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A speculative study of 2/3-order fractional Laplacian modeling of turbulence: Some thoughts and conjectures

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This study makes the first attempt to use the 2/3-order fractional Laplacian modeling of Kolmogorov −5/3 scaling of fully developed turbulence and enhanced diffusing movements of random turbulent particles. Nonlinear inertial interactions and molecular Brownian diffusivity are considered to be the bifractal mechanism behind multifractal scaling of moderate Reynolds number turbulence. Accordingly, a stochastic equation is proposed to describe turbulence intermittency. The 2/3-order fractional Laplacian representation is also used to model nonlinear interactions of fluctuating velocity components, and then we conjecture a fractional Reynolds equation, underlying fractal spacetime structures of Lévy 2/3 stable distribution and the Kolmogorov scaling at inertial scales. The new perspective of this study is that the fractional calculus is an effective approach to modeling the chaotic fractal phenomena induced by nonlinear interactions. © *2006 American Institute of Physics*. [DOI: 10.1063/1.2208452]

Turbulence occurs throughout nature, from the atmosphere to the oceans to electronics to inside stars and internal combustion chambers. Scaling methods are used to explore hidden structures in the random behavior of turbulent fluid flow even without a detailed solution of the equations of motion. Experimental measurement and direct numerical simulation data from finite Reynolds numbers turbulence, however, have observed a clear departure from the celebrated Kolmogorov scaling theory, also known as intermittency. This study makes the first attempt to develop an original stochastic intermittent equation and a Reynolds equation via the innovative fractional derivative approach. In addition, we propose a simple bifractal model to explain the well-known multifractal scaling of turbulence. We introduce a novel explanation of the fundamental mechanism behind intermittency, and we also provide a solution for the perplexing closure of the turbulence Reynolds equation.

I. INTRODUCTION

The Kolmogorov −5/3 scaling characterizes the statistical similarity of turbulent motion at small scales based on the argument of local homogeneous isotropy.[1] To some extent, the scaling law has been validated by numerous experimental and numerical data of sufficiently high Reynolds number turbulence.[1,2] However, a clear departure from the −5/3 scaling exponent is also often observed in various turbulence experiments at finite Reynolds numbers, i.e., the so-called intermittency. The consensus is that the intermittent property of turbulence calls for a power law of the energy spectrum having an exponent $-5/3-c$ ($c \geqslant 0$). There are a few theories in the derivation of the correction exponent c. For instance, the β model, the various multifractal model,[3,4] and Kolmogorov himself also refined his original −5/3 scaling by assuming that the kinetic energy dissipation rate ε is scale-dependent and obeys a log-normal distribution leading to the so-called intermittency correction.[1]

A school of researchers consider that the non-Gaussian distribution of turbulence velocity increments leads to a violation of the original Kolmogorov scaling, and in fact intermittency manifests a non-Gaussian velocity distribution.[4] This argument has been controversial since many believe that the Kolmogorov theory does not assume the velocity increment Gaussianity. In Sec. II, we revisit this issue and show that the Kolmogorov scaling indeed underlies the Gaussian distribution of velocity increments. It is noted that the Kolmogorov turbulence diffusion is consistent with Richardson's superdiffusion.[5–9] Then we propose a fractional Laplacian stochastic equation to describe the Richardson-like scaling of fully developed turbulence. It is worth stressing that the proposed model equation encounters the infinity of the second and higher moments. Thus, our model is physically significant only for finite domain turbulence. In other words, this model is based on the assumption of the sufficiently high but finite Reynolds number, which is weaker than the one for the original Richardson scaling. On the other hand, there exist quite a few statistical models of turbulence intermittency. To the best of my knowledge, however, little has been achieved in the partial differential equation modeling of the intermittency phenomenon. The addition of molecular diffusivity into the above-mentioned fractional Laplacian stochastic equation gives rise to a partial differential equation of intermittency. In Sec. III, by representing the nonlinear interactions of fluctuating velocity components with the 2/3 fractional Laplacian, we conjecture the fractional Reynolds equation underlying the Lévy 2/3 stable distribution of random turbulence displacements and the

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Kolmogorov scaling. Finally, Sec. IV concludes this paper with some remarks.

The profound understanding of turbulence has been regarded up to now as an unsolved problem. We consider that one major reason for this long-standing difficulty is the lack of an appropriate mathematical device. In this study, innovative fractional calculus modeling is attempted to describe the complicated random phenomena of turbulence. The work described here is of a speculative nature.

II. INTERMITTENT STATISTICAL EQUATION OF TURBULENT DIFFUSION

In Kolmogorov's view of local homogeneous isotropic turbulence, the second-order structure function of velocity increments $\Delta u = u(x+r) - u(x)$ over a distance r within the inertial range of scales is considered a stochastic variable and obeys a scaling law[5]

$$\langle (\Delta u)^2 \rangle \propto r^{2/3} \quad \text{for } \eta \leqslant r \leqslant L_0, \tag{1}$$

where the brackets represent the mean value of the random variable ensemble, ε denotes the kinetic energy dissipation rate per unit mass and is considered scale-independent, and $\eta = (\varepsilon^3/\upsilon)^{1/4}$ is the Kolmogorov dissipation length. The corresponding Kolmogorov scaling of turbulence kinetic energy transportation is

$$E(k) = C\varepsilon^{2/3} k^{-5/3}, \tag{2}$$

where $E(k)$ is the energy spectrum in terms of wave number k, and C denotes the Kolmogorov constant. On the other hand, it is well known that the diffusion of displacements in the Kolmogorov turbulence is consistent with Richardson's particle pair-distance superdiffusion[5–9] (enhanced diffusion) of a fully developed homogeneous turbulence, namely

$$\langle r^2 \rangle = \bar{C}\varepsilon \Delta t^3, \tag{3}$$

where Δt denotes time interval, and the experimental value of the dimensionless constant $\bar{C}$ is 0.5, given in Ref. 6. Equation (3) means particles move much faster than in normal diffusion ($\langle r^2 \rangle \propto \Delta t$). Through a dimensional analysis of Eqs. (1) and (3), we can derive

$$\langle (\Delta u)^2 \rangle \propto \Delta t. \tag{4}$$

Equations (3) and (4) show that the Kolmogorov turbulence in the inertial range of scales is of the normal diffusion of the velocity difference and the enhanced diffusion of displacements. Consequently, turbulence in the inertial range is considered to have a Gaussian velocity field and a non-Gaussian displacement field. Laboratory experiments and field observations have found that the statistics of the velocity increments in the inertial range of sufficiently high Reynolds number turbulence is often close to Gaussian.[6] The displacement diffusion equation (3) can be restated as

$$\langle r^2 \rangle = \bar{C}\varepsilon \Delta t^{2/\alpha}, \quad \alpha = 2/3. \tag{5}$$

Equation (5) can be interpreted as the displacement increments in turbulence that obey the Lévy α-stable distribution,[11] where α represents the stability index of the Lévy distribution. The rigorous mathematics proof shows that Lévy stability index α must be positive and not larger than 2 ($0 < \alpha \leqslant 2$) with the Gaussian distribution being its limiting $\alpha = 2$ case.[10,11] The non-Gaussian Lévy stable distribution of velocity difference has an algebraic decay tail. It is noted that the Gaussian distribution drastically underestimates the occurrence probability of the large events, while for heavy tailed statistics such as the Lévy stable distribution, the occurrence of extreme events is drastically enhanced.

The Lévy distribution has long been used to describe strong long-range spatiotemporal correlation, featuring heavy tails, of anomalous diffusion in turbulence.[12–14] To the best of my knowledge, the corresponding differential equation model, however, has been missing. The fractional Laplacian has been a popular approach in recent years to model the Lévy statistical superdiffusion in a variety of physical master equations such as the Fokker-Planck equation[15] and the anomalous diffusion equation.[10,16] Intuitively, we construct a linear phenomenological statistical equation within the inertial range of scales of fully developed isotropic homogeneous turbulence at sufficiently high Reynolds numbers

$$\frac{\partial P}{\partial t} + \gamma(-\Delta)^{\alpha/2} P = 0, \quad \alpha = 2/3, \tag{6}$$

where $P(x,t)$ is the probability density function (pdf) to find a particle at x at time instant t, which is initially situated at origin. $(-\Delta)^{\alpha/2}$ represents the homogeneous symmetric (isotropic) fractional Laplacian,[17,18] and γ is the turbulent diffusion coefficient. α can be understood as the fractal dimension in this study. In terms of the generalized Einstein dissipation-fluctuation theorem[19] and Eq. (3), we can derive $\gamma = (\bar{C}\varepsilon/2)^{1/3}$. The Green function of the Cauchy problem of Eq. (6) results in the time-dependent Lévy pdf, which naturally leads to Richardson's turbulence superdiffusion (3) through the so-called Lévy walk mechanism[7,12] while underlying the Gaussian velocity increments field and the Kolmogorov scaling in the inertial range of scales.

Unlike the Gaussian process, the Lévy process is known for infinite moments of second or higher order, which has been seen as a major drawback from a purely theoretical point of view. The truncated Lévy distribution was thus proposed in turbulence modeling,[14] in which the long fat tails of algebraic decay of the original Lévy distribution are truncated and replaced by the corresponding Gaussian distribution of exponential decay, and then the divergent second moments are cured.[20] It is noted that both the standard Lévy distribution and fractional Laplacian are defined under the infinite domain. However, this truncation is somewhat arbitrary and the Lévy distribution truncated in this way can no longer exactly underlie the fractional Laplacian of the infinite domain in the governing equation. The real-world turbulences all have finite Reynolds numbers, i.e., the finite size of the turbulence region. Thus, the moment of second or higher orders of the Lévy distribution in a finite domain must be finite, and the artificially truncated Lévy process is not necessary. It is noted that the standard definition of the fractional Laplacian under the infinite domain encounters hypersingularity,[18] corresponding to the infinite moment of the second and higher orders of Lévy distribution.[11,16]

Chen[17] recently introduced a definition of the fractional Laplacian under the finite domain that naturally includes boundary conditions and eliminates hypersingularity. Accordingly, the Lévy distribution corresponding to the fractional Laplacian of the finite domain in the fractional Laplacian NS equation (6) can be naturally truncated in terms of boundary conditions. Therefore, it is stressed that the stochastic model equation (6) is effective only in bounded domains underlying real-world turbulence flows of the finite Reynolds number.

As mentioned before, Refs. 8 and 12–14 have employed the Lévy distribution to analyze various turbulence data. Shlesinger *et al.*[12] also proposed a Lévy walk model to overcome the appearance of infinite moments. The novel contribution of this study is to make the first attempt to develop a stochastic partial differential equation via a fractional Laplacian to describe turbulence superdiffusion characterized by Richardson's third power of time (3).

It is known that the scalings [Eqs. (3) and (4)] are obtained for fully developed homogeneous turbulence under sufficiently high Reynolds numbers and reflect the statistical self-similarity of eddy structures generated from nonlinear inertial interactions. Therefore, the superdiffusion diffusion equation (6) actually describes the enhanced diffusivity originating from the coarse-grained average of the nonlinear inertial term in the Navier-Stokes equation. Also we call Eq. (6) the inertial diffusion equation, serving as a linear phenomenological model to characterize the fractal self-similarity of complicated nonlinear interactions.

On the other hand, for the finite Reynolds number turbulence, a clear deviation from the Gaussian velocity increment field and t^3 displacement superdiffusion at small scales has been observed in various turbulence experiments and numerical simulations, namely turbulence intermittency.[21,22] In the absence of molecular diffusion, model equation (6) cannot describe the intermittency. Otherwise, the addition of molecular diffusion will reflect intermittency, i.e.,

$$\frac{\partial P}{\partial t} + \gamma(-\Delta)^{1/3}P - \upsilon\Delta P = 0, \tag{7}$$

where Δ represents the Laplacian operator and υ denotes molecular viscosity. Equation (7) is called the intermittent stochastic equation in this study. It is noted that the two diffusion terms in Eq. (7) are induced by the inertial interactions and molecular viscosity in the Navier-Stokes equation of motion, respectively, reflecting the two inherent physical systems behind stochastic turbulence phenomena. A space Fourier transform of Eq. (7) results in

$$\frac{\partial P(k,t)}{\partial t} + \gamma k^{2/3}P(k,t) + \upsilon k^2 P(k,t) = 0. \tag{8}$$

Then we have the probability characteristic function

$$P(k,t) = \exp(-\gamma|k|^{2/3} - \upsilon|k|^2)t. \tag{9}$$

The pdf $P(x,t)$ can be evaluated by an inverse Fourier transform. It is apparent that the appearance of the molecular diffusivity in Eq. (6) destroys the Richardson displacement diffusions. Reference 32 makes detailed analyses and discussions on the multifractal nature of the solution of such fractional Laplacian equations as Eq. (7).

Compared with Kraichnan's direct-interaction approximation (DIA) theory,[23] Eq. (7) describes a phenomenological linear stochastic turbulence field in the presence of molecular diffusivity, while the DIA considers turbulence a nonlinear stochastic field. The Green functions of these two approaches are the statistical distribution of turbulence. However, the DIA is mathematically very complicated thanks to its nonlinearity, while the present fractional Laplacian model captures the major fractal feature of nonlinear inertial interactions via a mathematically far simpler approach, namely the fractional Laplacian.

To measure the intermittency in terms of Eq. (9), we introduce the dimensional ratio value

$$\theta = \gamma/\upsilon k^{4/3}. \tag{10}$$

The inertial diffusivity γ is considered much larger compared with the molecular diffusivity υ. However, we note from Eq. (10) that the extent of intermittency is also dependent on wave number and increases with it. Through a dimensional analysis, we find $\theta \propto \mathrm{Re}^{2/3}/(Lk)^{4/3}$, where Re and L are the Reynolds number and characteristic length of fluid flows, respectively. The larger the Reynolds number, the more dominant is the inertial diffusion. This conforms the consensus that the intermittency is of Reynolds-number dependency.

For fully developed isotropic homogeneous turbulence at sufficiently high Reynolds numbers, the intermittency parameter θ is very large for relatively low wave numbers in the so-called convective-inertial range. And the molecular diffusivity vanishes or is small enough that its effects are negligible. Consequently, Eq. (7) is reduced to the limiting equation (6) and the turbulence displacement is dominated by the third power of time law [Eq. (3)] and the Gaussian velocity increments field in the classical K41 inertial range. On another extreme limit when the value of θ is very small for very high wave numbers toward molecular scales, the inertial diffusivity is relatively weak and Eq. (7) is reduced to the normal diffusion equation for the molecular Brownian motion. And in this case, the turbulence displacement field is Gaussian,

$$\langle r^2 \rangle = 2\upsilon t, \tag{11}$$

while the velocity field is described by[4]

$$\langle (\Delta u)^2 \rangle = A\frac{\varepsilon}{\upsilon}r^2. \tag{12}$$

It is clear that the displacement fields vary from the 2/3 Lévy stable distribution to the Gaussian distribution. The nonlinear inertial interactions yield the Gaussian velocity increments field and the 2/3 Lévy stable distribution displacement, while the molecular viscosity is responsible for the non-Gaussian velocity field and the Gaussian displacement. The resulting velocity and displacement fields are a combined effect of these two contributing sources[24] to display varied degrees of intermittency. Therefore, the inertial range is split into the two parts: (i) the convective-inertial range, where the inertial interaction diffusion dominates and intermittency is less apparent, and (ii) the inertial-viscous range, where the molecular diffusion cannot be neglected and intermittency is sensibly observed.

The power spectral of kinetic energy of turbulence obeys the scaling law

$$E(k) \propto k^{-\beta}, \tag{13}$$

where the exponent parameter β has a simple relation with the exponents q of the corresponding second moment $[\langle(\Delta u)^2\rangle \propto r^q]$ of velocity random fields:[25] $\beta = q+1$. For the Kolmogorov Gaussian velocity field (1), $q=2/3$ and $\beta = 5/3$, while for the velocity field (12), $q=2$ and $\beta=3$. The value of β between these two extreme cases ranges from 5/3 to 3 as q varies from 2/3 to 2 as a function of wave number. For instance, $q=1$ and $\beta=2$. Thus, turbulence scaling is multifractal in nature. The stochastic essence of turbulence flows lies in its diffusion behaviors.

Turbulence intermittent stochastic equation (7) can be considered a special type of the fractional Laplacian Fokker-Planck equation (FFPE),[11,15,16] which has been used often in recent years to describe the evolution of the probability distribution function. The FFPE is known as the stochastic model equation of anomalous diffusion. Equation (7) is different from the standard FFPE in the literature in that it omits the convection term but instead contains the two diffusion terms. It is well known that the Fokker-Planck equation underlies the Navier-Stokes equation. And the present FFPE-like intermittent equation (7) may underlie the fractional Laplacian Navier-Stokes equations, a major subject to be discussed in the next section.

III. REYNOLDS EQUATION MODEL WITH FRACTIONAL DERIVATIVE

The intermittent statistical equation (7) can be considered the fractional Fokker-Planck equation (FFPE) with the fractional Laplacian to describe anomalous diffusion. It is well known that the classical Fokker-Planck equation underlies the classical Navier-Stokes equation. This inspires us to apply the fractional representation in the Navier-Stokes equation modeling of turbulence. The equations of motion of an incompressible fluid are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \upsilon\Delta\mathbf{u}, \tag{14a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{14b}$$

where $\mathbf{u}$ is the velocity vector and p represents pressure. Following Reynolds, velocity and pressure can be decomposed as a sum of mean flow components $\bar{u}, \bar{p}$ and small-scale fluctuating components $\tilde{u}, \tilde{p}$. The mean value of fluctuating quantities is considered to be zero. Substituting the decomposition of velocity and pressure into Eqs. (14), we have the following Reynolds equations:[26]

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j\cdot\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\nabla\bar{p} + \upsilon\Delta\bar{u}_i - \frac{\partial}{\partial x_j}\langle\tilde{u}_i\tilde{u}_j\rangle, \tag{15a}$$

$$\nabla\cdot\bar{u}_i = 0. \tag{15b}$$

The nonlinear fluctuation term $\partial\langle u_i u_j\rangle/\partial x_j$ gives rise to the controversial closure problem in the Reynolds equations. For the fully developed homogeneous isotropic turbulence, the fluctuating velocity components are considered to exhibit a variety of universal features, namely statistically homogeneous isotropy and self-similar eddy structures, corresponding to the Richardson and Kolmogorov picture of cascade transport of kinetic energy in the inertial range of scales. Intermittency is interpreted as the joint action of the mean zero random velocity field and molecular diffusion on the large scale and long times. By analogy with the previous statistical equation (7), we conjecture a representation of these universal characteristics of the Reynolds nonlinear fluctuation interactions by

$$\frac{\partial}{\partial x_j}\langle\tilde{u}_i\tilde{u}_j\rangle = \gamma(-\Delta)^{1/3}\bar{u}_i. \tag{16}$$

Equation (16) can be considered the turbulence diffusivity that leads to the enhanced diffusion. It is noted that Eq. (16) is different from the traditional eddy (effective) diffusivity of empirical turbulence models in that it underlies Gaussian velocity increments and agrees with Kolmogorov's key hypothesis that the small-scale structures of turbulence flows, away from boundaries, are independent of the large-scale configuration. Then we present the fractional derivative Reynolds equation

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j\cdot\frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\nabla\bar{p} + \upsilon\Delta\bar{u}_i - \gamma(-\Delta)^{1/3}\bar{u}_i. \tag{17}$$

Here the fractional Laplacian $(-\Delta)^{1/3}$ serves as a stochastic driver underlying statistical self-similarity in the inertial range and guarantees the positive definiteness of energy dissipation. The molecular diffusivity is a property of fluids, while the inertial diffusivity is a characteristic of flows[26] where the fractional Laplacian reflects the long-range correlation in chaotic turbulence motions, apparently resembling an inherent property of non-Newtonian fluids. In other words, the fractional Laplacian representation is to describe the complicated flow property rather than the complex fluid constitutive relationship.

The renormalization-group technique may be a plausible approach to derive Eq. (17) directly from the Navier-Stokes equation in a future study. It is worth mentioning that the naive numerical solution of the fractional Reynolds equation will be computationally expensive, since the fractional Laplacian is a nonlocal operator[17,18] and will result in the full matrix of numerical discretization.[27] The fast algorithms based on the preconditioning techniques, such as the fast multipole method, panel clustering, and the H-matrix method, will be of vital importance to perform effective numerical simulations.

Let T, L, V_∞, and P represent the characteristic time, length, velocity, and pressure of the fluid flow. We can then have the dimensionless expression of the fractional Reynolds equation (17)

$$\mathrm{St}\frac{\partial \bar{\mathbf{u}}^*}{\partial t} + \bar{\mathbf{u}}^* \cdot \nabla \bar{\mathbf{u}}^* = -\,\mathrm{Et}\,\nabla \bar{p}^* + \frac{1}{\mathrm{Re}}\Delta \bar{\mathbf{u}}^* - \frac{\hat{\gamma}}{\mathrm{Re}^{1/3}}(-\Delta)^{1/3}\bar{\mathbf{u}}^*, \quad (18)$$

where $\bar{\mathbf{u}}^*$ and $\bar{p}^*$ are dimensionless velocity and pressure, St is a constant, Et denotes the Euler number, and Re represents the Reynolds number. Equation (18) shows that the coefficient of the inertial chaos diffusion is three orders of magnitude greater than that of molecular diffusion. For instance, the inertial diffusion constant in Eq. (18) has a denominator only around 100 in a Reynolds number 10^6 flow.

The equation of motion (17) is deterministic, but its solution has many attributes of random processes thanks to both the Laplacian and fractional Laplacian viscous terms. We also find Eq. (17) satisfies the same scale invariance of the standard Navier-Stokes equation,[1,4]

$$x' = \lambda x, \quad t' = \lambda^{2/3} t, \quad u' = \lambda^{1/3} u,$$
$$(p/\rho)' = \lambda^{2/3}(p/\rho), \quad \upsilon' = \lambda^{4/3}\upsilon, \quad \gamma' = \gamma. \quad (19)$$

The very nature of the fractional Laplacian representation in the present Reynolds equations (18) also underlies the ballistic motion of turbulence particles under the Lévy walk picture[7] and implies that the turbulence diffusion is not fully irreversible, in between deterministic advection and fully random (irreversible) diffusion motions,[28] and may have stochastic and deterministic duality. In essence, this study conjectures a simple mathematical formulation of chaos in which the deterministic Newtonian dynamics generates the random thermoviscous behavior.

IV. CONCLUDING REMARKS

By using the fractional Laplacian, this paper proposes new statistical-mechanical descriptions of the dynamics of chaos-induced turbulence diffusion. The standard chaos dynamics models are mainly characterized by temporal complexity but spatial simplicity,[4] while the present fractional equations can be considered a kind of fractal continuum dynamics, complex both in time and space. The fractional calculus, fractal, and Lévy distribution are consistent mathematical concepts to describe complicated dissipation, transport, and diffusion phenomena of turbulence. One of the major new perspectives in this study is that the fractional calculus may be an effective approach to model the fractal phenomena resulting from chaotic nonlinear interactions. Although the nonlinear partial differential description and the fractional derivative representation are seemingly quite different mathematical approaches and the underlying relationship between them is still not explicit, their common feature is fractal in statistical physics, which leads to the present fractional calculus modeling of chaos-induced diffusions.

Turbulence intermittency has long been considered to possess multifractal structures.[3] However, to the best of my knowledge, an explicit partial differential multifractal equation does not exist and the multifractal mechanism is not well established. As discussed in Sec. II, the present dual diffusivity model equations provide a clear picture of how the displacement field distributions vary with wave number and fluid molecular viscosity. The bifractal model of 2/3 fractional Laplacian inertial diffusion and molecular viscosity generates multifractal in turbulence.

Warhaft[29] pointed out, "Apart from noting the presence of non-Gaussian tails, no deeper analysis of the shape of the *pdfs* has been made. Because the connection of these models to the Navier-Stokes equations is tenuous,…." In this study, an attempt was also made to explicitly connect non-Gaussian statistics of turbulence and the Reynolds equation, a variant of the Navier-Stokes equation, where the fractional Laplacian representation describes the kinetic energy transportation and dissipation induced by the complex nonlinear interactions. Section III actually presents a statistical and physics closure of the Reynolds equation.

It is worth mentioning that the fractional Laplacian[17,18] and Lévy stable distribution[11] can be asymmetric to describe the skewness of turbulence distributions and ballistic motion,[7] upon which this study does not touch. On the other hand, the stretched Gaussian[30,31] and Hausdorff derivatives[31] can also properly describe anomalous diffusion. Thus, the Lévy stable distribution and fractional derivatives may not be the only approaches in modeling turbulence.

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