Ultimate quantum bounds on mass measurements with a nano-mechanical oscillator

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Abstract

Nano-mechanical resonators have a large potential as sensors of very small adsorbed masses, down to the atomic level and beyond. Here I establish the fundamental lower bound on the mass that can be measured with a nano-mechanical oscillator in a given quantum state based on the quantum Cramér–Rao bound, limited only by the laws of quantum mechanics, and identify the quantum states which will allow the largest sensitivity for a given maximum energy.

High-quality nano-mechanical resonators can act as extremely sensitive sensors of adsorbed material. Impressive progress has been made in this direction over the last few years: In 2004, experiments reached a level of sensitivity of femto-grams [1], atto-grams [2], and two years later already zepto-grams [3]. Gas chromatography at the single molecular level was achieved a year ago [4], and brought a vast range of chemical and biological applications in reach. A mass sensitivity as small as half a gold atom has been demonstrated using a nano-mechanical resonator based on a carbon nano-tube [5]. At the same time, large efforts have been spent to cool down (at least one mode of) a nano-mechanical resonator to its ground state, with the ultimate goal of engineering arbitrary quantum states (see e.g. [6, 7]). Preparing an oscillator in a Fock state was demonstrated in [8]. The ground state was finally reached very recently for a piezo-electrical device, with no more than 0.07 vibrational quanta remaining [9]. It is therefore natural to ask whether the sensitivity of mass measurements could be increased further by engineering the quantum state of a nano-mechanical harmonic oscillator, and what would be the truly fundamental lower bound on the mass that can be measured based only on the laws of quantum mechanics. Early on, theoretical investigations tried to find the limitations of mass measurements with a nano-mechanical resonator [10–12]. But the bounds which were derived so far assume that one measures the linear response of the oscillator driven at its resonance frequency [10–13]. In the experiments, a variety of different read-out and/or cooling techniques (e.g. optical [14–20], through electrostatic effects [2, 21–23], mechanical [24], or even field emission in the case of a nano-tube [25]) were used, and it is not clear what would be the optimal measurement procedure.

The truly fundamental lowest (but achievable) bound on the mass sensitivity is a function of the quantum state of the oscillator, and optimized over all possible measurement procedures. It will be calculated below using quantum parameter estimation theory, which leads to the ultimate limit of sensitivity, the quantum Cramér-Rao bound [26]. It becomes relevant once all other limitations such as technical noise, adsorption-desorption noise, momentum exchange noise, etc. have been eliminated [13]. I will even assume a harmonic oscillator without any dissipation (and thus decoherence effects), as mixed states can only decrease the ultimate sensitivity compared to the pure states from which they are mixed [27]. Nevertheless, the bounds I calculate are attainable *in principle* if the idealized conditions are met, and therefore set an important benchmark to which the performance of existing sensors should be compared to. As a guide to further improving the sensitivity of mass-sensing using quantum-engineered states of a nano-oscillator, I determine the optimal quantum state for a given maximum number of excitation quanta in the oscillator.

Quantum parameter estimation theory

For small enough excitation amplitudes, the nano-mechanical oscillator can be modelled as a harmonic oscillator with mass M and effective spring constant D [5], resonance frequency $\omega = \sqrt{D/M}$, and hamiltonian

$$H = \hbar\omega(a_\omega^\dagger a_\omega + \frac{1}{2})\,, \tag{1}$$

with the usual raising (lowering) operators $a_\omega^\dagger$ (a_ω). If a small mass δM is added to the oscillator, its frequency changes to $\tilde{\omega} = \omega(1-\epsilon)$ with $\epsilon = (1/2)\delta M/M$, and we obtain the new hamiltonian $H_{\tilde{\omega}}$ from (1) by replacing $\omega \to \tilde{\omega}$ everywhere. An arbitrary initial quantum state ρ_0 is thus propagated to $\rho(\omega,t) = U(\omega,t)\rho_0 U^\dagger(\omega,t)$ (or $\rho(\tilde{\omega},t)$, respectively), if no mass (or the mass δM) is adsorbed at $t = 0$, where $U(\omega,t) = \exp(-\mathrm{i}H_\omega t/\hbar)$. Note that this assumes that the energy of the oscillator is conserved in the adsorption process, i.e. the additional mass is deposited with zero differential speed onto the oscillator. The distinguishability of the two states $\rho(\omega,t)$ and $\rho(\tilde{\omega},t)$ determines the smallest δM that can be measured. In general, for any density matrix $\rho(x)$ that depends on some parameter x, the smallest δx that can be resolved from N measurements of an observable A (starting always from an identically prepared state) is given by [26]

$$\delta x = \frac{\langle \delta A^2\rangle_x^{1/2}}{\sqrt{N}|\frac{\partial}{\partial x}\langle A\rangle_x|}\,. \tag{2}$$

It has the interpretation of the uncertainty of A in state $\rho(x)$ as judged by N measurements, renormalized by the "speed" by which the mean value of A changes as function of x. In other words, x has to change by an amount that moves the average value of A by at least its uncertainty. Optimizing (2) over all possible measurements leads to the quantum Cramér-Rao bound [26],

$$\delta x \geq \delta x_{\min} \equiv \frac{1}{2\sqrt{N}\frac{d_{\mathrm{Bures}}(\rho(x),\rho(x+dx))}{dx}}\,, \tag{3}$$

where $d_{\mathrm{Bures}}(\rho(x),\rho(x+dx))$ is the Bures distance between $\rho(x)$ and $\rho(x+dx)$ (also called Fisher information), defined as $d_{\mathrm{Bures}}(\rho_1,\rho_2) = \sqrt{2}\sqrt{1-\sqrt{F(\rho_1,\rho_2)}}$ through the fidelity $F(\rho_1,\rho_2) = \mathrm{tr}((\rho_1^{1/2}\rho_2\rho_1^{1/2})^{1/2})$. Thus, in our case, we obtain the minimal measurable mass

$\delta M_{\rm min}$ by evaluating the Bures distance between $\rho(\omega,t)$ and $\rho(\tilde\omega,t)$ in the limit $\epsilon\to 0$. It is important to note that (3) is, in the limit of large N, an *achievable* lower bound [26].

I. PURE STATES

In the case of two pure states, we have simply $F(|\psi\rangle\langle\psi|,|\phi\rangle\langle\phi|)^{1/2}=|\langle\psi|\phi\rangle|$. Starting from an initial state $|\psi(0)\rangle=\sum_{n=0}^\infty c_n|n\rangle_\omega$, expressed in the energy eigenbasis of the unperturbed oscillator, we have the overlap at time t,

$$|\langle\psi(t)|\tilde\psi(t)\rangle|=\left|\sum_{n,m,k}c_n^*c_k\mathrm{e}^{-i(\tilde E_m-E_n)t/\hbar}R_{\tilde\omega\omega}(m,k)R_{\tilde\omega\omega}(m,n)\right|\,, \tag{4}$$

where $R_{\tilde\omega\omega}(m,n)={}_{\tilde\omega}\langle m|n\rangle_\omega=R_{\omega\tilde\omega}(n,m)$ denotes the overlap matrix element between energy eigenstates of the two oscillators with frequency $\tilde\omega$ and ω. They are given by [28]

$$R_{\tilde\omega\omega}(m,n)=(2^{-(m+n)}qm!n!)^{1/2}\sum_{r=0,1}^{[m,n]}\frac{(2q)^r}{r!}\frac{y^{(m+n-2r)/2}(-1)^{(m-r)/2}}{(\frac12(n-r))!(\frac12(m-r))!}\,, \tag{5}$$

if m,n are both even or both odd (otherwise $R_{\tilde\omega\omega}(m,n)=0$), and $[m,n]$ denotes the smaller of the two integers m,n. The sum over r runs over even (odd) integers for $[m,n]$ even (odd), respectively, and $y=(\omega-\tilde\omega)/(\omega+\tilde\omega)$, $q=2(\omega\tilde\omega)^{1/2}/(\omega+\tilde\omega)$. We need $|\langle\psi(t)|\tilde\psi(t)\rangle|$ to second order in ϵ. A somewhat tedious calculation yields $F=1+\epsilon^2f(\{c_m\},t)$ with

$$\begin{aligned}f(\{c_m\},t) \;=\; & \Big[\sum_{m=0}^\infty\big\{\frac12\sqrt{(m+1)(m+2)}\Im(c_mc_{m+2}^*(\mathrm{e}^{2\mathrm{i}\tau}-1))+\tau m|c_m|^2\big\}\Big]^2\\ &+\sum_{m=0}^\infty\Big\{-\left(\frac12(m^2+m+1)\sin^2\tau+m^2\tau^2\right)|c_m|^2\\ &+\sqrt{(m+1)^3(m+2)}\tau\,\Im((1-\mathrm{e}^{2\mathrm{i}\tau})c_{m+2}^*c_m)\\ &+\frac18\sqrt{(m+1)(m+2)(m+3)(m+4)}\,\Re((1-\mathrm{e}^{2\mathrm{i}\tau})^2c_mc_{m+4}^*)\Big\}\end{aligned} \tag{6}$$

and $\tau=\omega t$. Inserting (6) in F, we find immediately $d_{\rm Bures}(|\psi(t)\rangle\langle\psi(t)|,|\tilde\psi(t)\rangle\langle\tilde\psi(t)|)=\epsilon|f(\{c_m\},t)|^{1/2}$, and thus

$$\frac{\delta M_{\rm min}}{M}=\frac{1}{\sqrt{N}|f(\{c_m\},t)|^{1/2}}\,. \tag{7}$$

Eq.(7) together with (6) constitutes the central result of this report which we now explore for a few particular cases.

A. Fock state

For $|\psi(0)\rangle = |n\rangle$, we have $f = -(1/2)(n^2+n+1)\sin^2\tau \equiv f_n^{\rm Fock}$ for all $n \geq 0$. The largest absolute value is achieved for $\tau = \pi/2(\mod 2\pi)$, and leads to

$$\frac{\delta M_{\rm min}}{M} = \sqrt{\frac{2}{N}}\frac{1}{\sqrt{n^2+n+1}} \sim \sqrt{\frac{2}{N}}\frac{1}{n} \text{ for } n \gg 1\,. \tag{8}$$

Thus, one can measure, at least in principle, arbitrarily small masses within the same fixed time interval by increasing the excitation of the harmonic oscillator. In reality, of course, non-harmonicities will start to arise at some level of excitation and the present analysis will then have to be extended to a more complicated hamiltonian. The ground state $n = 0$ of the harmonic oscillator allows to measure a mass which, for a single readout, can be of the order of the mass of the oscillator itself, $\delta M/M \geq \sqrt{2/N}/|\sin\tau|$. Increasing τ does not help beyond $\tau = \pi/2$, as $f_n^{\rm Fock}$ is periodic in τ.

B. Little Schrödinger cat states

Given that (7) depends on coherences between states $|n\rangle$ and $|n+2\rangle$ and $|n+4\rangle$, one might wonder whether the precision could be increased further by using superpositions of these states. The state $|\psi(0)\rangle = (|n\rangle + |n+2\rangle)/\sqrt{2} \equiv |\psi_{S1}\rangle$ leads to

$$f = \frac{1}{16}\left((n+1)(n+2)\sin^2(2\tau) - 8(n^2+3n+4)\sin^2\tau\right) - \tau^2\,. \tag{9}$$

The maximum of the periodic term is again achieved for $\tau = \pi/2\,(\mod 2\pi)$. For fixed $\tau \neq k\pi$ $(k \in \mathbb{N})$ and $n \gg \tau$, this term dominates and leads to $\delta M_{\rm min}/M \simeq \sqrt{2/N}/n$, just as for the Fock state. However, for fixed n, we can get an arbitrarily small $\delta M/M$ by increasing τ, as the last term in (9) leads to $\delta M_{\rm min}/M \simeq \sqrt{2/N}/\tau$ for $\tau \gg n$. Note that this improvement is beyond the usual factor $1/\sqrt{t}$ from increasing the measurement time. Indeed, the sensitivity $\delta M\sqrt{t}$ in $(\mathrm{g}/\sqrt{\mathrm{Hz}})$ still improves as $\propto 1/\sqrt{t}$.

The state $|\psi(0)\rangle = (|n\rangle + |n+4\rangle)/\sqrt{2} \equiv |\psi_{S2}\rangle$ gives

$$f = -\frac{1}{4}\left(2(n^2+5n+11)\sin^2\tau + \sqrt{(n+1)(n+2)(n+3)(n+4)}\cos(2\tau)\sin^2\tau\right) - 4\tau^2\,. \tag{10}$$

The maximum of the periodic terms (relevant for fixed τ and $n \gg \tau \gtrsim 1$) is close to $\tau = \pi/3$ with $|f| \simeq 9n^2/32$, which gives a $\delta M_{\rm min}$ 33% larger than for a Fock state with n excitations.

For fixed n and $\tau \gg n$, the factor 4 in front of τ^2 in (10) reduces $\delta M_{\rm min}/M$ by a factor 2 compared to $|\psi_{S1}\rangle$.

C. Coherent state

For a coherent state $|\alpha\rangle = \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}(\alpha^n/\sqrt{n!})|n\rangle$, and $\alpha \in \mathbb{R}$, we have

$$f = -\left(\frac{1}{2} + \alpha^2 + \alpha^4\cos^2\tau\right)\sin^2\tau - \alpha^2\tau\,(\tau + \sin(2\tau))\ . \tag{11}$$

For fixed τ and $\alpha^2 \gg \tau, \tau^2$, we find $f \simeq -\alpha^4\sin^2(2\tau)/4$, and hence $\delta M_{\rm min}/M \simeq 2/\alpha^2 = 2/\langle n\rangle$ for $\tau = \pi/4$. Thus, also with a coherent state, the sensitivity scales inversely with the (average) number of excitations in the oscillator, and one can, at least in principle, resolve arbitrarily small masses. Compared to a Fock state with $n = \alpha^2$, there is a factor $\sqrt{2}$ penalty. For $\alpha = 0$ we find $f = -\sin^2\tau/2$ which leads back to the result for the Fock state $n = 0$. For α fixed and $\tau \gg |\alpha|$ we obtain $\delta M_{\rm min}/M \simeq 1/(\sqrt{N}\alpha\tau)$. We see again that the minimal reduces faster than $1/\sqrt{t}$ with measurement time.

II. OPTIMAL STATE

While it is good news that the sensitivity can be improved by exciting the oscillator with a large number of quanta, one might wonder what would be the best sensitivity that can be achieved for a given maximum number L of excitations (i.e. $L+1$ basis states) and fixed measurement time. From (6) we see that for $\tau \gg 1$, the terms quadratic in t dominate and give simply $|f| = \tau^2(\langle n^2\rangle - \langle n\rangle^2)$. Hence, in this case the optimal pure state is the one which maximizes the excitation number fluctuations. One easily shows that this state has the form of an "ON" state (half a "NOON" state [29]), $|\psi_{\rm ON}\rangle = (|0\rangle + e^{i\varphi}|L\rangle)/\sqrt{2}$, where φ is an irrelevant phase which we will choose equal zero. It leads to $|f_{\rm ON}| \equiv |f(\psi_{\rm ON}, \tau)| \simeq \tau^2 L^2/4$, and thus a minimal mass $\delta M_{\rm min}/M = 2/(\sqrt{N}\tau L)$. Fig.1 shows a comparison of the (inverse) minimal mass for $\psi_{\rm ON}$ with the true minimal mass for given τ and the same L, obtained by numerically maximizing $|f|$, for $L = 3$. We see that $|f_{\rm ON}|$ approximates the best possible $|f|$ very well, even for $\tau \sim 1$. For $\tau = k\pi$, $k \in \mathbb{N}$, $|f_{\rm ON}|$ gives in fact the exact result, as is obvious from (6). Fig.1 also shows the result for a coherent state with the same *average* number of excitations as the ON state, $\langle n\rangle = 3/2$. It leads to comparable, sometimes even better sensitivity than the optimal state with $L = 3$. This is, of course, no contradiction, as the number of excitations in $|\psi_{\rm coh}\rangle$ is unbound. At $\tau = \pi/2$, the optimal pure state with $L = 3$ allows still a reduction of $\delta M_{\rm min}/M$ by 14% compared to the ON state, and by 27% compared to the Fock state with the same L.

Fig.2 shows the Wigner function, defined for a pure state $|\psi\rangle$ by [30]

$$W(x,p) = \frac{1}{\pi}\int_{-\infty}^{\infty} dy \psi^*(x-y)\psi(x+y){\rm e}^{-2{\rm i}yp}\,, \tag{12}$$

(with all lengths in units of the oscillator length $x_0 = \sqrt{\hbar/DM}$ and p in units of $\hbar/x_0$), for $|\psi_{\rm ON}\rangle$ and the optimal state $|\psi_{\rm opt}\rangle$, and $L = 4$. The optimal state is given explicitly by $|\psi_{\rm opt}\rangle \simeq (0.4037 - 0.1283{\rm i})|0\rangle + (-0.1269 + 0.3994{\rm i})|2\rangle + (0.7653 + 0.2432{\rm i})|4\rangle$. $|\psi_{\rm ON}\rangle$ and $|\psi_{\rm opt}\rangle$ have very similar Wigner functions, characterized by four lobes in azimuthal direction which guarantee minimal phase uncertainty, as is to be expected from the requirement of minimal noise and maximum uncertainty in the number of excitations. Rotations of $|\psi_{\rm opt}\rangle$ through evolution with the unperturbed hamiltonian before the adsorption of mass clearly leave $\delta M_{\rm min}$ invariant.

Among the pure states considered, the coherent states certainly come closest to the typical experimental situation, where the oscillator is cooled to low temperature and driven on resonance. Inserting typical numbers for micromachined resonators, $M = 10^{-16}$g, ω =1GHz, a detection bandwidth of kHz that translates into an evolution time $\tau = 10^6$, and an excitation with $\langle n\rangle \sim 10^{10}$ quanta (driving energy $E_d = 10^{-15}$J in [12]), we find $\delta M_{\rm min} \simeq 10^{-27}{\rm g}/\sqrt{N}$, or roughly the mass of an electron for a single readout, $N = 1$. This agrees with the prediction of [12], but the agreement appears to be a coincidence: The result in [12], based purely on noise considerations, still decreases as $1/\sqrt{Q}$ with the quality Q of the resonator, whereas (11) is independent of Q, taken as infinity in the present analysis. Also, while in the regime $\tau \gg \alpha$ relevant for the above numbers ($\alpha = 10^5$) both $\delta M_{\rm min}/M$ and the result in [12] scale as $1/\sqrt{\langle n\rangle}$, [12] predicts a proportionality to $1/\sqrt{\tau}$ if one identifies the inverse bandwidth $1/\Delta f$ with t, instead of the $1/\tau$ behavior that follows from (11).

Carbon nanotube resonators have typically much smaller masses than micro-engineered ones (of order $M \simeq 10^{-18}$g [5]) with comparable resonance frequency ($\omega = 2\pi \times 328.5$MHz in [5]), and can therefore resolve in principle even smaller masses. Assuming a coherent state with oscillation amplitude of about 10nm for the carbon nanotube oscillator in [5] and a sampling time of 100ms, $\delta M_{\rm min}$ according to (11) is of the order of a thousandth of an electron mass.

III. MIXED STATES

In general, due to the joint-convexity of the Bures distance, mixed states do not allow better sensitivities than the pure states from which they are mixed, as long as the weights are independent of the parameter to be measured and the evolution is linear in the state [27], but their study is justified by their practical relevance. For example, a finite quality of the resonator generates dissipation and thus decoherence and a mixed state, deteriorating therefore typically the fundamentally possible sensitivity. Evaluating the Bures distance between two mixed states is much more difficult than for pure states. Nevertheless, one can use upper and lower bounds on $d_{\rm Bures}$ [31] to get lower and upper bounds for $\delta M_{\rm min}/M$. Alternatively, one can evaluate the Bures distance numerically for a given initial state. One may also obtain an upper bound on $d_{\rm Bures}$ using the joint-convexity of $d_{\rm Bures}$ [27, 31], which leads to a (typically non-achievable) lower bound on $\delta M_{\rm min}$. An achievable upper bound on $\delta M_{\rm min}$ can be found by considering a particular measurement A in (2).

As an example, consider the thermal state $\rho=\sum_{n=0}^\infty p_n|n\rangle\langle n|$, with $p_n={\rm e}^{-nz}(1-{\rm e}^{-z})$, $z=\beta\hbar\omega$, $\beta=1/k_BT$ the inverse temperature, and k_B the Boltzmann constant. Using the invariance of the thermal state under the time evolution governed by H_ω and the joint convexity of $d_{\rm Bures}(\rho_1,\rho_2)$, one shows easily that $d_{\rm Bures}/dx\le\sum_n p_n|f_n^{\rm Fock}|^{1/2}\le|\sin\tau|/(\sqrt{2}(1-\exp(-z))$. For $z\to\infty$, this bound coincides with the exact result for the groundstate $n=0$. We may choose a measurement of the width $A=x^2$ as a way of measuring the change of mass. Thermal average $\langle x^2\rangle=(\hbar/(2M\omega))\coth(z)$ and fluctuations $\langle\delta(x^2)\rangle=(\langle x^4\rangle-\langle x^2\rangle^2)^{1/2}=(\hbar/\sqrt{2DM})\coth(z/2)$ give an achievable upper bound $\delta M_{\rm min}/M\le 2\sqrt{2/N}\sinh z/(\sinh z-z)$. For $z\to\infty$ this bound is only a factor 2 above the best possible value $\sqrt{2/N}$ for the groundstate $n=0$, whereas for $z\to 0$ the bound diverges. Fig.3 shows the exact $M/\delta M_{\rm min}$ obtained by evaluating the Bures distance numerically. We see that $\delta M_{\rm min}/M$ is periodic in τ, just as for the ground state. Increasing the temperature helps, as higher Fock states start to contribute, but at most a factor $\sqrt{2}$ can be gained, and the minimal resolvable mass remains bounded by the mass of the oscillator itself for all temperatures. A driving of the oscillator is in principle not necessary.

In summary, I have calculated the smallest measurable adsorbed mass $\delta M_{\rm min}$ on a nanomechanical harmonic oscillator for an arbitrary pure state of the oscillator, based on the fundamental quantum Cramér-Rao bound. The analysis shows that a coherent state allows to achieve a $\delta M_{\rm min}/M$ that scales as the inverse of the average number of excitations. For a given maximum number n of excitations, I found the optimal quantum state, which for large τ is an "ON" state $|\psi_{\rm ON}\rangle = (|0\rangle + |n\rangle)/\sqrt{2}$, with a sensitivity that scales as $1/n$. For τ different from integer multiples of π, the sensitivity can be further enhanced. Even with a coherent state with realistic experimental parameters for a carbon nano-tube resonator [5], the smallest resolvable mass should be of the order of a thousandth of an electron mass. If two more orders of magnitude could be gained (say by increasing τ and the number of excitations), in principle the regime could be reached where one can weigh the relativistic mass change due to the formation of a chemical bond or the absorption of a photon (energies of order 1 eV).

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[1] Lavrik, N. V., Sepaniak, M. J., & Datskos, P. G., Cantilever transducers as a platform for chemical and biological sensors, *Review of Scientific Instruments* **75**, 2229 (2004).

[2] Ilic, B., Attogram detection using nanoelectromechanical oscillators, *Journal of Applied Physics* **95**, 3694 (2004).

[3] Yang, Y. T., Callegari, C., Feng, X. L., Ekinci, K. L., & Roukes, M. L., Zeptogram-Scale Nanomechanical Mass Sensing, *Nano Letters* **6**, 583–586 (2006).

[4] Naik, A. K., Hanay, M. S., Hiebert, W. K., Feng, X. L., & Roukes, M. L., Towards single-molecule nanomechanical mass spectrometry, *Nature Nanotechnology* **4**, 445–450 (2009).

[5] Jensen, K., Kim, K., & Zettl, A., An atomic-resolution nanomechanical mass sensor, *Nature Nanotechnology* **3**, 533–537 (2008).

[6] Martin, I., Shnirman, A., Tian, L., & Zoller, P., Ground-state cooling of mechanical resonators, *Physical Review B* **69**, 125339 (2004).

[7] Rocheleau, T. *et al.*, Preparation and detection of a mechanical resonator near the ground state of motion, *Nature* **463**, 72–75 (2010).

[8] Buks, E., Segev, E., Zaitsev, S., Abdo, B., & Blencowe, M. P., Quantum nondemolition measurement of discrete Fock states of a nanomechanical resonator, *Europhysics Letters (EPL)* **81**, 10001 (2008).

[9] O'Connell, A. D. *et al.*, Quantum ground state and single-phonon control of a mechanical resonator, *Nature* **464**, 697–703 (2010).

[10] Cleland, A. N. & Roukes, M. L., Noise processes in nanomechanical resonators, *Journal of Applied Physics* **92**, 2758 (2002).

[11] Clerk, A., Quantum-limited position detection and amplification: A linear response perspective, *Physical Review B* **70**, 245306 (2004).

[12] Giscard, P. L., Bhattacharya, M., & Meystre, P., Quantum Mechanical Limits to Inertial Mass Sensing by Nanomechanical Systems, *0905.1081* (2009).

[13] Ekinci, K. L., Ultimate limits to inertial mass sensing based upon nanoelectromechanical systems, *Journal of Applied Physics* **95**, 2682 (2004).

[14] Karrai, K., Photonics: A cooling light breeze, *Nature* **444**, 41–42 (2006).

[15] Arcizet, O. *et al.*, High-Sensitivity Optical Monitoring of a Micromechanical Resonator with a Quantum-Limited Optomechanical Sensor, *Physical Review Letters* **97**, 133601 (2006).

[16] Regal, C. A., Teufel, J. D., & Lehnert, K. W., Measuring nanomechanical motion with a microwave cavity interferometer, *Nat Phys* **4**, 555–560 (2008).

[17] Groblacher, S. *et al.*, Demonstration of an ultracold micro-optomechanical oscillator in a cryogenic cavity, *Nat Phys* **5**, 485–488 (2009).

[18] Park, Y. & Wang, H., Resolved-sideband and cryogenic cooling of an optomechanical resonator, *Nat Phys* **5**, 489–493 (2009).

[19] Schliesser, A., Arcizet, O., Rivière, R., Anetsberger, G., & Kippenberg, T. J., Resolved-sideband cooling and position measurement of a micromechanical oscillator close to the Heisenberg uncertainty limit, *Nature Physics* **5**, 509–514 (2009).

[20] Aspelmeyer, M., Groeblacher, S., Hammerer, K., & Kiesel, N., Quantum Optomechanics - throwing a glance, *1005.5518* (2010), j. Opt. Soc. Am. B 27, A189-A197 (2010) - JOSA B Feature Issue on Quantum Optical Information Technologies, P. Grangier, A. Jordan, G. Morigi (Eds.).

[21] Poncharal, P., Wang, Z. L., Ugarte, D., & de Heer, W. A., Electrostatic Deflections and Electromechanical Resonances of Carbon Nanotubes, *Science* **283**, 1513–1516 (1999).

[22] LaHaye, M. D., Buu, O., Camarota, B., & Schwab, K. C., Approaching the Quantum Limit of a Nanomechanical Resonator, *Science* **304**, 74–77 (2004).

[23] Hertzberg, J. B. *et al.*, Back-action-evading measurements of nanomechanical motion, *Nat*

Phys **6**, 213–217 (2010).

[24] Garcia-Sanchez, D. *et al.*, Mechanical Detection of Carbon Nanotube Resonator Vibrations, *Physical Review Letters* **99**, 085501 (2007).

[25] Purcell, S. T., Vincent, P., Journet, C., & Binh, V. T., Tuning of Nanotube Mechanical Resonances by Electric Field Pulling, *Physical Review Letters* **89**, 276103 (2002).

[26] Braunstein, S. L. & Caves, C. M., Statistical distance and the geometry of quantum states, *Phys. Rev. Lett.* **72**, 3439–3443 (1994).

[27] Braun, D., Parameter estimation with mixed quantum states, *The European Physical Journal D* **59**, 3 (2010).

[28] Smith, W. L., The overlap integral of two harmonic-oscillator wave functions, *Journal of Physics B: Atomic and Molecular Physics* **2**, 1–4 (1969).

[29] Sanders, B. C., Quantum dynamics of the nonlinear rotator and the effects of continual spin measurement, *Phys. Rev. A* **40**, 2417–2427 (1989).

[30] Gardiner, C. W. & Zoller, P., *Qantum Noise, 3rd edition* (Springer, Berlin, Heidelberg, New York, 2004).

[31] Miszczak, J. A., Puchała, Z., Horodecki, P., Uhlmann, A., & K.Życzkowski, Sub– and super–fidelity as bounds for quantum fidelity, *Quantum Information and Computation* **9**, 0103 (2009).

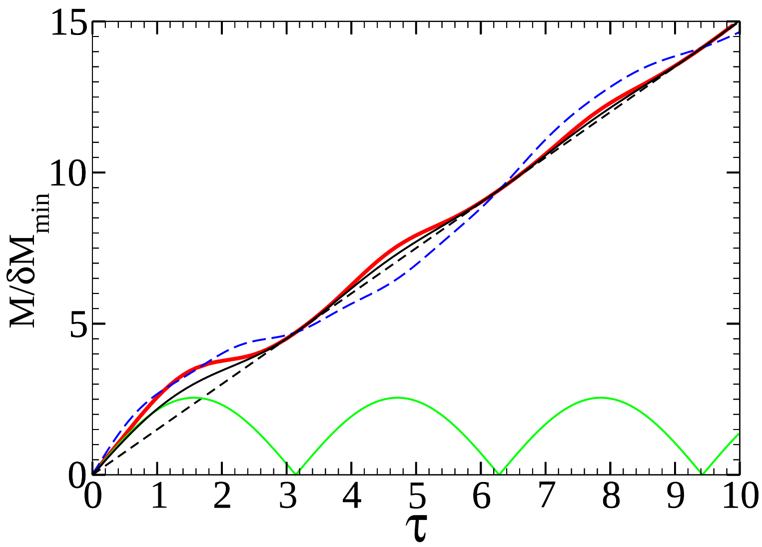

FIG. 1: Inverse minimal measurable mass $M/\delta M_{\rm min}$ (for $N=1$) as function of τ for pure states with at most $L=3$ quanta in the oscillator. Green line: Fock state $|n=3\rangle$; Full black line: ON state $|\psi_{\rm ON}\rangle=(|0\rangle+|3\rangle)/\sqrt{2}$; Dashed black line: asymptotic behavior $3\tau/2$ for $|\psi_{\rm ON}\rangle$; Red line: optimal state $|\psi_{\rm opt}\rangle$ (see text). For comparison, the result for a coherent state with the same *average* number of excitations as the ON state ($\langle n\rangle=3/2$) is shown as dashed blue line.

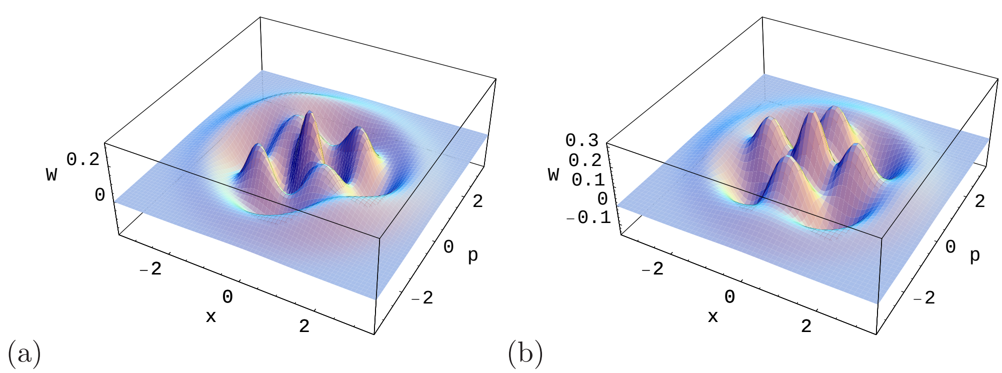

FIG. 2: Wigner function of (a) the optimal initial state $|\psi_{\rm opt}\rangle$ for $L=4$, and (b) for the ON state $|\psi_{\rm ON}(n=4)\rangle=(|0\rangle+|4\rangle)/\sqrt{2}$.

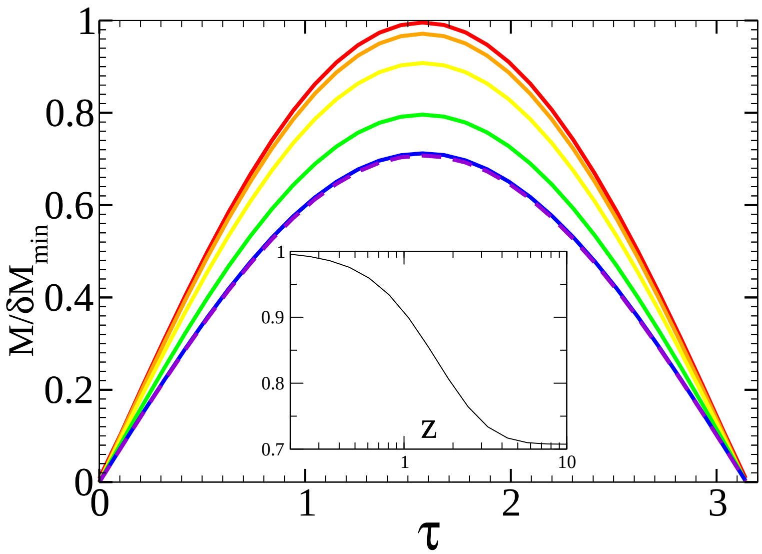

FIG. 3: Inverse minimal measurable mass $M/\delta M_{\rm min}$ in a thermal state as function of dimensionless time $\tau = \omega t$ for inverse dimensionless temperatures $z = \hbar\omega/(k_B T) = 0.2, 0.5, 1.0, 2.0, 5.0$ and 10.0 (red, orange, yellow, green, blue, and dashed purple line, respectively). Inset: $M/\delta M_{\rm min}$ as function of z for $\tau = \pi/2$.