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Meshfree boundary particle method applied to Helmholtz problems

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Abstract

This paper is concerned with the boundary particle method (BPM), a new boundary-only radial basis function collocation schemes. The method is developed based on the multiple reciprocity principle and applying either high-order nonsingular general solutions or singular fundamental solutions as the radial basis function. Like the multiple reciprocity BEM (MR-BEM), the BPM does not require any inner nodes for inhomogeneous problems and therefore is a truly boundary-only technique. On the other hand, unlike the MR-BEM, the BPM is meshfree, integration-free, symmetric, and mathematically simple technique. In particular, the method requires much less computational effort for the discretization than the MR-BEM. In this study, the accuracy and efficiency of the BPM are numerically demonstrated in some 2D inhomogeneous Helmholtz problems under complicated geometries.

Keywords: Multiple reciprocity principle; Boundary particle method; Radial basis function; Meshfree; Method of fundamental solution; Boundary knot method; Multiple reciprocity BEM

1. Introduction

In last decade the dual reciprocity BEM (DR-BEM) and multiple reciprocity BEM (MR-BEM) have been emerging as two most promising BEM techniques to handle inhomogeneous problems [1,2], where the dual reciprocity principle and multiple reciprocity principle are, respectively, used. It is well known that unlike the DR-BEM, the MR-BEM does not require inner nodes for general inhomogeneous problems and therefore is a truly boundary-only technique, especially attractive for high-dimension free surface and unbounded domain problems. The drawbacks of the MR-BEM, however, are that compared with the DR-BEM, the method in general requires more computational efforts [3] and is not easily applied to nonlinear problems [2]. Despite these disadvantages, in recent years the MR-BEM has been successfully used for a broad variety of problems [1,4].

In this study, we introduce the boundary particle method (BPM) [5,6], a new exact boundary-only discretization technique based on applying the radial basis function (RBF) and multiple reciprocity principle. The method is regarded as a boundary-type RBF collocation scheme since we use either high-order nonsingular general solutions or singular fundamental solutions as the RBFs to evaluate high-order homogeneous solutions, and then their sum approximates the particular solutions. Compared with the MR-BEM, the BPM is an easy-to-program, inherently meshfree, integration-free, and mathematically simple approach. The computational effort is also relatively reduced significantly. In particular, inspired by Fasshauer's Hermite RBF interpolation [7], we give the symmetric BPM scheme. This paper testifies this method to some Helmholtz problems under complicated geometry.

Developing the BPM was mainly motivated by some recent substantial advances on the method of fundamental solution (MFS), boundary knot method (BKM), and RBF. Golberg [8] combines the dual reciprocity principle and RBF with the MFS to solve some typical Poisson problems in a simple and efficient manner. The strategy is essentially meshfree and RBF-only method, where the fundamental solution can be regarded as the RBF for the evaluation of homogeneous solution. Very recently Chen and Tanaka [9, 10] further developed a boundary knot method, which applies the nonsingular general solution, RBF and dual reciprocity principle and therefore effectively eliminates the need of controversial fictitious boundary outside physical domain in the MFS. Both the MFS and BKM, however, need to use the inner nodes for inhomogeneous problems to guarantee the stability and accuracy of the solution. For more details on the MFS and BKM see Refs. [11,12]. On the other hand, since Kansa's pioneer work [13] in 1990, the research on the RBF method for PDEs has become very active. The methods based on radial basis function are inherently meshfree due to the fact that the RBF method

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uses the one-dimensional distance variable irrespective of dimensionality of problems [14]. Therefore, the RBF methods are independent of dimensionality and complexity of geometry. These recent developments on the MFS, BKM and RBF inspire the present author to present the boundary particle method which uses the multiple reciprocity principle, nonsingular general solution and fundamental solution.

In what follows, the BPM is introduced first in Section 2, followed by numerical validations in terms of some 2D inhomogeneous Helmholtz problems under complicated geometries in Section 3, and finally, Section 4 concludes some remarks based on the reported results.

2. Boundary particle methods

To clearly illustrate our idea, consider the following example without loss of generality

$$R\{u\} = f(x), \qquad x \in \Omega, \tag{1}$$

$$u(x) = R(x), \qquad x \in S_u, \tag{2a}$$

$$\frac{\partial u(x)}{\partial n} = N(x), \qquad x \in S_T, \tag{2b}$$

where R is a differential operator, x means multi-dimensional independent variable, and n is the unit outward normal. The solution of Eq. (1) can be expressed as

$$u = u_h^0 + u_p^0, \tag{3}$$

where u_h^0 and u_p^0 are the zero-order homogeneous and particular solutions, respectively. The multiple reciprocity method evaluates the particular solution in Eq. (3) by a sum of higher-order homogeneous solution, namely

$$u_p^0 = \sum_{m=1}^{\infty} u_h^m, \tag{4}$$

where superscript m is the order index of the homogeneous solution. Eq. (4) is also essential in the MR-BEM to evaluate the particular solution [1]. Thus, the solution of inhomogeneous equation (Eq. (1)) can be expressed as

$$u = u_h^0 + u_p^0 = \sum_{m=0}^{\infty} u_h^m. \tag{5}$$

It is noted that albeit the widespread use of the multiple reciprocity method, a general mathematical proof is not found in the literature.

The zero-order homogeneous solution has to satisfy both the governing equation and the boundary conditions, i.e.

$$R\left\{u_h^0\right\} = 0, \qquad x \in \Omega, \tag{6}$$

$$\begin{cases} u_h^0(x_i) = R(x_i) - u_p^0(x_i) & x \in S_u, \\ \dfrac{\partial u_h^0(x_j)}{\partial n} = N(x_j) - \dfrac{\partial u_p^0(x_j)}{\partial n} & x \in S_T, \end{cases} \tag{7}$$

where i and j indicate, respectively, the response knots located on the Dirichlet and Neumann boundary S_u, S_Γ. Eq. (7) is in fact the dual reciprocity method [1] formula without using inner nodes. In contrast, the multiple reciprocity method also involves the higher order homogeneous solution. For the first-order homogeneous solution, we have

$$R^1\left\{u_h^1\right\} = 0, \qquad x \in \Omega, \tag{8}$$

$$\begin{cases} R^0\left\{u_h^1(x_i)\right\} = f(x_i) - R^0\left\{u_p^1(x_i)\right\}, & x \in S_u, \\ \dfrac{\partial R^0\left\{u_h^1(x_j)\right\}}{\partial n} = \dfrac{\partial\left(f(x_j) - R^0\left\{u_p^1(x_j)\right\}\right)}{\partial n}, & x \in S_T, \end{cases} \tag{9}$$

where $R^1\{\ \} = R\{R\{\ \}\}$, $R^0\{\ \} = R\{\ \}$. Note that here we employed the external forcing function $f(x)$ and its normal derivative, respectively, as the corresponding Dirichlet and Neumann boundary conditions. Through a similar incremental differentiation operation over Eq. (9) via operator $R\{\ \}$, we have successive higher order boundary differential equations

$$R^n\{u_h^n\} = 0, \qquad x \in \Omega, \tag{10}$$

$$\begin{cases} R^{n-1}\{u_h^n(x_i)\} = R^{n-2}\{f(x_i)\} - R^{n-1}\left\{u_p^n(x_i)\right\}, & x \in S_u, \\ \dfrac{\partial R^{n-1}\left\{u_h^n(x_j)\right\}}{\partial n} = \dfrac{\partial\left(R^{n-2}\{f(x_j)\} - R^{n-1}\left\{u_p^n(x_j)\right\}\right)}{\partial n}, & x \in S_T, \end{cases}$$

$$n = 2, 3, \ldots, \tag{11}$$

where $R^n\{\ \}$ denotes the nth order operator of $R\{\ \}$, say $R^2\{\ \} = R\{R^1\{\ \}\}$. It is stressed that the inhomogeneous term $f(x)$ was repeatedly differentiated as the boundary conditions for higher-order operator equations.

In terms of the multiple reciprocity method, the nth order particular solution u_p^n is approximated by

$$u_p^n = \sum_{m=n+1}^{\infty} u_h^m. \tag{12}$$

The mth-order homogeneous solution can be analogized by

$$u_h^m(x) = \sum_{k=1}^{L} \beta_k^m u_m^{\#}(r_k), \tag{13}$$

where L is the number of boundary nodes, k is the index of source points on boundary; $u_m^{\#}$ is the nonsingular general solution of operator $R^m\{\ \}$. β_k are the desired coefficients, and $r_k = \|x - x_k\|$ represents the Euclidean distance norm. Note that we can use the fundamental solution u_m^* instead of general solution $u_m^{\#}$ in Eq. (13). However, the use of fundamental solution requires a fictitious boundary outside physical domain as encountered in the MFS.

Collocating Eqs. (7), (9) and (11) at all boundary knots in terms of the representation (13), we have the BPM boundary discretization equations:

$$\left.\begin{aligned} &\sum_{k=1}^{L} \beta_k^0 u_0^{\#}(r_{ik}) = R(x_i) - u_p^0(x_i) \\ &\sum_{k=1}^{L} \beta_k^0 \frac{\partial u_0^{\#}(r_{jk})}{\partial n} = N(x_j) - \frac{\partial u_p^0(x_j)}{\partial n} \end{aligned}\right\} = b^0, \tag{14}$$

$$\left.\begin{aligned}&\sum_{k=1}^{L}\beta_k^1 R^0\left\{u_1^{\#}(r_{ik})\right\}=f(x_i)-R^0\left\{u_{\mathrm{p}}^1(x_i)\right\}\\&\sum_{k=1}^{L}\beta_k^1\frac{\partial R^0\left\{u_1^{\#}(r_{jk})\right\}}{\partial n}=\frac{\partial\left(f(x_j)-R^0\left\{u_{\mathrm{p}}^1(x_j)\right\}\right)}{\partial n}\end{aligned}\right\}=b^1, \quad (15)$$

$$\left.\begin{aligned}&\sum_{k=1}^{L}\beta_k^n R^{n-1}\left\{u_n^{\#}(r_{ik})\right\}=R^{n-2}\{f(x_i)\}-R^{n-1}\left\{u_{\mathrm{p}}^n(x_i)\right\}\\&\sum_{k=1}^{L}\beta_k^n\frac{\partial R^{n-1}\{u_n^{\#}(r_{ik})\}}{\partial n}=\frac{\partial\left(R^{n-2}\{f(x_j)\}-R^{n-1}\left\{u_{\mathrm{p}}^n(x_j)\right\}\right)}{\partial n}\end{aligned}\right\}$$
$$=b^n, \qquad n=2,3,\dots \quad (16)$$

It is noted that the use of the naïve representation (13) will simply lead to the unsymmetric BPM interpolation matrix (14)–(16) due to the mixed boundary conditions. In order to get symmetric BPM scheme for self-adjoint operators, by analogy with Fasshauer's Hermite RBF interpolation [7], we presented the following RBF approximate expression to homogeneous solution instead of expression (13)

$$u_{\mathrm{h}}^m(x)=\sum_{s=1}^{L_{\mathrm{D}}}\beta_s u_m^{\#}(r_s)-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial u_m^{\#}(r_s)}{\partial n}, \quad (17)$$

where n is the unit outward normal as in boundary condition (2b), and L_{D} and L_{N} are, respectively, the numbers of knots at the Dirichlet and Neumann boundary surfaces. The minus sign associated with the second term is due to the fact that the Neumann condition of the first-order derivative is not self-adjoint. In terms of expression (17), the collocation analogue equations (Eqs. (14)–(16)) are rewritten as

$$\left.\begin{aligned}&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s u_0^{\#}(r_{is})-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial u_0^{\#}(r_{is})}{\partial n}=R(x_i)-u_{\mathrm{p}}^0(x_i)\\&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s\frac{\partial u_0^{\#}(r_{js})}{\partial n}-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial^2 u_0^{\#}(r_{js})}{\partial n^2}=N(x_j)-\frac{\partial u_{\mathrm{p}}^0(x_j)}{\partial n}\end{aligned}\right\}$$
$$=b^0 \quad (18)$$

$$\left.\begin{aligned}&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s R^0\left\{u_1^{\#}(r_{is})\right\}-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial R^0\{u_1^{\#}(r_{is})\}}{\partial n}=f(x_i)-R^0\left\{u_{\mathrm{p}}^1(x_i)\right\}\\&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s\frac{\partial R^0\left\{u_1^{\#}(r_{js})\right\}}{\partial n}-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial^2 R^0\left\{u_1^{\#}(r_{js})\right\}}{\partial n^2}=\frac{\partial\left(f(x_j)-R^0\left\{u_{\mathrm{p}}^1(x_j)\right\}\right)}{\partial n}\end{aligned}\right\}$$
$$=b^1 \quad (19)$$

The system matrix of the above equations is symmetric if operator $R\{\ \}$ is self-adjoint. Note that i and j are reciprocal indices of Dirichlet (S_u) and Neumann boundary (S_Γ) nodes.

In terms of the multiple reciprocity, the successive process is truncated at some order M, namely, let

$$R^{M-1}\left\{u_{\mathrm{p}}^M\right\}=0. \quad (21)$$

It is expected that as in the MR-BEM [1,3], the truncated order M in the BPM may not be large in a variety of practical uses if the problem has been properly scaled. However, a general strategy deciding M a prior is not available now.

The practical solution procedure of the BPM is a reversal recursive process:

$$\beta^M\to\beta^{M-1}\to\cdots\to\beta^0. \quad (22)$$

It is noted that due to

$$R^{n-1}\left\{u_n^{\#}(r)\right\}=u_0^{\#}(r), \quad (23)$$

the coefficient matrices of all successive equations (Eqs. (14)–(16) or Eqs. (7), (9) and (11)) are the same, i.e.

$$Q\beta^n=b^n, \qquad n=M,M-1,\dots,1,0. \quad (24)$$

Thus, the LU factorization algorithm is suitable for this task. Finally, after the solution of the above recursive algebraic equations (Eqs. (7), (9), (11) or (14)–(16)), we can employ the obtained expansion coefficients β to calculate the BPM solution at any knots. The unsymmetric and symmetric BPM solutions are given, respectively, by

$$u(x_i)=\sum_{n=0}^{M}\sum_{k=1}^{L}\beta_k^n u_n^{\#}(r_{ik}) \quad (25)$$

and

$$u(x_i)=\sum_{n=0}^{M}\left(\sum_{s=1}^{L_{\mathrm{D}}}\beta_s^n u_n^{\#}(r_{is})-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s^n\frac{\partial u_n^{\#}(r_{is})}{\partial n}\right). \quad (26)$$

The BPM can use either singular fundamental solution or nonsingular general solution, relative to the MFS and BKM, where the former requires an artificial boundary outside physical domain while the latter not. It is noted that the BPM with $M=1$ degenerates into the BKM or MFS without using the inner nodes. The only difference between the BKM (MFS) and BPM lies in how to evaluate the particular solution. The former applies the dual reciprocity principle, while the latter employs the multiple reciprocity

$$\left.\begin{aligned}&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s R^{n-1}\left\{u_n^{\#}(r_{is})\right\}-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial R^{n-1}\{u_n^{\#}(r_{is})\}}{\partial n}=R^{n-2}\{f(x_i)\}-R^{n-1}\left\{u_{\mathrm{p}}^n(x_i)\right\}\\&\sum_{s=1}^{L_{\mathrm{D}}}\beta_s\frac{\partial R^{n-1}\left\{u_n^{\#}(r_{js})\right\}}{\partial n}-\sum_{s=L_{\mathrm{D}}+1}^{L_{\mathrm{D}}+L_{\mathrm{N}}}\beta_s\frac{\partial^2 R^{n-1}\left\{u_n^{\#}(r_{js})\right\}}{\partial n^2}=\frac{\partial\left(R^{n-2}\{f(x_j)\}-R^{n-1}\left\{u_{\mathrm{p}}^n(x_j)\right\}\right)}{\partial n}\end{aligned}\right\}=b^n, \qquad n=2,3,\dots \quad (20)$$

principle. The advantage of the BPM over the BKM and MFS is that it dose not require interior nodes which may be especially attractive in such problems as moving boundary, inverse problems, and exterior problems. However, the BPM may be more mathematically complicated and computationally costly due to the iterative use of higher-order fundamental or general solutions.

The present form of the BPM uses the expansion coefficients rather than the direct physical variables in the approximation of boundary value. Therefore, such scheme is called as the indirect BPM. Chen [5] also gave the direct BPM with physical variable as basic variable.

3. Numerical experiments

The 2D irregular geometry tested is illustrated in Fig. 1, which includes a triangle cut-out. Except for Neumann boundary conditions shown in Fig. 1, the boundary conditions are all of the Dirichlet type. Equally spaced knots were applied on the boundary. The L_2 norms of relative errors are calculated based on the BPM numerical solutions at 364 inner and boundary nodes. The absolute error is taken as the relative error if the absolute value of the solution is less than 0.001. Note that different nodes are used for BPM coefficients and for L_2 norm of relative errors.

Helmholtz system equation is given by

$$\nabla^2 u + \gamma^2 u = 2\sin(dx)\cos(dx) + 4x\cos(dx)\cos(dy). \quad (27)$$

The accurate solution is

$$u = x^2 \sin(dx)\cos(dy) \quad (28)$$

for 2D inhomogeneous Helmholtz problem ($\gamma = d\sqrt{2}$). The

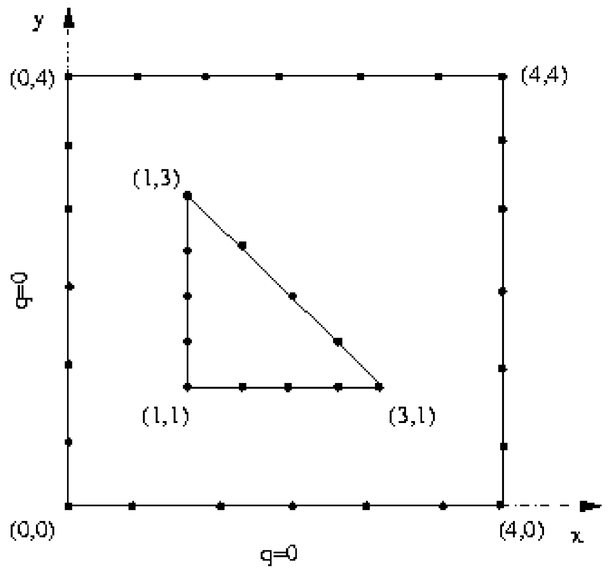

Fig. 1. Configuration of a square with a triangle cutout.

Table 1
L_2 norm of relative errors for 2D inhomogeneous Helmholtz problems by the unsymmetric and symmetric BPMs

	UBPM (26)	UBPM (40)	SBPM (26)	SBPM (40)
$\gamma = \sqrt{2}$	2.8×10^{-3}	4.5×10^{-4}	1.3×10^{-3}	2.9×10^{-4}
	UBPM (77)	UBPM (91)	SBPM (91)	SBPM (119)
$\gamma = 5\sqrt{2}$	1.5×10^{-3}	4.1×10^{-4}	6.7×10^{-3}	7.8×10^{-4}

corresponding high-order general solutions of Helmholtz is straightforward according to [15]

$$u_m^{\#}(r) = A_m(\gamma r)^{-n/2+1+m} J_{n/2-1+m}(\gamma r), \quad (29)$$

where $A_m = A_{m-1}/(2m\gamma^2)$, $A_0 = 1$; n is the dimension of the problem; m denotes the order of general solution; J represents the Bessel function of the first kind.

The experimental results are displayed in Table 1 where the UBPM and SBPM denote, respectively, the unsymmetric and symmetric BPM schemes. It is found that both BPM schemes produce very accurate solutions with a small number of boundary nodes for inhomogeneous Helmholtz problems. We also observed that the maximum relative errors occurred in random locations in difference cases. Note that we did not use any inner nodes here. Ref. [16] applied the BPM successfully to the steady convection–diffusion problems.

4. Concluding remarks

The BPM circumvents the troublesome singular integral inherent in the BEM and are very easy to learn and program. The method has advantage not requiring inner nodes for inhomogeneous problems, and therefore is very suitable for problems whose higher-order homogeneous solution quickly tends to zero. It is noted that unlike the MR-BEM, the BPM does not need to generate more than one interpolation matrix, which tremendously reduces computing effort and memory requirements. In addition, the BPM is essentially meshfree, spectral convergence and symmetric technique. It is expected that compared with other numerical techniques, the BPM may become more efficient for higher-dimensional complex-shape geometry problems since the general solutions of high-dimensional operators are simpler and radial basis function is independent of dimensionality and geometry complexity.

Similar to the BEM and MFS, the drawbacks of the method are the poor condition of the global interpolation and costly computing effort for full matrix. Some fast solvers based on the multipole [17] and wavelets may be promising to cure these inefficiencies. More investigations on the method are still under way and will be reported in subsequent papers.

Acknowledgments

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References

[1] Nowak AJ, Neves AC, editors. The multiple reciprocity boundary element method. Southampton: Computational Mechanics Publication; 1994.

[2] Partridge PW, Brebbia CA, Wrobel LW. The dual reciprocity boundary element method. Southampton: Computational Mechanics Publication; 1992.

[3] Nowak AJ, Partridge PW. Comparison of the dual reciprocity and the multiple reciprocity methods. Engng Anal Bound Elem 1992;10: 155–60.

[4] Ramachandran PA. Boundary elements methods in transport phenomena. Southampton: Computational Mechanics Publication; 1994.

[5] Chen W. New RBF collocation schemes and their applications. International workshop meshfree methods for partial differential equations, Bonn, Germany; September 2001.

[6] Chen W. RBF-based meshless boundary knot method and boundary particle method. Proceedings of China Congress on Computational Mechanics'2001, Guangzhou, China; December 2001.

[7] Fasshauer GE. Solving partial differential equations by collocation with radial basis functions. In: Mehaute A, Rabut C, Schumaker L. Proceedings of Chamonix; 1996. p. 1–8.

[8] Golberg MA. The method of fundamental solutions for Poisson's equation. Engng Anal Bound Elem 1995;16:205–13.

[9] Chen W, Tanaka M. A meshless, exponential convergence, integration-free, and boundary-only RBF technique. Comput Math Appl 2002;43:379–91.

[10] Chen W, Tanaka M. New insights into boundary-only and domain-type RBF methods. Int J Nonlinear Sci Numer Simul 2000;1(3): 145–51.

[11] Golberg MA, Chen CS. The method of fundamental solutions for potential, Helmholtz and diffusion problems. In: Golberg MA, editor. Boundary integral methods—numerical and mathematical aspects. Southampton: Computational Mechanics Publication; 1998. p. 103–76.

[12] Chen W, Tanaka M. Relationship between boundary integral equation and radial basis function. The 52nd Symposium of Japan Society for Computational Methods in Engineering (JASCOME) on BEM. Tokyo; 2000.

[13] Kansa EJ. Multiquadrics: a scattered data approximation scheme with applications to computational fluid-dynamics. Comput Math Appl 1990;19:147–61.

[14] Itagaki M. Higher order three-dimensional fundamental solutions to the Helmholtz and the modified Helmholtz equations. Engng Anal Bound Elem 1995;15:289–93.

[15] Mai-Duy N, Tran-Cong T. Mesh-free radial basis function network methods with domain decomposition for approximation of functions and numerical solution of Poisson's equations 2002;26(2):133–56.

[16] Chen W. High-order fundamental and general solutions of convection–diffusion equation and their applications with boundary particle method. Engng Anal Bound Elem 2002;. in press.

[17] Beatson RK, Cherrie JB, Ragozin DL. Fast evaluation of radial basis functions: methods for four-dimensional polyharmonic splines. SIAM J Math Anal 2001;32(6):1272–310.