An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 43 (2002) 379–391
www.elsevier.com/locate/camwa

A Meshless, Integration-Free, and Boundary-Only RBF Technique

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Abstract—Based on the radial basis function (RBF), nonsingular general solution, and dual reciprocity method (DRM), this paper presents an inherently meshless, integration-free, boundary-only RBF collocation technique for numerical solution of various partial differential equation systems. The basic ideas behind this methodology are very mathematically simple. In this study, the RBFs are employed to approximate the inhomogeneous terms via the DRM, while nonsingular general solution leads to a boundary-only RBF formulation for homogenous solution. The present scheme is named as the boundary knot method (BKM) to differentiate it from the other numerical techniques. In particular, due to the use of nonsingular general solutions rather than singular fundamental solutions, the BKM is different from the method of fundamental solution in that the former does not require the artificial boundary and results in the symmetric system equations under certain conditions. The efficiency and utility of this new technique are validated through a number of typical numerical examples. Completeness concern of the BKM due to the sole use of the nonsingular part of complete fundamental solution is also discussed.

Keywords—Boundary knot method, Dual reciprocity method, BEM, Method of fundamental solution, Radial basis function, Nonsingular general solution.

1. INTRODUCTION

It has long been claimed that the boundary element method (BEM) is a viable alternative to the domain-type finite element method (FEM) and finite difference method (FDM) due to its advantages in dimensional reducibility and suitability to infinite domain problems. However, nowadays the FEM and FDM still dominate science and engineering computations. The major bottlenecks in performing the BEM analysis have long been its weakness in handling inhomogeneous terms such as time-dependent and nonlinear problems. The recent introduction of the dual reciprocity BEM (DRBEM) by Nardini and Brebbia [1] greatly eases these inefficiencies. Notwithstanding, as was pointed out in [2], the method is still more mathematically complicated and requires strenuous manual effort compared with the FEM and FDM. In particular, the handling of singular integration is not easy to nonexpert users and often computationally expensive.

The use of the low-order approximation in the BEM also slows convergence. More importantly, just like the FEM, surface mesh or remesh in the BEM requires costly computation, especially

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Some valuable comments from a referee reshaped this paper into its present form. The authors also express grateful acknowledgment of helpful discussions with M. Golberg, Y.C. Hon, and A.H.D. Cheng. The first author was supported as a JSPS fellow by the Japan Society of Promotion of Science.

PII: S0898-1221(01)00293-0

for moving boundary and nonlinear problems. The method of fundamental solution (MFS) is shown an emerging technique to alleviate these drawbacks and is getting increasing attraction especially due to some recent works by Golberg *et al.* [2–5]. The MFS affords advantages of being integration-free, spectral convergence, and meshless.

However, the use of artificial boundary outside physical domain has been a major limitation of the MFS, which may cause severe ill-conditioning of the resulting equations, especially for complex boundary geometry [6,7]. These inefficiencies of the MFS motivate us to find an alternative technique, which keeps its merits but removes its shortcomings undermining its attractiveness.

Recently, Golberg *et al.* [4] established the DRBEM on the firm mathematical theory of the radial basis function (RBF). The DRBEM can be regarded as a two-step methodology. In terms of dual reciprocity method (DRM), the RBF is applied at first to approximate the particular solution of inhomogeneous terms, and then the standard BEM is used to discretize the remaining homogeneous equation. Chen *et al.* [3] and Golberg *et al.* [2,4] extended this RBF approximation of particular solution to the MFS, which greatly enhances its applicability. In fact, the MFS itself can also be considered a special RBF collocation approach, where the fundamental solution of the governing equation is taken as the radial function.

On the other hand, the domain-type RBF collocation is also now under intense study since Kansa's pioneer work [8]. The major charisma of the RBF-type techniques is their meshless inherence. The construction of mesh in high dimension is not a trivial work. Unlike the recently developed meshless FEM with the moving least square, the RBF approach is a truly cheap meshless technique without any difficulty applying boundary conditions [9,10]. The RBF is therefore an essential component in this study to construct a viable numerical technique.

Kamiya *et al.* [11] and Chen *et al.* [12] pointed out that the multiple reciprocity BEM (MRM) solution of the Helmholtz problems with the Laplacian plus its high-order terms is in fact to employ only the singular real part of complete complex fundamental solution of the Helmholtz operator. Power [13] simply indicated that the use of either the real or imaginary part of the Helmholtz Green's representation formula could formulate interior Helmholtz problems.

This study extends these ideas to general problems such as Laplace and convection-diffusion problems by a combined use of the nonsingular general solution, the dual reciprocity method, and RBF. This mixed technique is named as the boundary knot method (BKM) [14] due to its essential meshless property; namely, the BKM does not need any discretization grids for any dimension problems and only uses knot points. The inherent inefficiency of the MFS due to the use of the fictitious boundary is alleviated in the BKM, which leads to tremendous improvement in computational efficiency and produces the symmetric matrix structure under certain conditions.

It is noted that the BKM does not involve integration operation due to the use of the collocation technique. Just like the MFS, the method is very simple to implement. The nonsingular general solution for multidimensional problems can be understood as the nonsingular part of a complete fundamental solution of various operators. The preliminary numerical studies of this paper show that the BKM is a promising technique in terms of efficiency, accuracy, and simplicity. We also use the BKM with the response knot-dependent nonsingular general solutions to solve the varying parameter problems successfully.

This paper is organized as follows. Section 2 involves the procedure of the DRM and RBF approximation to a particular solution. In Section 3, we introduce a nonsingular general solution and derive the analogization equations of the BKM. Numerical results are provided and discussed in Section 4 to establish the validity and accuracy of the BKM. Completeness concern of the BKM is discussed in Section 5. Finally, Section 6 concludes with some remarks based on the reported results. In the Appendix, we give the nonsingular general solution of some 2D, 3D steady, and time-dependent operators.

2. RBF APPROXIMATION TO A PARTICULAR SOLUTION

Like the DRBEM and MFS, the BKM can be viewed as a two-step numerical scheme, namely, DRM and RBF approximation to particular solution and the evaluation of homogeneous solution. The latter is the emphasis of this paper. The former has been well developed [2–4]. For the sake of completeness, here we outline the basic methodology to approximate a particular solution. Let us consider the differential equation

$$L\{u(x)\} = f(x), \qquad x \in \Omega, \tag{1}$$

with boundary conditions

$$u(x) = b_1(x), \qquad x \subset \Gamma_u, \tag{2}$$

$$\frac{\partial u(x)}{\partial n} = b_2(x), \qquad x \subset \Gamma_T, \tag{3}$$

where L is a differential operator, $f(x)$ is a known forcing function, and n is the unit outward normal. $x \in R^d$, d is the dimension of geometry domain, which is bounded by a piecewise smooth boundary $\Gamma = \Gamma_u + \Gamma_T$. In order to facilitate discussion, it is assumed here that the operator L includes the Laplace operator, namely,

$$L\{u\} = \nabla^2 u + L_1\{u\}. \tag{4}$$

We should point out that this assumption is not necessary [15]. Equation (1) can be restated as

$$\nabla^2 u + u = f(x) + u - L_1\{u\}. \tag{5}$$

The solution of the above equation (5) can be expressed as

$$u = v + u_p, \tag{6}$$

where v and u_p are the general and particular solutions, respectively. The latter satisfies the equation

$$\nabla^2 u_p + u_p = f(x) + u - L_1\{u\}, \tag{7}$$

but does not necessarily satisfy boundary conditions (2) and (3). v is the homogeneous solution of the Helmholtz equation

$$\nabla^2 v + v = 0, \qquad x \in \Omega, \tag{8}$$

$$v(x) = b_1(x) - u_p, \qquad x \subset \Gamma_u, \tag{9}$$

$$\frac{\partial v(x)}{\partial n} = b_2(x) - \frac{\partial u_p(x)}{\partial n}, \qquad x \subset \Gamma_T. \tag{10}$$

The first step in the BKM is to evaluate the particular solution u_p by the DRM and RBF. After this, equations (8)–(10) can be solved by the boundary RBF methodology using the nonsingular general solution proposed in Section 3.

Unless the right side of equation (7) is rather simple, it is practically impossible to get an analytical particular solution in general cases. In addition, even if the analytical solutions for some problems are available, their forms are usually too complicated to use in practice. Therefore, we prefer to approximate these inhomogeneous terms numerically. The DRM with the RBF is a very promising approach for this task [1–5], which analogizes the particular solution by the use of a series of approximate particular solution at all specified nodes. The right side of equation (7) is approximated by the RBF approach, namely,

$$f(x) + u - L_1\{u\} \cong \sum_{j=1}^{N+L} \alpha_j \phi\left(\|x - x_j\|\right) + \psi(x), \tag{11}$$

where α_j are the unknown coefficients. N and L are, respectively, the numbers of knots on the boundary and the domain. $| \, |$ represents the Euclidean norm, and $\phi(\,)$ is the RBF. An additional polynomial term $\psi(x)$ is required to assure nonsingularity of the interpolation matrix if the RBF is conditionally positive definite such as multiquadratics (MQ) and thin plate spline (TPS) [8,16]. For example, in the 2D case with linear polynomial restraints, we have

$$f(x) + u - L_1\{u\} \cong \sum_{j=1}^{N+L} \alpha_j \phi(r_j) + \alpha_{N+L+1} x + \alpha_{N+L+2} y + \alpha_{N+L+3}, \tag{12}$$

where $r_j = |x - x_j|$. The corresponding side conditions are given by

$$\sum_{j=1}^{N+L} \alpha_j = \sum_{j=1}^{N+L} \alpha_j x_j = \sum_{j=1}^{N+L} \alpha_j y_j = 0. \tag{13}$$

By forcing equation (12) to exactly satisfy equations (7) and (13) at all nodes, we can get a set of simultaneous equations to uniquely determine the unknown coefficients α_j. In this procedure, we need to evaluate the approximate particular solutions in terms of the RBF ϕ. The standard approach is that ϕ in equation (11) is first selected, and then corresponding approximate particular solutions are determined by analytically integrating a differential operator. The advantage of this method is that it is a mathematically reliable technique. However, this methodology easily performs only for simple operators and RBFs. Recently, Muleskov *et al.* [5] made a substantial advance to discover the analytic approximate particular solutions for Helmholtz-type operators using the polyharmonic splines. But the analytical approximate particular solutions for general cases such as the MQ and other differential operators are not yet available now due to great difficulty involved.

Another scheme evaluating approximate solutions is a reverse approach [17,18]. Namely, the approximate particular solution is at first chosen, and then we can evaluate the corresponding ϕ by simply substituting the specified particular solution into a certain operator of interest. It is a very difficult task to mathematically prove under what conditions this approach is reliable, although it seems to work well so far for many problems [17–19]. This scheme is in fact equivalent to the approximation of particular solution using Kansa's method [8,9]. In this study, we use this scheme in terms of the MQ. The chosen approximate particular solution is

$$\varphi(r_j) = \left(r_j^2 + c_j^2\right)^{3/2}, \tag{14}$$

where c_j is the shape parameter. The corresponding MQ-like radial function is

$$\phi(r_j) = 6\left(r_j^2 + c_j^2\right) + \frac{3r^2}{\sqrt{r_j^2 + c_j^2}} + \left(r_j^2 + c_j^2\right)^{3/2}. \tag{15}$$

Finally, we can get particular solutions at any point by weighted summation of approximate particular solutions at all nodes with coefficients α_j. For more details on the procedure, see [1–5].

3. NONSINGULAR GENERAL SOLUTION AND BOUNDARY KNOT METHOD

One may think that the placement of source points outside domain in the MFS is to avoid the singularities of fundamental solutions. However, we found through numerical experiments that even if all source and response points were placed differently on physical boundary to circumvent the singularities, the MFS solutions were still degraded severely. In the MFS, the more distant the source points are located from physical boundary, the more accurate MFS solutions are obtained [2]. However, unfortunately, the resulting equations can become extremely ill conditioned which in some cases deteriorate the solution [2,6,7].

To illustrate the basic idea of the boundary collocation using a nonsingular general solution, we take the 2D Helmholtz operator as an illustrative example, which is the simplest among various often-encountered operators having nonsingular general solution. Note that the Laplace operator does not have a nonsingular general solution. For the other nonsingular general solutions, see the Appendix.

The 2D homogeneous Helmholtz equation (8) has two general solutions, namely,

$$v(r) = c_1 J_0(r) + c_2 Y_0(r), \tag{16}$$

where $J_0(r)$ and $Y_0(r)$ are the zero-order Bessel functions of the first and second kinds, respectively. In the standard BEM and MFS, the Hankel function

$$H(r) = J_0(r) + iY_0(r) \tag{17}$$

is applied as the fundamental solution. It is noted that $Y_0(r)$ encounters logarithm singularity, which causes the major difficulty in applying the BEM and MFS. Many special techniques have been developed to solve or circumvent this singular trouble.

The present BKM scheme discards the singular general solution $Y_0(r)$ and only uses $J_0(r)$ as the radial function to collocate the boundary condition equations (9) and (10). It is noted that $J_0(r)$ exactly satisfies the Helmholtz equation, and we can therefore get a boundary-only collocation scheme. Unlike the MFS, all collocation knots are placed only on physical boundary and can be used as either source or response points.

Letting $\{x_k\}_{k=1}^N$ denote a set of nodes on the physical boundary, the homogeneous solution $v(x)$ of equation (8) is approximated in a standard collocation fashion

$$v(x) = \sum_{k=1}^{N} \beta_k J_0(r_k), \tag{18}$$

where $r_k = \|x - x_k\|$. k is the index of source points. N is the number of boundary knots. β_k are the desired coefficients. Collocating equations (9) and (10) in terms of equation (18), we have

$$\sum_{k=1}^{N} \beta_k J_0(r_{ik}) = b_1(x_i) - u_p(x_i), \tag{19}$$

$$\sum_{k=1}^{N} \beta_k \frac{\partial J_0(r_{jk})}{\partial n} = b_2(x_j) - \frac{\partial u_p(x_j)}{\partial n}, \tag{20}$$

where i and j indicate Dirichlet and Neumann boundary response knots, respectively. If internal nodes are used, we need to constitute another set of supplement equations

$$\sum_{k=1}^{N} \beta_k J_0(r_{lk}) = u_l - u_p(x_l), \qquad l = 1, \ldots, L, \tag{21}$$

where l indicates the internal response knots and L is the number of interior points. Now we get the total $N + L$ simultaneous algebraic equations. It is stressed that the use of interior points is not always necessary in the BKM as in the DRBEM [15,17,20]. The term "boundary-only" is used here in the sense as in the DRBEM and MFS that only boundary knots are required, although internal knots can improve solution accuracy in some cases.

Before proceeding with the numerical experiments, we consider choosing the radial basis function. In general, RBFs are globally defined basis functions and lead to a dense matrix, which becomes highly ill conditioned if very smooth radial basis functions are used with a large number of interpolation nodes [16]. This causes severe stability problems and computationally inefficiency for a large size problem. A number of approaches have been proposed to remedy this problem such as domain decomposition and compactly supported RBFs (CS-RBFs). The latter was recently developed by Wendland [21], Wu [22], and Schaback [23]. Golberg *et al.* [2], Wong *et al.* [24], and Chen *et al.* [25], respectively, applied the CS-RBFs to the MFS, Kansa's method, and DRBEM successfully. However, in this study we will not use the CS-RBFs to focus on the illustration of the basic idea of the BKM with globally-supported RBFs.

The MQ [26], TPS [27], and linear RBF [1] are the most widely used globally-defined RBFs now. Among them, it is well known that the MQ ranks the best in accuracy [28] and is the only available RBF with the desirable merit of spectral convergence [4]. However, its accuracy is greatly influenced by the shape parameter [29,30]. So far, the optimal determination of shape parameter is still an open research topic. Despite this problem, the MQ is still most widely used in the RBF solution of various differential systems. For the numerical example of the Laplace equation shown in Section 4.2, the linear and generalized TPS RBFs show evidently slower convergence rate than the MQ. For example, the average relative error is 0.91% for the linear RBF with 11 knots, 0.39% for the TPS with 11 knots, and 0.023% for the MQ (shape parameter 2) with nine knots. We even got 0.5% relative average error by the MQ with only three knots. To simplify the presentation, this paper only uses the MQ with the DRM although the use of the TPS is also attractive in many cases.

4. NUMERICAL RESULTS AND DISCUSSIONS

In this paper, all numerical examples unless otherwise specified are taken from [20]. The geometry of a test problem is all an ellipse featured with a semimajor axis of length 2 and semiminor axis of length 1. These examples are chosen since their analytical and numerical solutions are obtainable to compare. More complicated problems can be handled in the same BKM fashion without any extra difficulty. The zero-order Bessel and modified Bessel functions of the first kind are evaluated via short subroutines given in [31]. The 2D Cartesian coordinates (x, y) system is used as in [20].

4.1. Helmholtz Equation

The 2D homogeneous Helmholtz equation is given by

$$\nabla^2 u + u = 0, \tag{22}$$

with inhomogeneous boundary condition

$$u = \sin x. \tag{23}$$

It is obvious that equation (23) is also a particular solution of equation (22). Numerical results by the present BKM are displayed in Table 1 together with those by the DRBEM for comparison.

The numbers in the brackets of Table 1 mean the total nodes used. It is found that the present BKM converges very quickly. This demonstrates that the BKM enjoys the super-convergent property as in the other types of collocation methods [32]. The BKM solutions using seven nodes are adequately accurate. In stark contrast, the DRBEM with 16 boundary and 17 interior points [20] produced a relatively less accurate solution due to the use of the Laplacian and the low order of BEM convergence ratio. Please note that, in this case, there is no particular solution to be approximated by using the RBF and DRM in the BKM.

Table 1. Results for the Helmholtz problem.

x	y	Exact	DRBEM (33)	BKM (7)	BKM (11)
1.5	0.0	0.997	0.994	0.999	0.997
1.2	−0.35	0.932	0.928	0.931	0.932
0.6	−0.45	0.565	0.562	0.557	0.565
0.0	0.0	0.0	0.0	0.0	0.0
0.9	0.0	0.783	0.780	0.779	0.783
0.3	0.0	0.296	0.294	0.289	0.296
0.0	0.0	0.0	0.0	0.0	0.0

4.2. Laplace Equation

Readers may argue that it is somehow unfair to choose the homogeneous Helmholtz equation to compare the BKM and DRBEM. The latter used the Laplace fundamental solution in the previous example. In the following, we will further justify the superconvergence of the BKM through a comparison with the BEM for Laplace equation

$$\nabla^2 u = 0, \tag{24}$$

with boundary condition

$$u = x + y. \tag{25}$$

Equation (25) is easily found to be a particular solution of equation (24). This homogeneous problem is typically well suited to be handled by the standard BEM technique. In contrast, there is an inhomogeneous term in the BKM formulation to apply the nonsingular general solution of the Helmholtz operator. Namely, equation (24) is rewritten as

$$\nabla^2 u + u = u, \tag{26}$$

where the right inhomogeneous term u is approximated by the DRM as shown in the Section 2. The numerical results are displayed in Table 2 where the BEM solutions come from [20].

The MQ shape parameter c is set 25 for both three and five boundary knots in the BKM. It is observed that the BKM solutions are not sensitive to the parameter c. It is seen from Table 2 that the BKM results using three boundary nodes achieve the accuracy of four significant digits and are far more accurate than the BEM solution using 16 boundary nodes. This striking accuracy of the BKM again validates its spectral convergence. In this case, only boundary points are employed to approximate the particular solution by the DRM and RBF. It is noted that the coefficient matrices of the BEM and BKM are both fully populated. Unlike the BEM, however, the BKM yields a symmetric coefficient matrix for all self-adjoint operators with one type of boundary conditions. This Laplace problem is a persuasive example to verify high accuracy and efficiency of the BKM vis-a-vis the BEM.

Table 2. Results for a Laplace problem.

x	y	Exact	BEM (16)	BKM (3)	BKM (5)
1.5	0.0	1.500	1.507	1.500	1.500
1.2	−0.35	0.850	0.857	0.850	0.850
0.6	−0.45	0.150	0.154	0.150	0.150
0.0	0.0	−0.450	−0.451	−0.450	−0.450
0.9	0.0	0.900	0.913	0.900	0.900
0.3	0.0	0.300	0.304	0.300	0.300
0.0	0.0	0.0	0.0	0.0	0.0

4.3. Convection-Diffusion Problems

The FEM and FDM encounter some difficulty to produce an accurate solution to the systems involving the first-order derivative of convection term. Special care needs to be taken to handle this problem with these methods. It is claimed that the BEM does not suffer a similar accuracy problem. In particular, the DRBEM was said to be very suitable for this type problem [17,20]. Let us consider the convection diffusion equation

$$\nabla^2 u = -\frac{\partial u}{\partial x}, \tag{27}$$

which is given in [20] to test the DRBEM. The boundary condition is stated as

$$u = e^{-x}, \tag{28}$$

which also constitutes a particular solution of this problem. By adding u on dual sides of equation (27), we have

$$\nabla^2 u + u = u - \frac{\partial u}{\partial x}. \tag{29}$$

The results by both the DRBEM and BKM are listed in Table 3.

Table 3. Results for $\nabla^2 - -\frac{\partial U}{\partial x}$.

x	y	Exact	DRBEM (33)	BKM(15)	BKM (18)
1.5	0.0	0.223	0.229	0.229	0.224
1.2	−0.35	0.301	0.307	0.301	0.305
0.0	−0.45	1.000	1.003	1.010	1.000
−0.6	−0.45	1.822	1.819	1.822	1.818
−1.5	0.0	4.482	4.489	4.484	4.477
0.3	0.0	0.741	0.745	0.744	0.743
−0.3	0.0	1.350	1.348	1.353	1.354
0.0	0.0	1.000	1.002	1.003	1.004

The MQ shape parameter is chosen as 4. The BKM employed seven boundary knots and eight or 11 internal knots. In contrast, the DRBEM [20] used 16 boundary and 17 inner nodes. It is stressed that unlike the previous examples, in this case, the use of the interior points can improve the solution accuracy evidently. This is due to the fact that the governing equation has a convection domain-dominant solution. Only by using boundary nodes, the present BKM with the Helmholtz nonsingular solution and the DRBEM with the Laplacian [20] cannot well capture convection effects of the system equation. It is found from Table 3 that both the BKM and DRBEM achieve the salient accurate solutions with inner nodes. The BKM outperforms the DRBEM in computational efficiency due to the super-convergent features of the MQ interpolation and global BKM collocation.

Further, consider the equation

$$\nabla^2 u = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \tag{30}$$

with boundary conditions

$$u = e^{-x} + e^{-y}, \tag{31}$$

which is also a particular solution of equation (30). The numerical results are summarized in Table 4.

In the BKM, the MQ shape parameter is taken as 5.5. We employed seven boundary knots and eight or 11 inner points in the BKM compared with 16 boundary nodes and 17 inner points

Table 4. Results for $\nabla^2 u = -\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$.

x	y	Exact	DRBEM (33)	BKM(15)	BKM (18)
1.5	0.0	1.223	1.231	1.225	1.224
1.2	−0.35	1.720	1.714	1.725	1.723
0.0	−0.45	2.568	2.557	2.546	2.551
−0.6	−0.45	3.390	3.378	3.403	3.405
−1.5	0.0	5.482	5.485	5.490	5.491
0.3	0.0	1.741	1.731	1.729	1.731
−0.3	0.0	2.350	2.335	2.349	2.350
0.0	0.0	2.000	1.989	1.992	1.993

in the DRBEM [20]. The BKM worked equally well in this case as in the previous ones. It is seen from Table 4 that the BKM with fewer points produced almost the same accurate solutions as the DRBEM. Considering the extremely mathematical simplicity and easy-to-use advantages of the BKM, the method is superior to the DRBEM in this problem.

4.4. Varying-Parameter Helmholtz Problem

Consider the varying-parameter Helmholtz equation

$$\nabla^2 u - \frac{2}{x^2} u = 0, \tag{32}$$

with inhomogeneous boundary condition

$$u = -\frac{2}{x}. \tag{33}$$

Equation (33) is also a particular solution of equation (32). This problem is the simplified Berger convection-diffusion problem given in [20]. Note that the origin of the Cartesian coordinates system is dislocated to the node $(3,0)$ to circumvent singularity at $x = 0$. The response knot-dependent nonsingular general solution of varying parameter equation (32) is given by

$$u(r_{ik}, x_i) = I_0\left(\frac{\sqrt{2}}{|x_i|} r_{ik}\right), \tag{34}$$

where I_0 is the zero-order modified Bessel function of the first kind. i and k, respectively, index the response and source nodes. In terms of the BKM, the problem can be analogized by

$$\sum_{k=1}^{N} \alpha_k I_0\left(r_{ik}\frac{\sqrt{2}}{x_i}\right) = -\frac{2}{x_i}. \tag{35}$$

Note that only the boundary nodes are used in equation (35). After evaluating the coefficients α, we can easily evaluate the value of u at any inner node p by

$$u_p = \sum_{k=1}^{N} \alpha_k I_0\left(r_{pk}\frac{\sqrt{2}}{x_p}\right). \tag{36}$$

Table 5 lists the BKM results against the DRBEM solutions. The BKM average relative errors under $N = 9, 13, 15$ are, respectively, 9.7e−3, 8.1e−3, and 7.6e−3, which numerically demonstrates its convergence. The accuracies of the BKM and DRBEM solutions are comparable. It is noted that the DRBEM used 33 nodes (16 inner and 17 boundary knots) in this case, while

Table 5. Relative errors for varying parameter Helmholtz problem.

x	y	DRBEM(33)	BKM (9)	BKM (15)
4.5	0.0	2.3e−3	3.3e−3	2.6e−3
4.2	−0.35	2.1e−3	4.1e−3	3.3e−3
3.6	−0.45	5.4e−3	6.8e−3	4.7e−3
3.0	−0.45	4.5e−3	1.1e−2	4.4e−3
2.4	−0.45	1.2e−3	1.4e−2	9.1e−4
1.8	−0.35	9.0e−4	5.2e−3	1.7e−2
1.5	0.0	*	9.4e−3	3.4e−2
3.9	0.0	3.9e−3	7.0e−3	5.3e−3
3.3	0.0	3.3e−3	1.1e−2	6.3e−3
3.0	0.0	4.5e−3	1.3e−2	5.6e−3
2.7	0.0	2.7e−3	1.5e−2	3.4e−3
2.1	0.0	3.2e−3	1.6e−2	8.8e−3

the BKM only employed much fewer boundary knots. The accuracy and efficiency of this BKM scheme are very encouraging. Note that the present BKM representation differs from the previous ones in that here we use the response point-dependent nonsingular general functions. Similarly, we can easily constitute response node-dependent fundamental solutions. Thus, the essential idea behind this work may be extended to the BEM and DRBEM solution of varying parameter problems. For example, unlike the DRBEM scheme for varying velocity convection-diffusion problems given in [17], the variable convection-diffusion fundamental solutions with response node-dependent velocity parameters may be employed to the BEM or the DRBEM formulations, which may be especially attractive for high Peclet number problems.

5. COMPLETENESS CONCERN

One major potential concern of the BKM is its completeness due to the fact that the BKM employs only the nonsingular part of fundamental solutions of differential operators. This incompleteness may limit its utility. Although the given numerical experiments favor the method, now we cannot theoretically ascertain of the general applicability of the BKM. On the other hand, Kamiya and Andoh [11] validated that similar incompleteness occurs in the multiple reciprocity BEM using the Laplacian for Helmholtz operators. Namely, if the Laplace fundamental solution plus its higher-order terms are used in the MRM for the Helmholtz problems as in its usual form, we actually employ only the singular part of Helmholtz operator fundamental solution. Although the MRM performed well in many numerical experiments, it is mathematically incomplete. It is interesting to note that the BKM and MRM, respectively, employ the nonsingular and singular parts of the complete complex fundamental solution. It should also be stressed that although Power [13] simply indicated that the singular or nonsingular parts of Green representation can formulate the interior Helmholtz problems; no related numerical and theoretical results are available from the published reports.

Kamiya and Andoh [11] also pointed out that the MRM formulation with the Laplacian cannot satisfy the well-known Sommerfeld radiation conditions at infinity. Chen *et al.* [12] addressed the issues relating to spurious eigenvalues of the MRM with the Laplacian. Some literature referred to in [12] also discussed the issues applying MRM with the Laplacian to problems with degenerate boundary conditions. These issues of the MRM raise some concerns about the applicability of the BKM which implements the nonsingular part of fundamental solution compared to the MRM using the singular part. Power [13] discussed the incompleteness issue of the MRM for the Brinkman equation and indicated that using one part of a complex fundamental solution of Helmholtz operator may fail to the exterior Helmholtz problems. However, now we cannot justify whether the BKM works for exterior problems since the method differs from the MRM in using the DRM approximation of particular solutions. Dai [33] successfully applied the dual reciprocity BEM with the Laplacian to waves propagating problems in an infinite or semi-infinite region. It is worth pointing out that the Laplace fundamental solution used in the DRBEM also does not satisfy the Sommerfeld radiation condition. Unlike the MRM, the BKM and DRBEM do not employ the higher-order fundamental solutions to approximate the particular solution. Our next work will investigate if the BKM with the DRM can analyze the unbounded domain problems.

In fact, all existing numerical techniques encounter some limits. The BKM is not exceptional. Power [13] pointed out that the incompleteness in the MRM is problem-dependent. Therefore, the essential issue relating to the concerns of BKM completeness is under what conditions the method works reliably and efficiently.

6. CONCLUDING REMARKS

The present BKM can be regarded as one kind of the Trefftz method [35] where the trial function is required to satisfy the governing equation. The BKM distinguishes from the other Trefftz techniques such as the MFS in that we employ nonsingular general solution. The shortcomings of the MFS using a fictitious boundary are eliminated in the BKM. The term "BKM" can be interpreted as a boundary modeling technique combining the DRM, RBF, and nonsingular general solution. In conclusion, the presented BKM inherently possesses some desirable numerical merits which include meshless, boundary-only, integration-free, and mathematical simplicity. The implementation of the method is remarkably easy. The remaining two concerns of the BKM are the possible incompleteness in solving some types of problems due to the only use of nonsingular general solution and solvability of Kansa's method for finding the particular solutions. More numerical experiments to test the BKM will be beneficial. This paper can be regarded as a starting point of a series of works.

APPENDIX

By its very basis, it is straightforward to extend the BKM to the nonlinear, three-dimensional, time-dependent partial differential systems. The following lists the nonsingular general solutions of some important steady and transient differential operators.

For the 3D Helmholtz-like operators

$$\nabla^2 u \pm \lambda^2 u = 0, \tag{A1}$$

we, respectively, have the nonsingular general solution

$$u^* = A\,\frac{\sin(\lambda r)}{r} \tag{A2}$$

and

$$u^* = A\,\frac{\sinh(\lambda r)}{r}, \tag{A3}$$

where sinh denotes the hyperbolic function, and A is constant. For the 2D biharmonic operator

$$\nabla^4 w - \lambda^2 w = 0, \tag{A4}$$

we have the nonsingular general solution

$$w^* = A_1 J_0(\lambda r) + A_2 I_0(\lambda r). \tag{A5}$$

The nonsingular general solution of the 3D biharmonic operator is given by

$$w^* = A_1\,\frac{\sin(\lambda r)}{r} + A_2\,\frac{\sinh(\lambda r)}{r}. \tag{A6}$$

For the 3D time-dependent heat and diffusion equation

$$\Delta u = \frac{1}{k}\frac{\partial u}{\partial t}, \tag{A7}$$

we have the nonsingular general solution

$$u^*(r,t,t_k) = Ae^{-k(t-t_k)}\frac{\sin r}{r}. \tag{A8}$$

Furthermore, considering the 3D transient wave equation

$$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \tag{A9}$$

we get the general solution

$$u^*(r,t,t_k) = \left\lfloor A_1\cos(c(t-t_k)) + \frac{A_2}{c}\sin(c(t-t_k))\right\rfloor \frac{\sin(r)}{r}. \tag{A10}$$

The 2D nonsingular general solutions of transient problems can be easily derived in a similar fashion. Two standard techniques handling time derivatives are the time-stepping integrators and the model analysis. The former involves some difficult issues relating to the stability and accuracy, while the latter is not very applicable for many cases, such as shock. The BKM using time-dependent nonsingular solutions may circumvent these drawbacks. The difficulty implementing such BKM schemes may lie in how to satisfy the inharmonic initial conditions inside domain as in the time-dependent BEM. On the other hand, the time-dependent nonsingular general solutions may be directly applied in the domain-type RBF collocation schemes such as Kansa's method. It is worth pointing out that the analogous method proposed by Katsikadelis *et al.* [15] may be combined with the BKM to handle the differential systems which do not include Laplace or biharmonic operators.

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