponding curve.

(events II and III of Table I) are discarded. Here, due to inefficiency, some of the events II and III will contribute to the data and cannot be locally discarded (see the second column of Table I).

Let us consider two observers, Alice and Bob, with dichotomic observables $a_i = \pm 1$ and $b_i = \pm 1$, respectively. Any theory assuming realism and locality must satisfy the CH inequality,

$$\begin{aligned} I_{\rm CH} =& p(a_1, b_1) + p(a_2, b_1) + p(a_1, b_2) \\ & - p(a_2, b_2) - p(a_1) - p(b_1) \leq 0, \end{aligned} \tag{1}$$

where $p(a_i, b_j)$ is the probability that Alice obtains $a_i = 1$ and Bob obtains $b_j = 1$, while $p(a_1)$ is the probability that Alice obtains $a_1 = 1$. A CH inequality test does not require four detectors (two in each side), but just one detector in each side. The advantage of CH inequality is that it is insensitive to any normalization. This implies that the "vacuum contribution" of standard SPDC sources does not contribute to $I_{\rm CH}$. When two particles are detected by Alice (Bob) we set $a_i = -1$ ($b_i = -1$) and these events do not contribute to Eq. (1).

Let us define Q as the value of I corresponding to the case when both particles are detected, M_A (M_B) the value of I when only particle A (B) is detected from the (1 1) events, T_A (T_B) the value of I when only particle A (B) is detected from (2 0) and (0 2) events, D_A (D_B) the value of I when two particles are detected at side A (B), and X the value when no particle is detected. Then, the average value of I will be

$$\begin{aligned} \langle I \rangle =& \frac{1-p}{2}[\eta_A^2(D_A - 2T_A + X) + \eta_B^2(D_B - 2T_B + X)] \\ & + p\eta_A\eta_B(Q - M_A - M_B + X) + X \\ & + \eta_A[pM_A + (1-p)T_A + X] \\ & + \eta_B[pM_B + (1-p)T_B + X]. \end{aligned} \tag{2}$$

It is easy to show that, for the singlet entangled state and choosing the observables a_i and b_i that maximally violate the inequality, we obtain the following values for the CH inequality: $Q = \frac{1}{\sqrt{2}} - \frac{1}{2}$, $M_A = M_B = -\frac{1}{2}$, and $X = 0$. When the two particles are detected by Alice (Bob), we have $a_i = -1$ ($b_i = -1$), which implies $D_A = D_B = 0$. In order to calculate T_A (T_B), it is necessary to know the particular two-photon state sent to Alice (Bob). In most experimental setups used to generate postselected entangled states, when two photons are sent to the same observer they have orthogonal polarizations. This implies that when only one photon is detected, we have $T_A = T_B = -\frac{1}{2}$. The local realistic bound is violated when $\langle I \rangle > 0$:

$$\frac{1-p}{2}(\eta_A^2 + \eta_B^2) + p\, \eta_A\eta_B(\frac{1}{2} + \frac{1}{\sqrt{2}}) - \frac{1}{2}(\eta_A + \eta_B) > 0. \tag{3}$$

Note that the (0 0), (2 0), and (0 2) events do not contribute to any term in (1).

In the symmetric case (η_A=η_B=η), the minimum detection efficiency is

$$\eta > \eta_{\rm crit} \equiv \frac{2}{2 + p(\sqrt{2} - 1)}. \tag{4}$$

For $p = 1$, we recover $\eta > \frac{2}{1+\sqrt{2}} \simeq 0.83$ [19]. For $p = 0.5$, we obtain $\eta > 0.90$. Postselection imposes a stricter constraint on the experimental setting, but still a loophole-free nonlocality test can be achieved.

For the fully asymmetric case ($\eta_A = 1$), we have $\eta_B > \eta_{\rm crit}$ with

$$\eta_{\rm crit} = \frac{-1 + p(1+\sqrt{2}) - \sqrt{4p(1-p) + (1 - p - \sqrt{2}p)^2}}{2(p-1)}. \tag{5}$$

In the limit $p \to 1$ we recover $\eta_B > \frac{1}{\sqrt{2}} \simeq 0.71$ [20]. In Fig. 3 we show the critical values of the efficiency in the symmetric and totally asymmetric cases. For the general case (3), Fig. 4 shows the values of η_A and η_B allowing a loophole-free Bell test for different values of p.

It is worth noting that a completely equivalent result is obtained by using the CHSH inequality,

$$I_{CHSH} = \langle a_1 b_1 \rangle + \langle a_2 b_1 \rangle + \langle a_1 b_2 \rangle - \langle a_2 b_2 \rangle - 2 \leq 0, \tag{6}$$

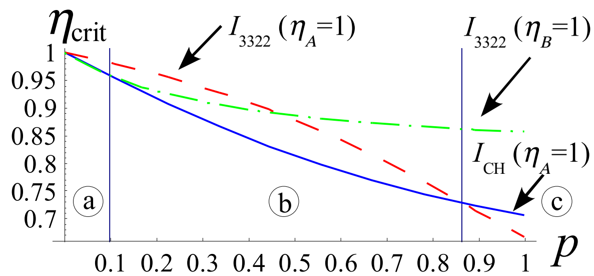

FIG. 5: Critical detection efficiencies in the completely asymmetric case for the CH and the Collins-Gisin inequalities. I_{3322} with $\eta_A = 1$ (dashed line) and with $\eta_B = 1$ (dash-dotted line), CH (Continuous line). Labels (a), (b), and (c) identify the three ranges where different inequalities lead to the lower efficiency threshold.

by using the arguments given in [21]. When one observer detects no particle or two particles, he sets a_i (b_i) $= +1$. When Alice detects two photons and Bob no photon [the (2 0) events], we have $p(a_1, b_1) = p(a_2, b_1) = p(a_1, b_2) = p(a_2, b_2) = p(a_1) = p(b_1) = 1$ [and similarly for the (0 2) events]. The same happens for the (0 0) events (where neither Alice nor Bob detects a particle). The same inequality (3) is obtained if the singlet state is generated with probability p.

Finally, we demonstrate that, in the asymmetric case $\eta_A \neq \eta_B$, an inequality with lower bound with respect to (5) does exist for some values of p. Consider the I_{3322} inequality [22],

$$\begin{aligned} I_{3322} =& p(a_1, b_1) + p(a_1, b_2) + p(a_1, b_3) + p(a_2, b_1) \\ &+ p(a_2, b_2) + p(a_3, b_1) - p(a_2, b_3) - p(a_3, b_2) \\ &- 2p(a_1) - p(a_2) - p(b_1) \leq 0. \end{aligned} \tag{7}$$

By setting $a_i = 1$ ($b_i = 1$) when Alice (Bob) detects zero or two photons it is possible to show that the inequality is violated if

$$\frac{1-p}{2}(\eta_A^2 + 3\eta_B^2) + p\ \eta_A\eta_B\frac{9}{4} - \frac{1}{2}(\eta_A + 3\eta_B) > 0. \tag{8}$$

Eq. (8) depends in a different way on η_A and η_B, hence we will consider separately the two conditions $\eta_A = 1$ and $\eta_B = 1$. We may compare the efficiency threshold in Fig. 5. The plot is divided in three regions (a–c), depending on which inequality leads to the lowest efficiency threshold.

For $\eta_A = 1$, the lower bound on η_B is $\eta_B > 4p/(9p - 6 + \sqrt{36 - 60p + 33p^2})$, which is better than (5) for any $p > 0.863$ (c). For $p = 1$, we obtain the same results presented in [23]. By setting $a_i = -1$ ($b_i = -1$), when Alice (Bob) detects zero or two photons, we obtain the same result with $\eta_A \leftrightarrow \eta_B$. In this case, for $\eta_A = 1$, the lower bound on η_B is better than the CH condition for any $p < 0.099$ (see Fig. 5) (a). In the central region (b), CH is still the optimal choice.

Due to the T_A and T_B terms in (2), the efficiency bound depends on the specific form of the source. Here we have calculated the bound for the case of two photons sent to the same observers with orthogonal polarization.

Finally let us comment on the degree of entanglement of the state generated by the source of Fig. 1. As shown in Fig. 3, the measurements obtained without postselection, that is, by considering also events II and III, violate Bell's inequalities, thus demonstrating that the state is always entangled. However, we note that a maximally entangled state may be obtained for $\eta_A = \eta_B = 1$ by postselecting the events I, hence leading to a higher violation of Bell's inequalities. Although we have consider maximally entangled states, the same approach can be applied to nonmaximally entangled states [24, 25].

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