Journal of Sound and Vibration (1997) **206**(5), 745–748

ON THE DQ ANALYSIS OF GEOMETRICALLY NON-LINEAR VIBRATION OF IMMOVABLY SIMPLY-SUPPORTED BEAMS

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(*Received* 15 *October* 1996, *and in final form* 28 *May* 1997)

1. INTRODUCTION

Feng and Bert [1] applied the differential quadrature (DQ) method to analyze geometrically non-linear vibrations of beams. The results for the clamped–clamped (C–C) case agreed very well with existing FEM solutions by Mei [2]. However, the results for the simply-supported simply-supported (SS–SS) case showed a somewhat large discrepancy with FEM [2] and analytical solutions [3]. This is because the conventional approach of applying boundary conditions in the DQ method was not very successful for SS–SS boundary conditions. Recently, a new approach was presented by Wang and Bert [4] and proven to be very efficient for the DQ analysis of linear structural components with SS–SS boundary conditions. The purpose of the present communication is to improve the accuracy and efficiency of the DQ analysis of the geometrically non-linear vibration of SS–SS beams by using this new approach. A straightforward and intuitive procedure is also given to obtain the analytical solution in this case. References [5, 6] showed that Chebyshev grid points in the DQ method are superior to even spacing. In this case the DQ method using Chebyshev grid points also had a faster rate of convergence than the equally spaced points. By applying the Hadamard product of matrices, the non-linear formulation is greatly simplified.

2. HADAMARD PRODUCT AND NON-LINEAR FORMULATION

For details on the DQ method, see references [1, 4]. The governing equation for the geometrically non-linear vibration of beams can be normalized as [1]

$$\frac{d^4v}{d\xi^4}-\frac{3}{8}\frac{a^2}{r^2}\left(\int_0^1\left(\frac{dv}{d\xi}\right)^2 d\xi\right)\frac{d^2v}{d\xi^2}-\bar{\omega}^2v=0, \tag{1}$$

where $r^2 = I/A$, I is the centroidal moment of inertia of the beam, A is the area of the beam cross-section, a is the amplitude, v is a non-linear normal mode; $\xi = x/L$, L is the length of the beam, x is the axial position co-ordinate; $\bar{\omega}^2 = \omega^2 mL^4/EI$ is the dimensionless frequency, ω is the non-linear frequency, m is the mass per unit length, and E is the modulus of elasticity. For more details, see reference [1]. In what follows the notation of the Hadamard product of matrices is introduced. The Hadamard product has been proven

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0022–460X/97/400745 + 04 $25.00/0/sv971136

to be a very powerful concept in non-linear computations by the DQ method [7]. In this study, the Hadamard product is applied to simplify the formulation.

Definition 1. If matrices $\mathbf{Q} = \{\mathbf{q}_{ij}\} \in \mathbf{C}^{n \times m}$ and $\mathbf{P} = \{\mathbf{p}_{ij}\} \in \mathbf{C}^{n \times m}$, the Hadamard product of these two matrices is defined as $\mathbf{Q} \circ P = \{\mathbf{q}_{ij}\, \mathbf{p}_{ij}\} \in \mathbf{C}^{n \times m}$, where $\circ$ represents the Hadamard product of matrices and $\mathbf{C}^{n \times m}$ denotes the set of $n \times m$ dimension matrices.

The essence of the new approach applying boundary conditions, proposed by Wang and Bert [4], is that the DQ weighting coefficient matrices for the inner grid points are modified by boundary conditions in advance. The detailed description of this new approach can be found in reference [4].

The non-linear formulation for equation (1) can be expressed in Hadamard product form as

$$\bar{\mathbf{D}}\vec{\mathbf{V}} - \frac{3}{8}\frac{a^2}{r^2}\{\vec{\mathbf{G}}[(A\tilde{V}) \circ (\mathbf{A}\tilde{\mathbf{V}})]\}\bar{\mathbf{B}}\vec{\mathbf{V}} - \bar{\omega}^2\vec{\mathbf{V}} = 0, \tag{2}$$

where $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$ are the DQ weighting coefficient matrices, modified by the respective boundary conditions using Wang and Bert's new approach, for the second and fourth order derivatives, respectively. The order of matrices $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$ is $n - 2$ in which n is the number of grid points. The boundary conditions have been already used in the formulation of $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$, and are no longer considered. $\vec{\mathbf{V}}$ is the (n-2) $\times$ 1 mode vector at inner grid points: $\tilde{\mathbf{V}} = \{0, \vec{\mathbf{V}}^{\mathrm{T}}, 0\}^{\mathrm{T}}$. $\mathbf{A}$ is the original $n \times n$ DQ weighting coefficient matrix for the first order derivative. Since the upper and lower bounds of the integral in equation (2) are constants, it is not necessary to utilize the DQ method for numerical integration as in reference [1]. One herein uses the Newton–Cotes numerical integration approach for simplicity. $\vec{\mathbf{G}}$ is a $1 \times n$ vector composed of the Cotes coefficients for numerical integration. It is noted that the DQ formulation equation (2) has an explicit, compact and simple matrix form, and is obviously easier to program than the conventional one given by Feng and Bert [1].

It is known that, in the new approach, there exists

$$\bar{\mathbf{D}} = \bar{\mathbf{B}}^2 \tag{3}$$

for SS–SS boundary conditions [4]. Therefore, $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$ are orthogonally similar and both have the same eigenvectors. Thus, the beam oscillates at the same mode as the one in the linear case, and the iterative procedure for the solution of equation (2) used in reference [1] are not necessary for the SS–SS case. In this study, one first solves for the eigenvalues and eigenvectors of $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$, and then obtains the non-linear coefficient of equation (2) by using these eigenvectors. The resulting dimensionless non-linear frequency can be obtained by

$$\bar{\omega} = \sqrt{\lambda_{\bar{\mathbf{D}}} - \eta\lambda_{\bar{\mathbf{B}}}} \tag{4}$$

where $\lambda_{\bar{\mathbf{B}}}$ and $\lambda_{\bar{\mathbf{D}}}$ are the eigenvalues of $\bar{\mathbf{B}}$ and $\bar{\mathbf{D}}$, and η is the non-linear coefficient. If the minimum values of $\lambda_{\bar{\mathbf{B}}}$ and $\lambda_{\bar{\mathbf{D}}}$ are chosen, the fundamental dimensionless non-linear frequency is obtained. Also, it is noted that there is a typographical error in the non-linear formulation equation (16) in reference [1]: namely, the first operation in that equation should be minus rather than plus.

The mth order normal mode of a linear SS–SS beam is

$$v(\xi) = \sin(m\pi\xi). \tag{5}$$

Based on the assumption that the non-linear SS–SS beam has the same vibrational mode as the linear SS–SS beam, one obtains the non-linear frequency of the mth order mode

TABLE 1

The ratios (ω/ω_1) of the non-linear frequencies to the linear frequencies for a SS–SS beam

a/r	Elliptical integral [8]	Analytical	Present DQ	DQ [1]	FEM [2]
0·1	1·0009	1·0009	1·0009	1·0010	1·0009
0·2	1·0037	1·0037	1·0037	1·0043	1·0037
0·4	1·0149	1·0149	1·0149	1·0170	1·0148
0·6	1·0031	1·0332	1·0332	1·0384	**1·0329**
0·8	1·0580	1·0583	1·0582	1·0673	1·0578
1·0	1·0892	1·0897	1·0896	1·1030	1·0889
1·5	1·1902	1·1924	1·1922	1·2045	1·1902
2·0	1·3178	1·3229	1·3225	1·3170	**1·3183**
3·0	1·6257	1·6394	1·6389	–	1·6260
4·0	1·9760	2·0000	1·9991	–	1·9715
5·0	2·3501	2·3848	2·3836	–	2·3341

Results in bold are incorrectly typed in reference [1].

for a geometrically non-linear SS–SS beam by directly substituting equation (5) into equation (1), namely,

$$\bar{\omega} = (m\pi)^2\sqrt{1 + \tfrac{3}{16}\, a^2/r^2}. \tag{6}$$

The above solution is coincident with that obtained by using the perturbation method [3] and is regarded as the analytical solution of the mth order mode. Thus,

$$\bar{\omega}/\omega_1 = \sqrt{1 + \tfrac{3}{16}\, a^2/r^2}, \tag{7}$$

where $\omega_1 = (m\pi)^2$ is the linear frequency.

3. RESULTS AND DISCUSSIONS

Seven equally spaced grid points are used in the present DQ computation. Table 1 shows the remarkable agreement between the analytical, finite element, and present DQ solutions. Amplitude–frequency curves are plotted in Figure 1. Obviously, the new approach of

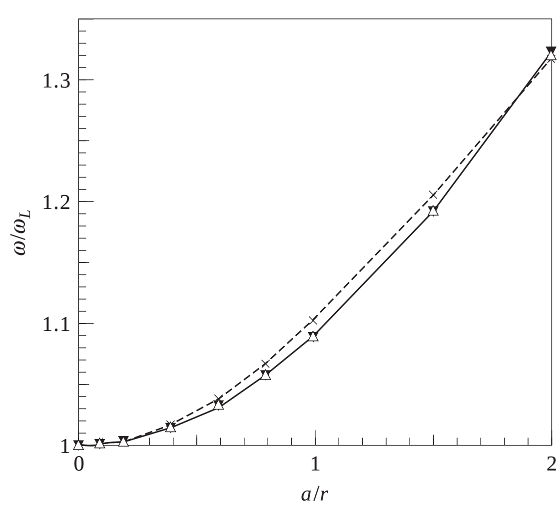

Figure 1. Dimensionless amplitude-frequency curves of a geometrically non-linear SS–SS beam. Key: —, analytical; ▼, present DQ (equally spaced grid points); △, FEM; --x--, DQ (Feng and Bert [1]).

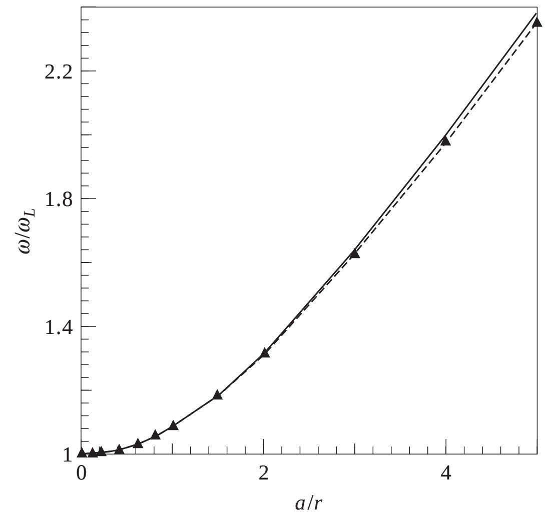

Figure 2. Comparison of ASM and ATM solutions for a geometrically non-linear SS–SS beam. Key: —, ATM; --▲--, ASM.

applying the DQ method by Wang and Bert [4] gives more accurate results than the conventional one by Feng and Bert [1]. As is expected, the DQ solutions using the zeros of the Chebyshev polynomial of seventh order are more accurate than using equally spaced grid points and in this case, are coincident with the analytical results. Therefore, the DQ results using the Chebyshev points are not presented here for the sake of brevity. Compared with FEM, the DQ method yields far more exact results, is easier to use, and requires much less computational effort and storage.

The elliptic integral solutions for the assumed space model (ASM) in this case by Woinowsky-Krieger [8] are also listed in Table 1 and compared with the analytical solutions of the governing equation (1) in Figure 2. It is noted that both agree well especially when a/r is less than 2·0. Therefore, it is concluded that the governing equation (1), e.g., the so-called assumed time model (ATM), provides a rather accurate description for the geometrically non-linear vibration of SS–SS beam.

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