Fractional Laplacian time-space models for linear and nonlinear lossy media exhibiting arbitrary frequency power-law dependency

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Frequency-dependent attenuation typically obeys an empirical power law with an exponent ranging from 0 to 2. The standard time-domain partial differential equation models can describe merely two extreme cases of frequency-independent and frequency-squared dependent attenuations. The otherwise nonzero and nonsquare frequency dependency occurring in many cases of practical interest is thus often called the anomalous attenuation. In this study, a linear integro-differential equation wave model was developed for the anomalous attenuation by using the space-fractional Laplacian operation, and the strategy is then extended to the nonlinear Burgers equation. A new definition of the fractional Laplacian is also introduced which naturally includes the boundary conditions and has inherent regularization to ease the hypersingularity in the conventional fractional Laplacian. Under the Szabo's smallness approximation, where attenuation is assumed to be much smaller than the wave number, the linear model is found consistent with arbitrary frequency power-law dependency. [DOI: 10.1121/1.1646399]

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I. INTRODUCTION

Frequency-dependent attenuation has been observed in a wide range of important engineering areas such as acoustics (Blackstock, 1985; Szabo, 1994; Wojcik *et al.*, 1995), viscous dampers in seismic isolation of buildings (Makris and Constantinou, 1991), structural vibration (Enelund, 1996; Rusovici, 1999; Adhikari, 2000), seismic wave propagation (Caputo, 1967; Caputo and Mainardi, 1971), anomalous diffusions occurring in porous media (Hanyga, 1999), just to mention a few. This frequency dependency is described by

$$E=E_0 e^{-\alpha(\omega)z}, \tag{1}$$

where E denotes the amplitude of an acoustic field variable such as velocity or pressure, and ω represents angular frequency. Coefficient $\alpha(\omega)$ is often characterized with an empirical power law

$$\alpha(\omega)=\alpha_0|\omega|^y, \quad y\in[0,2], \tag{2}$$

for a wide range of frequencies of practical interest, in which α_0 and y are media-specific attenuation parameters obtained through a fitting of measured data.

The most straightforward strategy in computer simulation of the power-law lossy behavior is to do both mathematical and numerical modeling in the frequency domain via the Laplace transform (Ginter, 2000). The drawbacks of this approach are that the frequency-domain methods are often ineffective for nonlinear problems and the numerical inverse Laplace transform is very tedious and expensive. The time-domain simulation, in contrast, is feasible for general nonlinear problems and relatively easier to implement and less costly (Wismer and Ludwig, 1995). In addition, the time-domain models also outperform the frequency-domain models as they allow numerical simulation of various initial and boundary value problems (Hanyga, 2001a).

However, it has long been noted that common time-domain partial differential equations (PDE) can model merely two extreme cases of frequency-independent $(y=0)$ and frequency-squared dependent $(y=2)$ absorption behaviors. In many cases of practical interest such as acoustics in biomedical materials and fractal rock layers, $0<y<2$ mostly appears and the standard time-domain PDE modeling methodology does not apply (Blackstock, 1985; Nachman *et al.*, 1990; Szabo, 1994). In contrast to the $y=0,2$ attenuations well described by the standard PDEs, the attenuations obeying $0<y<2$ power law are thus often called the anomalous diffusion (Hanyga, 2001c), nonexponential relaxation, inelastic damping (Adhikari, 2000), hysteretic damping (Gaul, 1999), singular hereditary or singular memory media (Hanyga, 1999), originating from different engineering applications.

The recent decade has witnessed increasing attention to accurate time-domain mathematical modeling of such anomalous $(0<y<2)$ attenuation phenomena, due to a dramatic increase in computer simulation of acoustic wave propagation through human tissues and irregular porous random media. Among these existing models are the adaptive proportional damping model (Wojcik *et al.*, 1995, 1999), the time-domain model via finite frequency decomposition (He, 1998; Chen and Holm, 2002a), the Z-transform model (Wismer and Ludwig, 1995), the multiple relaxation model

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(Nachman *et al.*, 1990; Mast *et al.*, 2001; Yuan *et al.*, 1999), the fractional time derivative models (Caputo 1967; Bagley and Torvik, 1983; Ochmann and Makarov, 1993), and Szabo's model via the singular convolution kernel (Szabo, 1993, 1994). As mentioned by Blackstock (1985), the space or space-time modeling of thermoviscous behavior is often replaced by a pure-time operation in the above models under the condition that the thermoviscous term is relatively small. Ochmann and Makarov (1993) further elaborate that this replacement is impossible in the general case where the interaction between two oppositely traveling sound waves cannot be neglected. The preference of the time-only expression is mostly due to its ease of analysis. For instance, the time-space representation $\partial^3 p/\partial t \partial z^2$ in the one-dimension thermoviscous wave equation, where t and z are, respectively, time and space variables [see Eq. (19) further below], is approximated by a triple time derivative $\partial^3 p/\partial t^3$ (Blackstock 1967; Pierce, 1989; Szabo, 1994). However, numerical implementations of the time-only models are still uncommon, and most research is now restricted to the related mathematical analysis partly due to great numerical difficulties involved. In addition, when $1<y$, fractional time derivative involves the initial condition of the second-order derivative which is unavailable in most practical problems. It is well known that anomalously attenuative and dispersive media often establish complicated microstructures in space; the spatial fractional derivative models may therefore instead be more suitable as a modeling approach, where the initial condition of second-order derivative is never required (Hanyga, 2001b).

The purpose of this study is to employ the spatial fractional Laplacian, also sometimes called the fractional Laplace operator and the Riesz derivative, instead of the Szabo's time convolutional integral and the time fractional derivative to develop linear and nonlinear mathematical models of anomalous thermoviscous behaviors characterized by nonzero and nonquadratic frequency dependency. It is known (Samko *et al.*, 1987) that the standard definition of the fractional Laplacian leads to a hypersingular convolution integral as in the Riemann–Liouville fractional derivative. We present the new definition of the fractional Laplacian which naturally includes the boundary conditions and has inherent regularization operation to ease the hypersingularity of the convolution kernel function. Therefore, it is more useful for engineering modeling.

In what follows, the new definition of the fractional Laplacian is introduced first in Sec. II, followed by a presentation and analysis of the linear fractional Laplacian thermoviscous models of wave equation in Sec. III. The corresponding nonlinear models are then developed in Sec. IV. Conclusions are presented in Sec. V. In the Appendix, a finite-element numerical model is briefly discussed.

II. FRACTIONAL LAPLACIAN

It is worth pointing out that the fractional Laplacian and the fractional derivative are two related but different mathematical concepts. Both are defined through a singular convolution integral, but the former is guaranteed to be the positive definition like the standard Laplace operator, while the latter (Diethelm, 2000) is not.

The conventions of Fourier transform used in Pierce (1989) and Szabo (1994) are also employed in this study. Namely

$$F_-\left(\frac{\partial^n \phi}{\partial z^n}\right) = (ik)^n \Phi(k,\omega), \tag{3}$$

$$F_+\left(\frac{\partial^n \phi}{\partial t^n}\right) = (-i\omega)^n \Phi(k,\omega), \tag{4}$$

where $\Phi(k,\omega)$ is the time and space two-dimensional Fourier transforms of a sufficiently good function $\phi(z,t)$

$$\Phi(k,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(z,t) e^{-i(kz-\omega t)} dz\, dt, \tag{5}$$

where k is the wave number. The inverse of the space Fourier transform is designated as F_-^{-1}, and the inverse of the time Fourier transform F_+^{-1}.

A common interpretation of the fractional Laplacian is to employ the inverse of its Fourier transform (e.g., see Samko *et al.*, 1987; Jespersen, 1999), i.e.,

$$F_-\{(-\nabla^2)_*^{s/2}\varphi\} = k^s \Phi, \quad 0<s<2, \tag{6}$$

$$(-\nabla^2)_*^{s/2}\varphi = F_-^{-1}\{k^s\Phi\} = \frac{1}{2\pi}\int \Phi k^s e^{ikx} dk. \tag{7}$$

The fractional Laplacian is also often called the Riesz fractional derivative in terms of the Riesz potential (Gorenflo and Mainardi, 1998). The Riesz potential I_d^s of order s of d dimensions reads (Zahle, 1997; Samko *et al.*, 1987)

$$I_d^s \varphi(x) = \frac{\Gamma[(d-s)/2]}{\pi^{s/2} 2^s \Gamma(s/2)} \int_\Omega \frac{\varphi(\xi)}{\|x-\xi\|^{d-s}} d\Omega(\xi), \quad 0<s<2, \tag{8}$$

where Γ denotes the Euler's gamma function, Ω is integral domain. The traditional definition of the fractional Laplacian involves the approximate finite difference expression (Samko *et al.*, 1987) and is not well suited for multidimensional irregular domain. By analogy with the fractional time derivative, we give an analytical definition below

$$(-\nabla^2)_*^{s/2}\varphi(x) = -\nabla^2[I_d^{2-s}\varphi(x)]. \tag{9}$$

It is known that the Laplacian operator has the expression

$$\nabla^2\varphi(x) = \frac{d^2\varphi}{dr^2} + \frac{d-1}{r}\frac{d\varphi}{dr}, \tag{10}$$

where $r=\|x-\xi\|$. Equation (9) can then be reduced to

$$(-\nabla^2)_*^{s/2}\varphi(x)$$
$$= -\frac{\Gamma[(d-2+s)/2]}{\pi^{2-s/2}2^{2-s}\Gamma[(2-s)/2]}\nabla^2\int_\Omega \frac{\varphi(\xi)}{\|x-\xi\|^{d-2+s}}d\Omega(\xi)$$
$$= -\frac{(d-2+s)s\Gamma[(d-2+s)/2]}{\pi^{(2-s)/2}2^{2-s}\Gamma[(2-s)/2]}\int_\Omega \frac{\varphi(\xi)}{\|x-\xi\|^{d+s}}d\Omega(\xi). \tag{11}$$

It is noted that (11) encounters the detrimental hypersingularity, which means the singularity order $d+s$ is larger than the topological dimension d. An alternative way is thus presented below to define the fractional Laplacian

$$(-\nabla^2)^{s/2}\varphi(x) = -I_d^{2-s}[\nabla^2\varphi(x)]$$
$$= -\frac{\Gamma[(d-2+s)/2]}{\pi^{(2-s)/2}2^{2-s}\Gamma[(2-s)/2]} \times \int_\Omega \frac{\nabla^2\varphi(\xi)}{\|x-\xi\|^{d-2+s}}d\Omega(\xi). \tag{12}$$

It is noted that the definition (12) has a weak singularity of order $d-2+s$ compared with the hypersingularity of order $d+s$ in (11). The Green's second identity is useful to connect (12) and (11), and can be stated as

$$\int_\Omega v\nabla^2\varphi\, d\xi = \int_\Omega \varphi\nabla^2 v\, d\Omega(\xi) - \int_S\left(\varphi\frac{\partial v}{\partial n} - v\frac{\partial\varphi}{\partial n}\right)dS(\xi), \tag{13}$$

where S represents the surface of the domain, and n is the unit outward normal. Let

$$v = 1/\|x-\xi\|^{d-2+s}, \tag{14}$$

and

$$\varphi(x)|_{x\in S} = D(x), \tag{15}$$
$$\left.\frac{\partial\varphi(x)}{\partial n}\right|_{x\in S} = N(x), \tag{16}$$

With the Green's second identity, the definition (12) is then reduced to

$$(-\nabla^2)^{s/2}\varphi(x) = -\frac{(d-2+s)s\Gamma[(d-2+s)/2]}{\pi^{(2-s)/2}2^{2-s}\Gamma[(2-s)/2]}\int_\Omega \frac{\varphi(\xi)}{\|x-\xi\|^{d+s}}d\Omega(\xi) + h\int_S\left[\varphi(\xi)\frac{\partial}{\partial n}\left(\frac{1}{\|x-\xi\|^{d+s-2}}\right) - \frac{1}{\|x-\xi\|^{d+s-2}}\frac{\partial\varphi(\xi)}{\partial n}\right]dS(\xi)$$
$$= (-\nabla^2)_*^{s/2}\varphi(x) + h\int_S\left[D(\xi)\frac{\partial}{\partial n}\left(\frac{1}{\|x-\xi\|^{d+s-2}}\right) - \frac{N(\xi)}{\|x-\xi\|^{d+s-2}}\right]dS(\xi), \tag{17}$$

where

$$h = \frac{\Gamma[(d-2+s)/2]}{\pi^{(2-s)/2}2^{2-s}[(2-s)/2]}. \tag{18}$$

It is seen from (17) that the fractional Laplacian definition $(-\nabla^2)^{s/2}$ is considered the fractional Laplacian derivative $(-\nabla^2)_*^{s/2}$ augmented with the boundary integral, which is a parallel to the fractional time derivatives in the Caputo sense relative to that in the Riemann–Liouville sense.

The above two definitions $(-\nabla^2)_*^{s/2}$ and $(-\nabla^2)^{s/2}$ involve only the symmetric fractional Laplacian for isotropic media. To simplify the illustration of the basic idea of this study without loss of generality, we only consider isotropic media in this paper. For the traditional definition of the anisotropic fractional Laplacian see Feller (1971) and Hanyga (2001). By analogy with the definitions (11) and (12), it will be straightforward to have the corresponding new expression of the anisotropic fractional Laplacian.

Albeit a long history, the research on the space fractional Laplacian still appears fairly poor in the literature (Gorenflo and Mainardi, 1998). In recent years, some interest has arisen from anomalous diffusion problems. Readers are advised to find more detailed description of the fractional Laplacian from Samko *et al.* (1987), Zaslavsky (1994), Gorenflo and Mainardi (1998), Hanyga (2001), and references therein. In the Appendix, we briefly discuss the finite-element numerical model of the fractional Laplacian.

III. LINEAR FRACTIONAL LAPLACIAN THERMOVISCOUS MODEL

Szabo (1994) started his time-domain model building and causality analysis of the attenuation power law with the thermoviscous wave equation (Blackstock, 1967; Lighthill, 1980; Pierce, 1989), also known as the augmented wave equation (Johnson and Dudgeon, 1993), which governs the propagation of sound through a viscous fluid and can be stated as

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\mu}{c_0^2}\frac{\partial}{\partial t}(-\nabla^2 p), \tag{19}$$

where c_0 is the small signal sound speed, and $\mu = [4\eta/3 + \eta_B + \kappa(\gamma_h - 1)/c_p]/\rho_0$ the collective thermoviscous coefficient, η and η_B the shear and bulk viscosity coefficients, respectively, ρ_0 the ambient density, κ thermal conductivity, γ_h ratio of specific heats, and c_p special heat at constant pressure. Equation (19) describes both dispersion (waveform alternation with respect to frequency) and attenuation behaviors. Szabo (1994) pointed out that the low-frequency approximation of (19) leads to a square dependence of attenuation on frequency with constants

$$\alpha_0 = \mu/2c_0^3, \quad y=2, \tag{20}$$

in terms of the power law (2). Szabo (1994) noted that the exponent y and the differential order of the lossy term in the generalized wave equation are related. Namely, the time derivative order of the lossy term is higher than y by 1. By analogy with this relationship, we generalize (19) via the space fractional Laplacian and intuitively get

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0^{1-y}}\frac{\partial}{\partial t}(-\nabla^2)^{y/2} p, \tag{21}$$

where $(-\nabla^2)^{y/2}$ is the fractional Laplace. When $y=2$, (21) is equivalent to the model equation (19). When $y=0$, (21) is reduced to the standard damped wave equation

$$\nabla^2 p = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0}\frac{\partial p}{\partial t}, \tag{22}$$

which describes the frequency-independent attenuation (Szabo, 1994).

In order to verify that our intuitive fractional Laplacian wave equation (21) reflects the frequency power-law attenuation (2), the corresponding dispersion equation is derived below and analyzed. To facilitate the analysis without loss of generality, let us consider the 1D case of model equation (21)

$$\frac{\partial^2 p}{\partial z^2} = \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{2\alpha_0}{c_0^{1-y}}\frac{\partial}{\partial t}\left(-\frac{\partial^2}{\partial z^2}\right)^{y/2} p. \tag{23}$$

The Fourier transforms of the time and space derivatives are given by

$$F_-\left\{\frac{\partial^2 p}{\partial z^2}\right\} = (ik)^2 P = -k^2 P, \tag{24}$$

$$F_-\left\{\left(-\frac{\partial^2}{\partial z^2}\right)^{y/2} p\right\} = (-(ik)^2)^{y/2} P = k^y P, \tag{25}$$

$$F_+\left\{\frac{\partial^2 p}{\partial t^2}\right\} = (-i\omega)^2 P = -\omega^2 P. \tag{26}$$

Applying the time and space Fourier transforms (24)–(26) to (23), we have the dispersion equation

$$k^2 - \omega^2/c_0^2 - i2\alpha_0 \omega k^y / c_0^{1-y} = 0. \tag{27}$$

Since $k=\beta + i\alpha$ and $\beta = \omega/c$, it is straightforward to have

$$\beta^2 - \alpha^2 - \omega^2/c_0^2 + i2\alpha\beta - i2\alpha_0\omega\beta^y(1+i\alpha/\beta)^y/c_0^{1-y} = 0. \tag{28}$$

To move the analysis further, the Szabo conservative smallness approximation [Eqs. (17) and (18) of Szabo (1994)] is crucial, i.e.,

$$\alpha/\beta \approx \alpha_0|\omega|_{\lim}^{y}/\beta_0 = \alpha_0|\omega|_{\lim}^{y-1} c_0 \leqslant 0.1, \tag{29}$$

where $\omega_{\lim}$ is the frequency limit corresponding to 0.1. As discussed in Szabo (1994), the limit frequency range in terms of (29) is adequate enough to cover the frequency spectrum of practical interest in medical ultrasound applications. In terms of (29), (28) is then approximated by the binomial expansion as

$$\beta^2 - \alpha^2 - \omega^2/c_0^2 + i2\alpha\beta - i2\alpha_0\omega\beta^y/c_0^{1-y} + 2\alpha_0 y\omega\alpha\beta^{y-1}/c_0^{1-y} = 0. \tag{30}$$

Then, separating the real and imaginary parts of the above Eq. (30) produces

$$\alpha = \alpha_0\omega(\beta c_0)^{y-1}, \tag{31a}$$

$$\beta^2 = \omega^2/c_0^2 + (1-2y)\alpha^2. \tag{31b}$$

With the further help of the smallness approximation (29) and $\beta_0 = \omega/c_0$, from (31a) we derive

$$\alpha \approx \alpha_0|\omega|^y, \tag{32}$$

It is noted that (31a) matches the power law (2). By now we have shown that the fractional Laplacian model (23) does have a power-law attenuation under the Szabo smallness approximation condition (29) corresponding to the time convolutional integral model (Szabo, 1994) and the time fractional models (Baglegy and Torvik, 1983). Szabo (1994) demonstrated his wave equation is causal indirectly by verifying its parabolic counterpart is causal. In the latter section III, we will show that the fractional Laplacian wave equation (21) can be reduced to the well-known parabolic anomalous diffusion equation. The causality of the latter is guaranteed (Hanyga, 2001c). We are therefore convinced that equation (21) is also causal. It is noteworthy that the fractional Laplacian is a positive definit operator (Gorenflo and Francesco, 1998), and the Duhamel's principle applies to the fractional Laplacian equations.

IV. NONLINEAR LOSSY MEDIA

Most nonlinear acoustic equations are only useful for describing lossy media with a quadratic dependence or independence on frequency (Szabo, 1993), and thus have limited practical utility. In this section, we are concerned with the extension of the previous linear fractional Laplacian modeling methodology to nonlinear media obeying the dissipative power law of arbitrary exponents. It is noted that the thermoviscous representation in the standard nonlinear acoustic PDE models, which characterizes the effects of absorption and dispersion, is mostly the same as in the corresponding linear models. Thus, Blackstock (1985) presented a straightforward strategy constructing the nonlinear anomalous attenuation model by simply replacing its attenuation term with that in the corresponding linear model, while keeping all other linear and nonlinear terms unchanged. The methodology is justified by a perturbation analysis (Blackstock, 1985). Following this strategy, Szabo (1993) extended his linear convolution integral modeling of arbitrary power law exponents (Szabo, 1994) to the Burgers, KZK, and Westervelt equations. By analogy with Blackstock (1985) and

Szabo (1993), we can generalize these nonlinear acoustic models by replacing their thermoviscous term with the fractional Laplacian lossy terms given in Eqs. (21) and (23).

It is stressed that the smallness approximation condition (29), crucial in the dispersion analysis of the preceding linear fractional Laplacian models, is also the foundation of the nonlinear acoustics modeling (Hamilton and Blackstock, 1998; Szabo, 1993). Thus, all derivations here are consistent.

The Burgers equation may be the best-known simple nonlinear acoustic model which describes the combined effects of nonlinearity and dissipation. The one-dimensional Burgers equation for plane progressive waves is stated as

$$\frac{\partial p}{\partial t}+Bp\,\frac{\partial p}{\partial z}-\varepsilon\,\frac{\partial^2 p}{\partial z^2}=0, \tag{33}$$

where B denotes the nonlinear coefficient (Sugimoto, 1991), and ε is a constant proportional to the coefficients of viscosity and heat conduction (Blackstock, 1985). It is known (Blackstock, 1985; Szabo, 1993) that the Burgers equation (31) describes lossy acoustic propagation of square frequency dependence. To extend the Burgers equation to accommodating power-law media of arbitrary exponent y, Blackstock (1985) suggests and verifies to some extent that only the third term of (33) needs to be modified, which involves the absorption, while keeping all others the same.

In terms of an approach detailed by Szabo (1994), the hyperbolic wave equation (23) can be approximated to the parabolic equation by removing the left-hand side term, namely

$$\frac{1}{c_0^2}\,\frac{\partial^2 p}{\partial t^2}+\frac{2\alpha_0}{c_0^{1-y}}\,\frac{\partial}{\partial t}\left(-\frac{\partial^2}{\partial z^2}\right)^{y/2}p=0. \tag{34}$$

And then, integrating (34) with respect to time t and multiplying by c_0^2, we have

$$\frac{\partial p}{\partial t}+2\alpha_0 c_0^{1+y}\left(-\frac{\partial^2}{\partial z^2}\right)^{y/2}p=0. \tag{35}$$

The model (35) is a generalized diffusion equation and corresponds to the Burgers equation (33) without the nonlinear convection term. The analytical solution of Eq. (35) can be found in Hanyga (2001c). Applying the spatial Fourier transform (25) to (35), we have

$$\frac{dP}{dt}+2\alpha_0 c_0^{1+y}k^y P=0. \tag{36}$$

Thus, the transformation solution is

$$P(k,t)=Ce^{-2\alpha_0 c_0^{1+y}tk^y}, \tag{37}$$

where C depends on the initial condition. When $y=2$, (37) exhibits the normal frequency-squared diffusion. It is clear that Eq. (37) is a parabolic model originating from Eq. (23) while holding the capability describing arbitrary power (y) law attenuation.

By analogy with the generalizing methodology presented by Blackstock (1985) and Szabo (1993), the fractional Laplacian Burgers equation is presented below by simply adding the nonlinear convection term of Burgers equation (33) to Eq. (35), i.e.,

$$\frac{\partial p}{\partial t}+Bp\,\frac{\partial p}{\partial z}+2\alpha_0 c_0^{1+y}\left(-\frac{\partial^2}{\partial z^2}\right)^{y/2}p=0. \tag{38}$$

Note that the nonlinear term in (38) can be considered the source term in the sense of an inhomogeneous equation (Szabo, 1993).

The nonlinear equation model (38) belongs to the so-called fractal Burgers equations or the fractional advection–dispersion equation. A detailed analysis of such equations is given in Biler *et al.* (2001). In higher dimensional cases, (37) is restated as

$$\frac{\partial p}{\partial t}+Bp\cdot\nabla p+2\alpha_0 c_0^{1+y}(-\nabla^2)^{y/2}p=0, \tag{39}$$

where ∇p represents the pressure gradient vector, and the dot stands for a scalar product. Ochmann and Makarov (1993) also developed the time fractional derivative Burgers equation to describe the power-law absorptions with arbitrary y. By using the same strategy, Chen and Holm (2002b) also developed the fractional Laplacian KZK, Westervelt, general second-order approximation model, incompressible Naiver–Stokes, and Boussinesq shallow-water wave equation to incorporate arbitrary power-law frequency-dependent dissipations.

V. CONCLUDING REMARKS

Attenuation plays an essential part in many acoustics applications, for instance, the ultrasound second harmonic imaging and high-intensity focused ultrasound beam for therapeutic surgery. Compared with the Szabo's time convolutional integral model of the power-law attenuation, the present fractional Laplacian time-space model has a uniform and simpler expression. On the other hand, it is also not a simple task for the fractional time derivative models to obtain the initial conditions of the second-order derivative when $y>1$, since most physical systems only provide the zero- and first-order initial conditions. More importantly, most anomalous thermoviscous attenuations occur in spatially inhomogeneous environments (Henry and Wearne, 1999), notably biomaterials and geological random media, whose microgeometry largely have fractal dimension structures in space. The power-law formula (2) also shows that y is independent of frequency ω (time scale). It is therefore reasonable to think that y may in fact underlie the spatial fractal. For example, y varies with different human body tissues, which have different spatial microstructures. We thus conclude that a spatial representation of the dissipation via the fractional Laplacian is physically more valid than the fractional time derivative representations.

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APPENDIX: FEM DISCRETIZATION OF THE LINEAR FRACTIONAL LAPLACIAN LOSSY WAVE EQUATION

Let the FEM approximate discretization of a Laplacian operator be expressed as

$$-\nabla^2 p \Rightarrow K\mathbf{p}, \tag{A1}$$

where $\mathbf{p}$ represents the pressure value vector at the discrete nodes. Since the Laplacian is the positive definite operator, it is well known that its FEM discretization matrix K is also positive definite. The corresponding FEM formulation of the fractional Laplacian of $y/2$ order is then obtained by

$$(-\nabla^2)^{y/2} p \Rightarrow K^{y/2}\mathbf{p}. \tag{A2}$$

It is worth noting here that despite the fact that the FEM discretization matrix K is sparse, $K^{y/2}$ will be a full matrix underlying the non-local property of the fractional Laplacian, which models the global interactions in space. The FEM discretization of the fractional Laplacian wave equation model (21) is thus stated as

$$\mathbf{p}_{tt} + 2\alpha_0 c_0^{1+y} K^{y/2}\mathbf{p}_t + c^2 K\mathbf{p} = g(t), \tag{A3}$$

where the subscript t represents the temporal derivative, and $g(t)$ is due to the source term. It is obvious that (A3) readily takes into account frequency-dependent viscous effects for a multitude of frequency components (broadband signal) with empirical coefficients α_0 and y of the power-law attenuation.

For a little complicated fitting of measurement fitting, the empirical formula (2) of frequency dependent attenuation can be technially replaced by (He, 1998)

$$\alpha(\omega) = \alpha_1 + \alpha_0|\omega|^y, \quad y \in [0,2], \tag{A4}$$

where α_1 is an empirical parameter. Thus, equation (A3) can be accordingly restated as

$$p_u + 2(\alpha_1 c_0 I + \alpha_0 c_0^{y+1} K^{y/2}) p_t + c_0^2 K p = g(t). \tag{A5}$$

When $y=2$, the semi-discrete model (A5) brings out the square frequency dependence and is reduced to the classical Rayleigh proportional damping model.

The temporal discretization of (A3) can easily be done via the standard finite difference time integrators. The major issue here is about computer resource requirements in the evaluation of $K^{y/2}$. The orthodox analytical approach for this task is costly singular value decomposition, i.e.,

$$K^{y/2} = \Phi^T \left(\sum \lambda_i^2 \right)^{y/2} \Phi, \tag{A6}$$

where Φ is the orthogonal matrix, the superscript T represents the matrix transpose, and Σ denotes a diagonal matrix with eigenvalues λ. The popular numerical methods for evaluating $K^{y/2}$ are the Schur decomposition, Padé approximation, and iterative method, which usually require $O(n^3)$ operations (Lu, 1998). In order to overcome such enormous cost, a parallel numerical model is under study. The results will be reported in a separate subsequent paper. As a matter of fact, the numerical solution of fractional Laplacian equations has not been researched well, and very few related reports are known to the authors.

It is also interesting to mention that the matrix power function of the fractional order [e.g., (A2)] can be considered a clear algebraic correspondence to the fractional calculus in analysis and the fractal in geometry. All three of these methodologies may consist of a complete set of mathematic apparatus in modeling, analyzing, simulating, and visualizing complex phenomena, where the traditional mathematic methods of integer order do not work well.

Adhikari, S. (**2000**). ''Damping Models for Structural Vibration,'' Ph.D. thesis, Cambridge University, Cambridge.

Baglegy, R. L., and Torvik, P. J. (**1983**). ''A theoretical basis for the application of fractional calculus to viscoelasticity,'' J. Rheol. **27**, 201–210.

Biler, P., Karch, G., and Woyczynski, W. A. (**2001**). ''Asymptotics and high dimensional approximations for nonlinear pseudodifferential equations involving Levy generators,'' Demonstratio Math. **34**(2), 403–413.

Blackstock, D. T. (**1967**). ''Transient solution for sound radiated into a viscous fluid,'' J. Acoust. Soc. Am. **41**, 1312–1319.

Blackstock, D. T. (**1985**). ''Generalized Burgers equation for plane waves,'' J. Acoust. Soc. Am. **77**(6), 2050–2053.

Caputo, M. (**1967**). ''Linear models of dissipation whose *Q* is almost frequency independent. II,'' Geophys. J. R. Astron. Soc. **13**, 529–539.

Caputo, M., and Mainardi, F. (**1971**). ''A new dissipation model based on memory mechanism,'' Pure Appl. Geophys. **91**, 134–147.

Chen, W., Holm, S., Bounaim, A., Ødegård, Å., and Tveito, A. (**2002a**). ''Frequency decomposition time-domain model of broadband frequency-dependent absorption,'' The 9th Workshop of the Finite Element Method in Biomedical Engineering, Biomechanics and Related Fields, Ulm, Germany, pp. 40–48.

Chen, W., and Holm, S. (**2002b**). ''Fractional Laplacian time-space models and Lévy stable distribution for linear and nonlinear frequency-dependent lossy media,'' Research Report of Simula Research Laboratory, No. 2002-09.

Diethelm, K. (**2000**). *Fractional Differential Equations, Theory and Numerical Treatment* (preprint).

Enelund, M. (**1996**). ''Fractional Calculus and Linear Viscoelasticity in Structural Dynamics,'' Ph.D thesis, Chalmers University of Technology, Sweden.

Feller, W. (**1971**). *An Introduction to Probability Theory and its Applications*, 2nd ed. (Wiley, New York), Vol. 2.

Gaul, L. (**1999**). ''The influence of damping on waves and vibrations,'' Mech. Syst. Signal Process. **13**(1), 1–30.

Ginter, S. (**2000**). ''Numerical simulation of ultrasound-thermotherapy combining nonlinear wave propagation with broadband soft-tissue absorption,'' Ultrasonics **27**, 693–696.

Gorenflo, R., and Mainardi, F. (**1998**). ''Random walk models for space-fractional diffusion processes,'' Fract. Calc. Appl. Anal. **1**, 167–191.

Hamilton, M. F., and Blackstock, D. T., Editors (**1998**). *Nonlinear Acoustics* (Academic, New York).

Hanyga, A. (**1999**). ''Simple memory models of attenuation in complex viscoporous media,'' in Proceedings of the 1st Canadian Conference on Nonlinear Solid Mechanics, Victoria, BC, 16–20 June.

Hanyga, A. (**2001a**). ''Multi-dimensional solutions of time-fractional diffusion-wave equations,'' Proc. R. Soc. London, Ser. A (in press).

Hanyga, A. (**2001b**). ''Fractional diffusion and wave equations,'' Conference of ''Modelli Matematici e Problemi Analitici per Materiali Speciali,'' Cortona.

Hanyga, A. (**2001c**). ''Multi-dimensional solutions of space-fractional diffusion equations,'' Proc. R. Soc. London, Ser. A **457**, 2993–3005.

He, P. (**1998**). ''Simulation of ultrasound pulse propagation in lossy media obeying a frequency power law,'' IEEE Trans. Ultrason. Ferroelectr. Freq. Control **45**(1), 114–125.

Henry, B. I., and Wearne, S. L. (**1999**). ''Fractional reaction-diffusion,'' *Elsevier preprint*.

Jespersen, S. (**1999**). ''Anomalous diffusion,'' Progress Report of Institute of Physics and Astronomy, University of Aarhus.

Johnson, D., and Dudgeon, D. (**1993**). *Array Signal Processing* (Prentice-Hall, Englewood Cliffs, NJ).

Lighthill, J. (**1980**). *Waves in Fluids* (Cambridge University Press, Cambridge).

Lu, Y. Y. (**1998**). ''A Padé approximation method for square roots of symmetric positive definite matrices,'' Research Report, City University of Hong Kong.

Makris, N., and Constantinou, M. C. (**1991**). ''Fractional-derivative Maxwell model for viscous dampers,'' J. Struct. Eng. **117**(9), 2708–2724.

Mast, T. D., Souriau, L. P., Liu, D. L., Tabei, M., Nachman, A. I., and Wagg, R. C. (**2001**). ''A *k*-space method for large-scale models of wave propagation in tissue,'' IEEE Trans. Ultrason. Ferroelectr. Freq. Control **48**(2), 341–354.

Nachman, A. I., Smith, J., and Waag, R. C. (**1990**). ''An equation for acoustic propagation in inhomogeneous media with relaxation losses,'' J. Acoust. Soc. Am. **88**(3), 1584–1595.

Ochmann, M., and Makarov, S. (**1993**). ''Representation of the absorption of nonlinear waves by fractional derivative,'' J. Acoust. Soc. Am. **94**(6), 3392–3399.

Pierce, A. D. (**1989**). *Acoustics, an Introduction to its Physical Principles and Applications* (Acoustical Society of America, Woodbury, NY).

Rusovici, R. (**1999**). ''Modeling of Shock Wave Propagation and Attenuation in Viscoelastic Structures,'' Ph.D. thesis, Virginia Polytechnic Institute and State University.

Samko, S. G., Kilbas, A. A., and Marichev, O. I. (**1987**). *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach Science, New York).

Sugimoto, N. (**1991**). ''Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves,'' J. Fluid Mech. **225**, 631–653.

Szabo, T. L. (**1993**). ''Time Domain Nonlinear Wave Equations for Lossy Media,'' in *Advances in Nonlinear Acoustics: Proceedings of 13th ISNA*, edited by H. Hobaek (World Scientific, Singapore), pp. 89–94.

Szabo, T. L. (**1994**). ''Time domain wave equations for lossy media obeying a frequency power law,'' J. Acoust. Soc. Am. **96**(1), 491–500.

Wismer, M., and Ludwig, R. (**1995**). ''An explicit numerical time domain formulation to simulate pulsed pressure waves in viscous fluids exhibiting arbitrary frequency power law attenuation,'' IEEE Trans. Ultrason. Ferroelectr. Freq. Control **42**(6), 1040–1049.

Wojcik, G., Mould, J., Lizzi, Jr., F., Abboud, N., Ostromogilsky, M., and Vaughan, D. (**1995**). ''Nonlinear modelling of therapeutic ultrasound,'' 1995 IEEE Ultrasonics Symposium Proceedings, 1617–1622.

Wojcik, G., Mould J., Carcione, L. M., Ostromogilsky, M., and Vaughan, D. (**1999**). ''Combined transducer and nonlinear tissue propagation simulations,'' 1999 International Mechanical Engineering Congress & Exposition Proceedings.

Yuan, X., Borup, D., and Wiskin, J. (**1999**). ''Simulation of acoustic wave propagation in dispersive media with relaxation losses by using FDTD method with PML absorbing boundary condition,'' IEEE Trans. Ultrason. Ferroelectr. Freq. Control **46**(1), 14–23.

Zähle, M. (**1997**). ''Fractional differentiation in the self-affine case. V. The local degree of differentiability,'' Math. Nachr. **185**, 279–306.

Zaslavsky, G. M. (**1994**). ''Fractional kinetic equation for Hamiltonian chaos,'' Physica D **76**, 110–122.