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Coupling dual reciprocity BEM and differential quadrature method for time-dependent diffusion problems

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Abstract

This paper presents the very first combined application of dual reciprocity BEM (DRBEM) and differential quadrature (DQ) method to time-dependent diffusion problems. In this study, the DRBEM is employed to discretize the spatial partial derivatives. The DQ method is then applied to analogize temporal derivatives. The resulting algebraic formulation is the known Lyapunov matrix equation, which can be very efficiently solved by the Bartels–Stewart algorithms. The mixed scheme combines strong geometry flexibility and boundary-only feature of the BEM and high accuracy and efficiency of the DQ method. Its superiority is demonstrated through the solution of some benchmark diffusion problems. The DQ method is shown to be numerically accurate, stable and computationally efficient in computing dynamic problems. In particular, the present study reveals that the DRBEM is also very efficient for transient diffusion problems with Dirichlet boundary conditions by coupling the DQ method in time discretization.

Keywords: Dual reciprocity BEM; Differential quadrature method; Transient diffusion; Lyapunov matrix equation; Time integration

1. Introduction

In recent years, the BEM has become increasingly popular in the numerical solution of time-dependent partial differential equations occurring in many branches of science and engineering. Transformation of the domain integrals has been a central task in the BEM solution of such problems to preserve its boundary-only merits. There are several different approaches available now for this purpose. However, as was pointed out by Partridge et al. [1] and Tanaka et al. [2] the dual reciprocity BEM (DRBEM) stands out as the method of choice in engineering computations due to its ease of implementation, intrinsically meshless and strong flexibility of applying fundamental solutions. On the other hand, the choice of time integration scheme is an essential part to accurately assess the performance of the BEM solution of time-dependent problems. Recently, Singh and Kalra [3] provided a comprehensive comparative study of various different time integration algorithms (see Table 1) in context of the DRBEM differential–algebraic formulations

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Table 1
Abbreviations of various time schemes

Algorithm	Abbreviation
Differential quadrature	DQ
Least squares family	
One step least squares	LS11
Two step least squares	LS21
Cubic Hermitian family	
The midstep rule	CHMS
Fully implicit algorithm	CHFI
SSp1	
One step methods	SS11
Crank–Nicholson	SS11CN
Galerkin	SS11GN
Backward difference	SS11BD
Two step methods	
Backward difference	SS21BD
Three step methods	
Backward difference	SS31BD
Optimum	SS31OP

of transient diffusion problems. A one step least squares algorithm was concluded the most accurate and efficient technique among all methods assessed [3]. In particular, when Dirichlet boundary condition is involved, the one step backward difference method is recommended as the method of choice for short term response [3]. However, as was found therein, all of those algorithms encountered great drop of accuracy and efficiency in the solution of problems with Dirichlet boundary conditions. Therefore, an alternative algorithm is in demand, which triggers the present work.

The objective of this paper is to apply the differential quadrature (DQ) method to approximate time derivative in the DRBEM formulation of transient diffusion problems. Due to its recent origin, the DQ method may not be well-known in computational community. The method can be regarded as the "direct approach" of the traditional collocation (pseudo-spectral) methods in that the governing equations are analogized in terms of practical physical variables instead of usually fictitious expansion (spectral) coefficients. The salient advantages of the DQ method over the normal collocation method are its ease in implementation and more flexibility in choosing the sampling points. In the literature, the DQ method has been usually applied to approximate spatial derivatives and shows high efficiency and accuracy through the solution of a broad range of problems with regular boundary shape. Recent studies have also launched the geometry flexibility of the DQ applications by means of the coordinate mappings and element techniques. Although some preliminary successes were achieved, the flexibility of complex geometry problems is still a major deterrence in the broad application of the method to the practical engineering problems [4,5].

The strength of the DQ method lies in its fast rate of convergence and high accuracy, namely, so-called spectral accuracy. It is noted that the time variable has the simplicity of geometry. Therefore, the DQ analogue of temporal derivative is expected to perform well, while it is well known that the BEM enjoys the strong geometry flexibility and boundary-only feature. So a combined use of both DRBEM and DQ method will be very attractive. In this study, we investigate transient diffusion problems by this mixed methodology. It is found that the resultant algebraic system is a Lyapunov matrix equation. By using the Bartels–Stewart algorithm, [6] the computing effort of solving such matrix equations is greatly reduced. The detailed solution procedure including analysis of computing stability and efficiency is next explained, and some conclusions are finally drawn based on the present study.

2. DRBEM discretization of spatial variables of diffusion problems

The equation governing transient diffusion problems can be expressed as

$$\frac{\partial u(x,t)}{\partial t} - \nabla^2 u(x,t) = 0, \quad x \subset \Omega \tag{1}$$

with the initial conditions

$$u(x,0) = u_0(x), \tag{2}$$

and the Dirichlet, Neumann and Linear Radiation boundary conditions are given by

$$u(x,t) = \overline{u}(x,t), \quad x \subseteq \Gamma_u, \tag{3}$$

$$q(x,t) = \overline{q}(x,t), \quad x \subseteq \Gamma_q, \tag{4}$$

$$q(x,t) = -h(x,t)\{u(x,t) - u_r(x,t)\}, \quad x \subseteq \Gamma_r, \tag{5}$$

where variable domain $\Omega \subset R^2$ is bounded by a piece-wise smooth boundary $\Gamma = \Gamma_u + \Gamma_q + \Gamma_r$, $q = \partial u/\partial n$, n is the unit outward normal.

Eq. (1) can be weighted by the fundamental solution u^* of Laplace operator

$$\int_\Omega \left(\frac{\partial u}{\partial t} - \nabla^2 u\right) u^* \,\mathrm{d}\Omega = 0. \tag{6}$$

Applying Green's second identity to Eq. (6) yields

$$c_i u_i + \int_\Gamma (q^* u - u^* q)\,\mathrm{d}\Gamma = -\int_\Omega \frac{\partial u}{\partial t} u^* \,\mathrm{d}\Omega, \tag{7}$$

where subscript i denotes the source point, $q^* = \partial u^*/\partial n$, and $c_i = \int_\Omega \delta(\zeta, x)\,\mathrm{d}\Omega$. The dual reciprocity method transforms the domain integral in Eq. (7) by means of a set of coordinate functions $f^j(x)$

$$\dot{u}(x,t) \approx \sum_{j=1}^{N+L} f^j(x)\dot{\alpha}(t), \tag{8}$$

where the upper dot represents the time derivative, the α^j are unknown functions of time, N and L are the numbers of the boundary and selected internal nodes, respectively. The coordinate function f^j given by Wrobel and Brebbia [7] are here exploited as in Singh and Kalra [3] to facilitate later numerical comparisons. These functions are also linked with $\psi^j(x)$ through

$$\nabla^2 \psi^j = f^j. \tag{9}$$

Therefore, we have

$$\int_\Omega \frac{\partial u}{\partial t} u^* \,\mathrm{d}\Omega = \sum_{j=1}^{N+L} \int_\Omega u^* \nabla^2 \psi^j \,\mathrm{d}\Omega. \tag{10}$$

Eq. (7) can finally be reduced to

$$c_i u_i + \int_\Gamma (q^* u - u^* q)\,\mathrm{d}\Gamma = \sum_{j=1}^{N+L} \left[c_i \psi_i^j + \int_\Gamma \left(q^* \psi^j - u^* \eta^j\right) \mathrm{d}\Gamma \right] \dot{\alpha}, \tag{11}$$

where $\eta^j = \partial \psi^j/\partial n$. Note that ψ^j and f^j are known functions. The resulting DRBEM formulation is

$$C\dot{u} + Hu - Gq = 0, \tag{12}$$

where $C = (GE - H\Psi)F^{-1}$; H and G denote the whole matrices of boundary element with kernels q^* and u^*, respectively; F, Ψ and E comprise the coordinate function column vectors f^j, ψ^j and η^j. The discretization procedure in detail can be found in [1]. Eq. (12) is a differential algebraic system for problems with Dirichlet boundary conditions. Singh and Kalra [3] presented an approach to partition Eq. (12) in differential and algebraic parts in such a way that the standard form of the first-order initial problem is obtained, namely

$$\dot{u} + Bu = f, \tag{13}$$

where B and f are known coefficient matrix and vector. The remaining algebraic components can be easily calculated after the solutions of the above differential system are accomplished. For details of this methodology, see [3]. In this study, we applied this partitioned approach to the diffusion problems with Dirichlet conditions.

3. DQ approximation of time derivative

The DQ analog of the first-order derivative of function $g(t)$ can be expressed as

$$\left.\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right|_{t_i} = \sum_{j=1}^{N} A_{ij} g(t_j), \quad i = 1, 2, \ldots, N, \tag{14}$$

where t_js are the discrete points in the temporal variable domain. $g(t_j)$ and A_{ij} are the function values at these points and the related DQ weighting coefficients, respectively. It is worth pointing out that the calculation of these weighting coefficients need be done only once for specified grid points by means of an accurate and efficient formula. For more details, see [4]. Some new developments of this method were included in [5].

Eq. (13) can be recognized as the standard form of the first-order initial value problems. By using the transformation

$$\overline{u} = u - u(0), \tag{15}$$

Eq. (13) can be restated as

$$\dot{\overline{u}} + B\overline{u} = \bar{f}, \tag{16}$$

where $\bar{f} = f + Bu(0)$. The initial conditions are reduced to

$$\overline{u}(0) = 0. \tag{17}$$

In terms of approximate formula (14), Eq. (16) can be analogized as

$$\overline{U}\overline{A}^{\mathrm{T}} + B\overline{U} = C, \tag{18}$$

where $\overline{A}^{\mathrm{T}}$ is a transpose of $(N-1)\times(N-1)$ matrix $\overline{A}$ obtained by removing the first row and column of the DQ weighting coefficient matrix A in Eq. (14). The DQ discretization at multiple temporal gird points simultaneously causes the matrix algebraic equation (18). Similar situations are often encountered in the optimal control modeling. It is worth stressing that the initial conditions specified in Eq. (17) have been built into the modified coefficient matrix $\overline{A}$. The $\overline{U}$ and C here denote rectangular matrices, namely,

$$\overline{U}_{M\times(N-1)} = \begin{bmatrix} \overline{u}_{12} & \overline{u}_{13} & \cdots & \overline{u}_{1N} \\ \overline{u}_{22} & \overline{u}_{23} & \cdots & \overline{u}_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{u}_{M2} & \overline{u}_{M3} & \cdots & \overline{u}_{MN} \end{bmatrix} \quad \text{and} \quad C_{M\times(N-1)} = \begin{bmatrix} \bar{f}_1 & \bar{f}_1 & \cdots & \bar{f}_1 \\ \bar{f}_2 & \bar{f}_2 & \cdots & \bar{f}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{f}_M & \bar{f}_M & \cdots & \bar{f}_M \end{bmatrix}, \tag{19}$$

where M denotes the order of differential system. It is observed that Eq. (18) is in fact a Lyapunov matrix equation often encountered in control engineering. It need also be pointed out that the above procedure discretizes the time variable in the element way, which is somehow different from the standard step by step integration algorithms. In other words, the present methodology is to advance element by element and the multiple grid points are employed in each time element in such a way that high accuracy of solution is achieved.

3.1. Solver of Lyapunov matrix equations

Several very efficient methods of solving the Lyapunov algebraic matrix equation (18) have been well developed in literature. Their performances are stable and accurate. These solution procedures generally include the following four steps:

Step 1: Reduce B and $\overline{A}^{\mathrm{T}}$ of Eq. (18) into certain simple form via the similarity transformations $G = -P^{-1}BP$ and $H = V^{-1}\overline{A}^{\mathrm{T}}V$.

Step 2: $Q = P^{-1}CV$ for the solution of Q.

Step 3: Solve the transformed equation $GY + YH = Q$ for Y.

Step 4: $\overline{U} = PYV^{-1}$.

In the present study, the so-called Bartels–Stewart algorithm [6] is utilized. The major time-consuming calculation occurs in Step 1, where $\mathrm{O(M^3)}$ scalar multiplications are required. While all implicit step methods also demand $\mathrm{O(M^3)}$ operations. Therefore, computing effort of the present scheme is the same magnitude as in the common implicit methods. Moreover, it should be pointed out that operation in Step 1 need be done only once irrespective of the adjustable time stepping employed. In contrast, if the time step size is changed in certain steps, the common implicit methods do need anew LU decomposition of $\mathrm{O(M^3)}$ multiplications.

On the other hand, Steps 2–4 of the present method need be performed repeatedly in each time element. The computing effort of these steps is $\mathrm{O(M^2)}$ multiplications. Under the same time step size, the ratio of multiplication operations between the DQ and normal step integration methods is about $1 + (N-1)/M$, where N is the number of grid points in the DQ method. N is taken 3 and 5 in this study, while differential system order M is comparatively by far bigger. Therefore, one can conclude that the computing effort in the DQ method is nearly the same as that in the step methods.

3.2. Error estimation and accuracy

In what follows, we will address another important issue of the error estimates in the DQ method. The error estimator of the DQ approximation of the first-order derivative of function $f(x)$ is given by [5]

$$|R_i| \leqslant K \cdot \mathrm{err}_i \cdot \Delta t^{N-1}, \quad i = 1, 2, \ldots, N, \tag{20}$$

where $K = \max|f^{(N)}(x)|$; err_i denotes the error constants at various grids dependent on grid spacing and can be obtained easily. Δt represents the time step size, and N is the number of grid points in the DQ time element.

According to formula (20), the accuracy of the DQ method is $O(\Delta t^{N-1})$. So the accuracy of solutions will be two order under $N = 3$ and four order under $N = 5$. The method can earn higher order of accuracy than the standard integration techniques. Therefore, in some cases, the DQ method can consume comparatively less computing effort by using larger time step while still producing accurate solutions.

3.3. A-Stability

It is centrally important whether or not an algorithm is stable in the solution of temporal ordinary differential equations. It is known that the collocation method is A-stable, [8] in the terminology of structural dynamics, unconditionally stable. Therefore, the DQ method is also unconditionally stable due to the actual equivalence with the collocation method. It is worth pointing out that the present DQ method is superior to the traditional collocation methods due to the twofold reasons. First, the practical physical values are directly computed in the DQ method instead of the indirect expansion (spectral) variables in the collocation methods. This greatly simplifies the engineering implementations and manifests the DQ method in easy-to-choose starting solutions of nonlinear iterations, while, in contrast, the fictitious expansion variables in the collocation methods usually have no physical meanings and are therefore difficult to do so. Also, the DQ method provides more flexibility to choose grid points [9]. Second, the fast solver of the Lyapunov equation drastically reduces the formulation and computing effort as well as storage requirements in the solution of initial value problems.

Dahlquist [10] presented the famous Barrier theorem which restricts the maximum order of all linear multistep methods up to two in order to preserve the A-stability. The DQ method is not a traditional multistep algorithm and therefore circumvents this rigorous limitation of solution accuracy while still attaining the desirable A-stability merits.

4. Applications and discussions

In the present study, the Chebyshev–Gauss–Lobatto collocation points are used in each time element of the DQ method, namely,

$$t_i = \frac{\Gamma}{2}\left[1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right], \quad i = 1, 2, \ldots, N, \tag{21}$$

where Γ denotes the length of DQ time element. The L_2 relative error norm is a standard approach to assess the accuracy of the solutions and defined as

$$\eta\% = \frac{\|e\|_2}{\|u_{\text{exact}}\|_2} \times 100, \tag{22}$$

where $\| \ \|_2$ represents the L_2 norm operator, e and u_{exact} are the absolute error and the analytical solution at boundary point, respectively. In this section, the mixed technique of the DRBEM and DQ method was applied to four typical diffusion problems provided by Singh and Kalra [3], where the DRBEM was exploited in coupling of several time integration techniques. Table 1 lists these time integration schemes along with the present DQ method. The spatial variable domains of the test problems are square and the linear element $(\Delta\Gamma = 0.1)$ was employed in the DRBEM. The initial conditions of all tested numerical examples are taken as

$$u(x, 0) = 1. \tag{23}$$

The performances are measured via error estimate formula (22). The DQ solutions are compared with those given in [3] by using the other integration schemes listed in Table 1.

4.1. Dirichlet problem

All boundary conditions are specified as

$$u(x,t) = 0, \quad x_1, x_2 = -1, 1. \tag{24}$$

The analytical solution is given by

$$u(x,t) = v(x_1,t)v(x_2,t), \tag{25}$$

where

$$v(z,t) = \frac{4}{\pi}\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}\cos\left(\frac{2i+1}{2}\pi z\right)\exp\left[-(2i+1)^2\frac{\pi^2 t}{4}\right]. \tag{26}$$

Some internal points are necessary in this case due to the fact that all potential values of u at four edges are known. We employed 33 interior points in the DRBEM, the same number as in [3] for this case. The L_2 relative errors of the solutions ($t = 1.0$ s) are displayed in Table 2. It is seen that the DQ method produces the strikingly accurate solutions by comparing with all other step schemes. The one step least square method was found the most accurate in all other methods except for the present DQ method, while, in contrast, the DQ method can achieve much more accurate solutions by using an evidently bigger time step. So the computing efficiency and accuracy of the DQ method is superior to all others in this case.

4.2. Dirichlet–Neumann problem

This case is a reduced form of the previous Dirichlet problem and thus holds the same analytical solution formulas (25) and (26). The Dirichlet conditions are given by

$$u(x,t) = 0, \quad x_1, x_2 = 1. \tag{27}$$

The Neumann conditions are specified as

$$q(x,t) = 0, \quad x_1, x_2 = 0. \tag{28}$$

One internal point is placed in the center of domain. Table 3 shows the L_2 errors of all methods included in Table 1. It is observed that the DQ method performs more excellently against all other

Table 2
Dirichlet problem – L_2 relative error norm η (%) of boundary fluxes at $t = 1.0$ s

Algorithms	$\Delta t = 1/8$	$\Delta t = 1/16$	$\Delta t = 1/32$	$\Delta t = 1/64$
DQ ($N = 3$)	7.70	0.61	3.07	4.16
LS11	68.71	29.62	8.00	4.52
SS11CN	52.79	9.31	10.96	11.35
SS11GN	45.13	30.67	21.76	16.93
SS11BD	145.42	78.81	45.26	28.78
CHMS	14.21	12.28	11.75	11.56
CHFI	12.96	12.13	11.73	11.56
LS21	165.05	26.82	5.65	6.41
SS21BD	53.97	6.04	14.98	16.32
SS31BD	51.66	13.76	11.30	11.02
SS31OP	40.09	14.73	13.41	12.95

Table 3
Dirichlet–Neumann problem – L_2 relative error norm η (%) of boundary fluxes at $t = 1.0$ s

Algorithms	$\Delta t = 1/16$	$\Delta t = 1/32$	$\Delta t = 1/64$	$\Delta t = 1/128$
DQ ($N = 3$)	0.65	2.66	3.74	4.15
LS11	51.54	11.80	1.37	4.77
SS11CN	19.70	10.51	5.06	5.14
SS11GN	23.97	15.24	10.44	7.94
SS11BD	70.91	37.88	21.64	13.64
CHMS	5.78	5.36	5.23	5.19
CHFI	5.61	5.34	5.23	5.19
LS21	28.52	18.20	12.03	8.94
SS21BD	8.80	5.41	7.16	7.26
SS31BD	11.14	5.95	5.30	5.12
SS31OP	7.04	6.50	6.02	5.82

methods. It is also interesting to note that the DQ method reaches the least L_2 errors with time step $\Delta t = 1/16$. After this, the less time step decreases the accuracy of the solution. The results in Table 2 have similar behavior. Also, it was found that the other methods also present such phenomena. It appears that the accumulation of step errors dominates the solution accuracy in such situations. The less time step size increases the number of solution steps.

In order to illustrate the short and long term behaviors of various methods, Tables 4 and 5 respectively illustrate propagation of the L_2 error of the back difference, one step least square and DQ methods with regard to the previous Dirichlet problem and present Dirichlet–Neumann problem. Table 5 only involves the DQ and one step square methods. The accuracies of the other methods in both cases are too low for long term response and are therefore not included there. The data at $t = 2.0$ s is at most available from Singh and Kalra [3] to compare in these cases. For short term response, Singh and Kalra [3] suggested that the backward difference method is the most preferred among all methods assessed by them when the Dirichlet boundary conditions are involved. Table 4 shows that in such cases the DQ method is more efficient especially for pure

Table 4
Comparison of L_2 relative error norm η (%) of boundary fluxes in Dirichlet problem and boundary potentials in Dirichlet–Neumann problem (short term response)

t	Dirichlet			Dirichlet–Neumann		
	DQ[a]	LS11[b]	SS11BD[b]	DQ[a]	LS11[b]	SS11BD[b]
0.0625	15.45	36	22	4.13	1.8	2.4
0.125	0.06	17	12	3.7	2.8	0.2
0.25	0.89	4.8	3.6	2.8	3.0	2.2
0.375	0.69	4.9	8.4	2.4	3.3	4.2

[a] $\Delta t = 1/16$.
[b] $\Delta t = 1/64$.

Table 5
Comparison of L_2 relative error norm η (%) of boundary fluxes in Dirichlet problem and boundary potentials in Dirichlet–Neumann problem (long term response)

t	Dirichlet		Dirichlet–Neumann	
	DQ ($\Delta t = 1/16$)	LS11 ($\Delta t = 1/64$)	DQ ($\Delta t = 1/64$)	LS11 ($\Delta t = 1/64$)
0.5	0.72	3.55	1.81	2.9
1.0	0.67	4.52	0.65	4.8
2.0	0.50	11.13	5.74	7.2

Dirichlet problems. On the other hand, one can find from Table 5 that the DQ method using much bigger time step can achieve remarkably higher accuracy than one step least square method for long term response. This is well in agreement with our observations in Tables 2 and 3. Singh and Kalra [3] concluded that one step least square method for long term and the backward difference method for short term are the optimal time integration method of choice for the problems with Dirichlet boundary conditions through a comprehensive comparison of solution accuracy. According to the present results, one can easily see that among all these methods, the DQ method has the best accuracy and fastest convergence rate for the cases involving Dirichlet boundary.

4.3. Linear radiation problem

In this case, the spatial domain is all bounded by the linear radiation boundary conditions

$$q(x,t) = -hu(x,t), \quad x_1, x_2 = -1, 1, \tag{29}$$

where h is the radiation constant and is set 2. The analytical solution is given by

$$u(x,t) = w(x_1,t)w(x_2,t), \tag{30}$$

where

$$w(z,t) = 2h\sum_{i=1}^{\infty} \frac{\cos(\beta_i z)\sec(\beta_i)}{h(h+1)+\beta_i^2}\exp(-\beta_i^2 z). \tag{31}$$

Table 6 summarizes the L_2 errors of this case. All schemes of high order accuracy yield accurate solutions. In comparison to the other methods, the DQ method performs well but not exceptionally excellently as in the foregoing cases involving the Dirichlet boundary.

4.4. Neumann-linear radiation problem

By utilizing the symmetry of boundary and geometry, the previous linear radiation problem can be simplified to a problem with Neumann and linear radiation boundary conditions:

$$q(x,t) = 0, \quad x_1, x_2 = 0, \tag{32}$$

$$q(x,t) = -hu(x,t), \quad x_1, x_2 = 1, 1, \tag{33}$$

where h is taken 2. The analytical solution is the same as in the previous linear radiation problem. The L_2 errors are shown in Table 7. The performances of various high order methods are roughly

Table 6
Linear radiation problem – L_2 relative error norm η (%) of boundary fluxes at $t = 1.0$ s

Algorithms	$\Delta t = 1/4$	$\Delta t = 1/8$	$\Delta t = 1/16$	$\Delta t = 1/32$
DQ ($N = 5$)	5.77	2.16	2.13	2.14
LS11	2.95	2.41	2.12	2.09
SS11CN	5.61	2.42	2.07	2.14
SS11GN	10.91	7.18	4.88	3.58
SS11BD	26.82	16.87	10.23	6.39
CHMS	2.21	2.21	2.15	2.15
CHFI	1.81	2.11	2.14	2.15
LS21	16.63	2.36	0.36	1.07
SS21BD	12.34	2.02	1.08	1.89
SS31BD	34.08	5.67	2.92	2.34
SS31OP	1.92	2.69	2.05	2.02

Table 7
Neumann-linear radiation problem – L_2 relative error norm η (%) of boundary fluxes at $t = 1.0$ s

Algorithms	$\Delta t = 1/4$	$\Delta t = 1/8$	$\Delta t = 1/16$	$\Delta t = 1/32$
DQ ($N = 5$)	3.29	0.94	0.90	0.90
LS11	1.77	0.64	0.60	0.69
SS11CN	5.30	1.07	0.71	0.77
SS11GN	10.25	6.15	3.70	2.32
SS11BD	27.06	16.54	9.44	5.33
CHMS	1.93	1.25	0.95	0.85
CHFI	1.09	0.98	0.89	0.83
LS21	9.44	3.80	1.10	0.23
SS21BD	13.10	3.96	0.43	0.51
SS31BD	17.38	4.61	1.61	0.99
SS31OP	0.98	1.30	0.70	0.66

Table 8
Comparison of L_2 relative error norm η (%) of boundary potential in radiation and Neumann-radiation problems ($\Delta t = 1/32$)

t	Radiation		Neumann-radiation	
	DQ	LS11	DQ	LS11
0.0625	2.0	2.7	1.5	3.4
0.125	4.1	4.0	0.83	1.2
0.25	4.3	4.2	0.45	0.75
0.375	2.9	2.8	0.24	0.41
0.5	1.7	1.6	0.3	0.4
1.0	2.1	2.09	0.9	0.7
2.0	10.1	9.1	2.2	1.4
4.0	27.9	26.3	4.9	2.9

similar to those in the linear radiation problems. For both linear radiation problem and Neumann-linear radiation problem, Table 8 further lists the L_2 errors of the short and long term in applying the one step least square and DQ methods. It is found that the one step least square method yields slightly better solutions in long term response, while the DQ method performs a little more accurately in short term response.

Based on the above-given four numerical examples, the DQ method is found to perform continuously well. In the problems involving Dirichlet boundary, all other algorithms behave much below the DQ method. In terms of constantly good accuracy and efficiency, the DQ method is considered the method of choice for the tested diffusion problems.

5. Concluding remarks

In this paper, the DRBEM and DQ method were coupled to solve four time-dependent diffusion problems. The high accuracy, efficiency and stability of the mixed method are demonstrated. The proposed methodology retains the boundary-only feature and strong geometry flexibility of the DRBEM and high accuracy and efficiency of the DQ method. Based on the foregoing discussions of numerical results, the DQ method is found to perform continuously well. For the diffusion problems with Dirichlet boundary, the DQ method outperforms remarkably all other time algorithms investigated in [3]. For the problems without Dirichlet boundary, the method performs well. In terms of constantly good accuracy and efficiency, the DQ method is considered the best one among all methods listed in Table 1 for solving the DRBEM formulation of diffusion problems. The method also attains the following attractive merits in computing dynamic problems:

1. A-stable, in other words, unconditional stability.
2. The computing effort is nearly the same as that of common implicit time step methods. $O(M^3)$ operations need be done only once irrespective of adjustable time step is used.
3. High order accuracy $O(\Delta t^{N-1})$, where N denotes the number of collocation points in time element.
4. Self-starting.

More numerical experiments may be beneficial. At least, the present study shows that the DQ method is the most preferred alternative to solve the differential–algebraic systems resulting from the DRBEM discretization of the time-dependent diffusion problems. It was seen from Singh and Kalra [3] that the DRBEM seems not to perform well for the problems involving Dirichlet boundary conditions. The present work reveals that such inefficiencies to the Dirichlet problems are due to the integration schemes used therein rather than the DRBEM itself. It should be mentioned that Garica and Power [11] got the similar DRBEM solutions by using three-level integration scheme together with active polynomial Richardson extrapolation. The DRBEM are here demonstrated as very attractive for use in time-dependent diffusion problems with Dirichlet boundary conditions. The extension of the present coupled DRBEM and DQ method to elastodynamic problems, which usually involve the second-order time derivative, is the subject of current research.

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