Persistent currents in Bose gases confined in annular traps

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We examine the problem of stability of persistent currents in a mixture of two Bose gases trapped in an annular potential. We evaluate the critical coupling for metastability in the transition from quasi-one to two-dimensional motion. We also evaluate the critical coupling for metastability in a mixture of two species as function of the population imbalance. The stability of the currents is shown to be sensitive to the deviation from one-dimensional motion.

I. INTRODUCTION

Bose-Einstein condensates of dilute vapors of atoms offer a very promising testing ground for questions associated with superfluidity for a number of reasons. Firstly, these gases are dilute as opposed, for example, to the "traditional" superfluid liquid helium. Furthermore, the atomic gases can be manipulated in many different ways, including the shape of the confining potential, the strength and the sign of the effective interatomic interaction, the number of different species in multicomponent systems, etc.

While the term "superfluidity" covers a whole collection of many different phenomena [1], we focus in the present study on the metastability of superflow in annular traps [2–6]. Persistent flow has been observed recently in a Bose-Einstein condensate of sodium atoms confined in a toroidal trap [4]. In this experiment, an initial angular momentum of $\hbar$ per particle was transferred to the atoms and the rotational flow was observed to persist for up to ten seconds, limited only by the trap lifetime and other experimental imperfections. Persistent currents with two units of angular momentum were also observed in the same experiment.

Theoretical studies have examined the existence of metastable states in Bose-Einstein condensed gases trapped in single- [7–9] and double-ring-like [10] confining potentials, as well as other phenomena in double-ring-like traps [11, 12]. As shown in Ref. [8], mixtures of two components that are confined in a strictly one-dimensional single ring support persistent currents, but the interaction strength necessary for metastability increases with the admixture of the second component. For comparable populations of the two components, the critical value of the coupling becomes infinite. Finally, persistent currents with a value of the circulation higher than one unit were shown to be stable only in single-component systems.

In the present study we investigate the effect of the deviations from purely one-dimensional motion on the metastability of the currents. We also examine two-component Bose-Einstein condensates, and find that it is possible to have metastable superflow for sufficiently small admixtures of the second component. One of the novel results of our study is that in this latter case the deviation from one-dimensional motion gives rise to persistent currents with circulation higher than one unit, as opposed to the purely one-dimensional case.

The paper is organized as follows. We describe in Sec. II our model, and in Sec. III the two methods that we have employed to solve it, i.e., the mean-field approximation and numerical diagonalization of the Hamiltonian. In Sec. IV we present our results. More specifically, in Sec. IV.A we consider the single-component case and study the existence of metastable states in the transition from quasi-one-dimensional to two-dimensional motion. In Sec. IV.B we examine the effect of the admixture of a second component on the stability of the persistent currents, comparing the obtained results with the ones of strictly one-dimensional motion. Finally, we present our summary and conclusions in Sec. V.

II. MODEL

Let us consider two distinguishable species of bosonic atoms, labelled as A and B, with populations N_A and N_B respectively; without loss of generality we assume below that $N_B \leq N_A$. We also assume two-dimensional motion, which in an experiment corresponds to the case of very tight confinement in the perpendicular direction. In the plane of motion we model the annular trap by a displaced harmonic potential,

$$V(\rho) = \frac{1}{2}M\omega^2(\rho - R_0)^2, \qquad (1)$$

which is plotted in Fig. 1. Here ρ is the usual radial variable in cylindrical polar coordinates, R_0 is the radius of the annulus, ω is the trap frequency, and M is the atom mass, which we assume for simplicity to be equal for both species. We stress that the results presented below with the potential of Eq. (1) are – at least qualitatively – similar to the ones that we have obtained using a harmonic-plus-Gaussian trapping potential.

In order to investigate the effects due to the transition from quasi-one-dimensional to two-dimensional motion, we vary the confinement strength ω in the radial direction. For large ω, the confinement is strong enough to
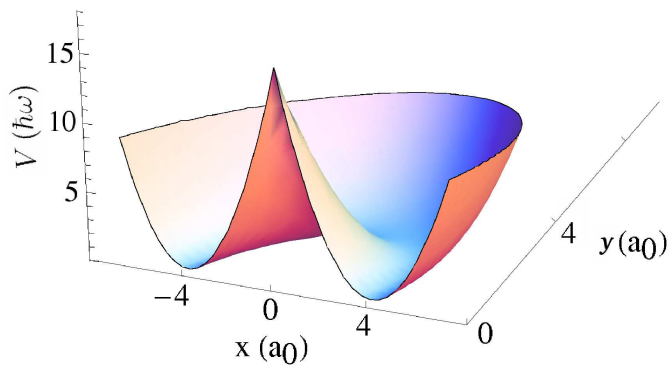
FIG. 1: (Color online) The confining potential V of Eq. (1) as a function of the cartesian coordinates (x, y), for $R_0/a_0 = 4$ and $\omega/\omega_0 = 1$.

freeze out the motion in this direction, which makes the system quasi-one-dimensional. As ω decreases, the confinement becomes weaker and, as a result, the motion becomes two-dimensional. In all the calculations presented below we fix the radius of the annulus to $R_0/a_0 = 4$, where $a_0 = \sqrt{\hbar/(M\omega_0)}$ is the ususal oscillator length corresponding to some fixed frequency ω_0.

The interatomic interactions are modelled via the usual effective contact potential, which is assumed to be repulsive. Therefore, the Hamiltonian H of the system is $H = H_{\text{sp}} + H_{\text{int}}$, where H_{sp} is the single-particle part

$$H_{\text{sp}} = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2M}\nabla_i^2 + V(\mathbf{r}_i) \right), \quad (2)$$

and H_{int} is the interaction part, given by

$$H_{\text{int}} = \frac{u_{AA}}{2}\sum_{i\neq j=1}^{N_A} \delta(\mathbf{r}_i - \mathbf{r}_j) + \frac{u_{BB}}{2}\sum_{i\neq j=1}^{N_B} \delta(\mathbf{r}_i - \mathbf{r}_j)$$
$$+ u_{AB}\sum_{i=1,j=1}^{N_A,N_B} \delta(\mathbf{r}_i - \mathbf{r}_j). \quad (3)$$

Here the parameters u_{kl} are proportional to the s-wave scattering lengths for zero-energy elastic atom-atom collisions. For simplicity, we take $u_{AA} = u_{BB} = u_{AB} \equiv u$, which is also experimentally relevant in several cases; furthermore, since the interaction is assumed repulsive, $u > 0$.

III. METHOD

We attack this problem using both the mean-field Gross-Pitaevskii approximation, as well as diagonalization of the many-body Hamiltonian.

Within the diagonalization approach, we fix the particle numbers N_A and N_B and the total angular momentum $L\hbar$, and use the Lanczos method [13] to obtain the

eigenenergies and the corresponding eigenvectors. This can be done for a range of the values of L, yielding the dispersion relation $E(L)$, i.e., the smallest eigenvalue for some given L, allowing us to identify the possible local minima associated with the metastable states that give rise to persistent currents in the system.

The basis states that we choose in this approach are the eigenfunctions of the single-particle problem, which are of the form

$$\psi_{n,m}(\rho, \theta) = \frac{e^{im\theta}}{\sqrt{2\pi}}R_{n,m}(\rho). \quad (4)$$

The quantum number n corresponds to the radial excitations, m is the quantum number associated with the angular momentum, which results from the rotational symmetry of our problem, and θ is the usual angular variable in cylindrical polar coordinates. The radial wavefunctions $R_{n,m}(\rho)$ are evaluated numerically. If $E_{n,m}$ are the corresponding eigenvalues, we work under the assumption that the typical value of the interaction energy V_{int} is weak enough, so that $V_{\text{int}} \ll E_{1,0}$. Thus, we set $n = 0$ and the only quantum number that remains is m, whose highest value is chosen so that $V_{\text{int}} \lesssim E_{0,m} \ll E_{1,0}$. Having evaluated the single-particle states, these are then combined in all possible ways to form a basis of the Fock states for the many-body problem for some fixed N_A, N_B, and L.

Turning to the mean-field approximation, within this approach the system is described via the two order parameters of the two components, ϕ_A and ϕ_B. From Eqs. (2) and (3) one obtains the two coupled Gross-Pitaevskii-like equations for these two order parameters

$$-\frac{\hbar^2\nabla^2}{2M}\phi_A + V(r)\phi_A + g_{AA}|\phi_A|^2\phi_A + g_{AB}|\phi_B|^2\phi_A$$
$$= \mu_A\phi_A,$$
$$-\frac{\hbar^2\nabla^2}{2M}\phi_B + V(r)\phi_B + g_{BB}|\phi_B|^2\phi_B + g_{AB}|\phi_A|^2\phi_B$$
$$= \mu_B\phi_B. \quad (5)$$

In the above equations the two order parameters are normalized as $\int|\phi_A|^2 d^2\mathbf{r} = 1$, and $\int|\phi_B|^2 d^2\mathbf{r} = N_B/N_A$. The parameters g_{ij} are defined as $g_{AA} = N_A u_{AA}$, $g_{BB} = N_A u_{BB}$, and $g_{AB} = N_A u_{AB}$. Finally, μ_A and μ_B are the chemical potentials of the two components.

To solve Eqs. (5) we make use of a fourth-order split-step Fourier method within an imaginary-time propagation approach [14]. Starting with an initial state with some given angular momentum, the imaginary-time propagation drives the system along the dispersion relation $E(L)$ until it finds a local minimum. If $E(L)$ has such local minima, the final state may be a metastable one. Otherwise, the initial state decays to the non-rotating ground state of the system. One should note that the dissipation of the angular momentum L is inherent to the diffusive character of this method, and it does not contradict the fact that the Hamiltonian commutes with the

operator of the angular momentum, since the imaginary-time propagation operator is not unitary, and therefore does not satisfy Heisenberg's equation of motion [15].

The two methods mentioned above complement each other: the diagonalization gives the full dispersion relation $E(L)$ for any fixed angular momentum $L\hbar$, but due to numerical limitations in diagonalizing matrices of large size, one has to restrict the number of particles, as well as the values of the interaction strength. The mean-field Gross-Pitaevskii approach, on the other hand, does not have these limitations, but does not allow us to evaluate the whole dispersion relation $E(L)$ – at least in a straightforward way, but rather its local/absolute minima. Finally, due to its mean-field character, possible correlations between the atoms in certain limiting cases are not captured within this approximation, although they are not expected to be important in the problem that we consider here.

IV. RESULTS

A. Deviation from one-dimensional motion and metastability

We start with the case of a single component, i.e., when the total number of atoms $N = N_A + N_B$ is equal to N_A. We evaluate the minimum interaction strength g_{min} necessary for the system to support persistent currents as a function of the width of the annulus, or equivalently as a function of the confinement frequency ω. As we argued earlier, for large values of ω, the motion becomes quasi-one-dimensional, while as ω decreases, the motion becomes two-dimensional. We solve Eqs. (5) and also diagonalize the Hamiltonian numerically for several values of ω. The results of both calculations are shown in Fig. 2, represented by (black) circles (mean-field) and (red) crosses (diagonalization). As seen from this plot, the results obtained from the two approaches are in good agreement with each other.

At this point it is instructive to see how these data compare with the analytical result of a purely one-dimensional model. To reduce the two-dimensional problem into an effectively one-dimensional one, at least in the limit where the width of the annulus (set by the oscillator length) is much smaller than its radius R_0, one may start from the initial expression for the energy and integrate over the radial degrees of freedom [16]. To simplify this calculation, we assume that the density in the transverse direction is nonzero only for $R_0 - a_{osc}/2 \leq \rho \leq R_0 + a_{osc}/2$, where R_0 is the radius of the annulus and a_{osc} is the oscillator length corresponding to ω. In this limit, the order parameter may be written in a product form,

$$\phi_A(\rho, \theta) = R_A(\rho)\,\Theta_A(\theta). \qquad (6)$$

Assuming the above factorization, one may start from the two-dimensional problem and derive the following one-dimensional nonlinear equation

$$-\frac{\hbar^2}{2MR_0^2}\frac{\partial^2\Theta_A(\theta)}{\partial\theta^2} + \frac{g}{a_{osc}R_0}|\Theta_A(\theta)|^2\Theta_A(\theta) = \mu_A\Theta_A(\theta), \qquad (7)$$

where the coefficient of the nonlinear term involves the integral $\int |R_A(\rho)|^4\,\rho d\rho$. According to previous studies [17, 18], Eq. (7) implies that the minimum value of the coupling g for metastability of currents with one unit of circulation, i.e., with $2\pi\hbar/M$, is (in the limit $R_0 \gg a_{osc}$)

$$g_{min} \approx \frac{3\pi}{2}\frac{\hbar^2}{M}\frac{a_{osc}}{R_0} = \frac{3\pi}{2}\frac{\hbar^{5/2}}{M^{3/2}}\frac{1}{R_0\sqrt{\omega}}, \qquad (8)$$

i.e., it scales as $\omega^{-1/2}$. Clearly the numerical prefactor in the above expression depends on the form of the order parameter in the transverse direction $R(\rho)$. If we define $g_0 \equiv \hbar^{5/2}/(M^{3/2}\omega_0^{1/2}R_0)$, this expression may be written in the more convenient form

$$\frac{g_{min}}{g_0} = \frac{3\pi}{2}\sqrt{\frac{\omega_0}{\omega}}. \qquad (9)$$

Within this simplified model, the obtained power-law dependence of g_{min} scaling as $\omega^{-1/2}$ is in agreement with the numerical results plotted in Fig. 2. This figure shows that g_{min} indeed increases linearly as a function of $(\omega/\omega_0)^{-1/2}$ for strong confinement in the transverse direction, when the motion is quasi-one-dimensional. From the numerical data we also see that for $\omega/\omega_0 \lesssim 1$, g_{min} grows faster, as the extra degrees of freedom associated with the motion in the transverse direction start to play a role. We also notice that for small values of ω, when the confinement becomes weak, the maximum value of the confining potential at the center of the trap decreases. For example, for $\omega/\omega_0 \sim 0.25$, the density of the system is already substantial at the center of the trap, and the deviations from quasi-one-dimensional motion are very pronounced.

B. Population imbalance and metastability

We now turn to the question of metastability in a two-component system. Again, we study the minimum interaction strength for the existence of metastable flow when the atoms are confined in a (two-dimensional) annular trap. To do this, we take $R_0/a_0 = 4$ as before, and consider a fixed and relatively weak confinement $\omega/\omega_0 = 1$, varying $x_A \equiv N_A/(N_A + N_B)$. This can be done easily within the Gross-Pitaevskii scheme, where one can choose any value for the relative population N_B/N_A, but not within the diagonalization approach, where one has to specify the (integer) particle numbers N_A and N_B, and therefore the values of x_A are more restricted. In addition, in order to achieve small steps in x_A, the number of particles has to be fairly large, which results in Hamiltonian matrices of large size, making it hard to diagonalize numerically.
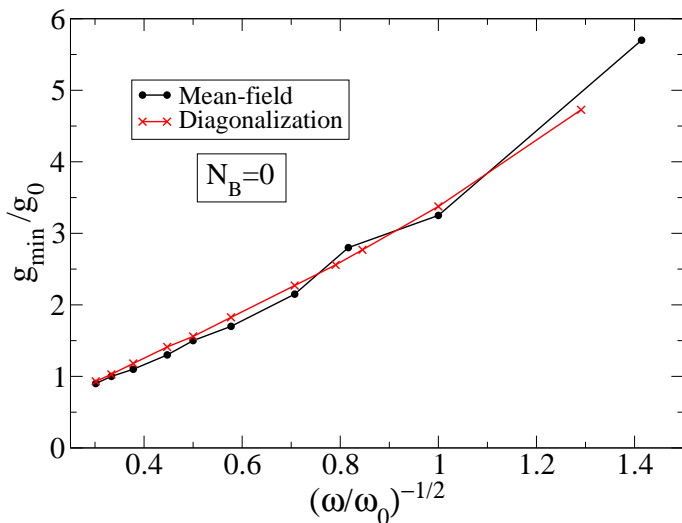
FIG. 2: (Color online) Minimum interaction strength $g_{\min}/g_0$ for the existence of persistent currents, as a function of $(\omega/\omega_0)^{-1/2}$ in an annular trap with $R_0/a_0 = 4$. The (black) circles show the result from the mean-field approximation, and the (red) crosses from the diagonalization of the Hamiltonian.
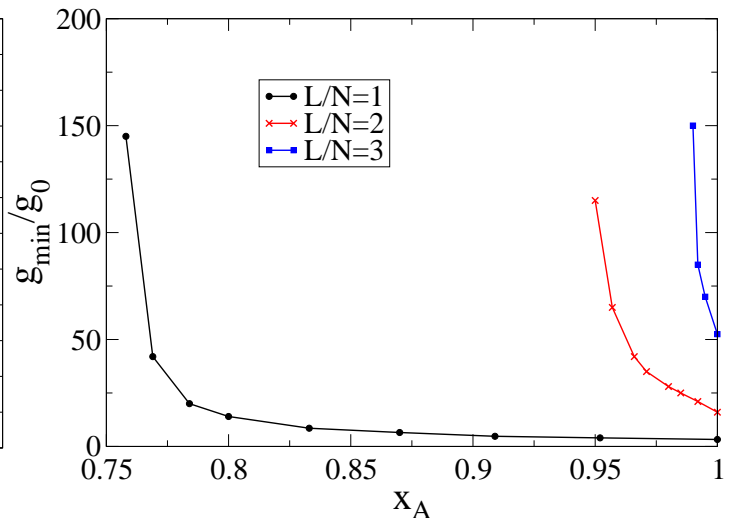

FIG. 3: (Color online) The minimum interaction strength $g_{\min}/g_0$ for stability of persistent currents in an annular trap with $R_0/a_0 = 4$ and $\omega/\omega_0 = 1$, as a function of the relative population x_A, obtained within the mean-field approximation. The (black) circles correspond to $L/N = 1$, the (red) crosses to $L/N = 2$, and the (blue) squares to $L/N = 3$.

As before, one can obtain a simple analytic result for $g_{\min}$ from the result of a strictly one-dimensional trapping potential [8]. Following the same arguments as in the previous subsection, then Eq. (7) implies that, with the assumption of a step function for the density of the gas in the transverse direction,

$$\frac{g_{\min}}{g_0} = \frac{3\pi}{2(4x_A - 3)}\sqrt{\frac{\omega_0}{\omega}}, \qquad (10)$$

which clearly reduces to Eq. (9) when $x_A = 1$. According to this simplified model, $g_{\min}$ diverges at $x_A = 3/4$ and thus metastability cannot exist for $x_A \leq 3/4$.

In Fig. 3 we show the numerical results that we obtain for the minimum interaction strength necessary for the existence of metastable states as a function of x_A, within the mean-field approximation. The (black) circles correspond to a value of the angular momentum per particle L/N equal to unity. Our calculations indicate a divergence of $g_{\min}$ for sufficiently small values of x_A, pretty much like the one-dimensional case, where this divergence occurs at $x_A = 3/4$. On the other hand, metastability takes place at a stronger interaction strength than in the purely one-dimensional case. Therefore, the deviation from this limit works against the stability of the currents. This result is consistent with the general statement that metastability of the superflow is not possible in trapping potentials which decrease monotonically with the distance from the center of the trap [19].

What has been mentioned so far has to do with values of the circulation equal to one unit, i.e., equal to $2\pi\hbar/M$. According to Ref. [8], in the strictly one-dimensional problem persistent currents with circulation higher than one unit are stable only in the single-component case, i.e.,

for $x_A = 1$ (or equivalently $N_B = 0$). Interestingly, unlike the one-dimensional case, our mean-field calculations show that persistent currents with a value of the circulation which is higher than one unit are stable for finite admixtures of a second component. This result is shown in Fig. 3 in the two higher curves, represented as (red) crosses, and (blue) squares, corresponding to $L/N = 2$ and $L/N = 3$, respectively.

The two limiting cases of strong and weak confinement are worth commenting on: as one approaches the one-dimensional limit, i.e., when ω increases, the two curves corresponding to higher values of the circulation approach infinity more rapidly, eventually becoming vertical, as in the purely one-dimensional case. Thus, metastability with $L/N = 2$ or $L/N = 3$ is not possible in this limit, in agreement with the results of Ref. [8]. Metastability is not possible, when the confinement becomes very weak, either, i.e., when the width of the annulus gets large, and the gas has a significantly nonzero density at the center of the trap. In other words, the two curves have an infinite slope in the two limiting cases (of small and large values of ω), but have a finite slope for intermediate values, indicating the stability of persistent currents in this regime.

While the behavior of the system looks similar in both limits, the underlying physical origin of the non-monotonic behavior of the slope of these curves is different in the two extremes. In the limit of small ω, this is due to the fact that metastability is absent in an almost homogeneous gas [19]. In the opposite limit of large ω, it is the one-dimensionality of the problem that causes this effect [8].

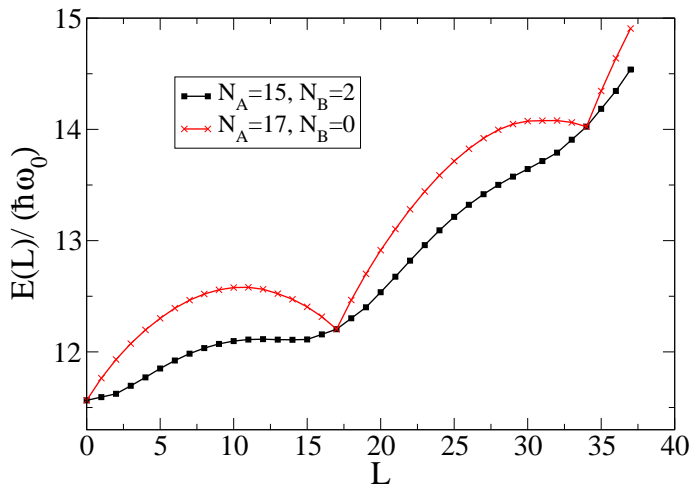The results that we have obtained from the numerical

FIG. 4: (Color online) The dispersion relation $E(L)$ obtained from the diagonalization of the Hamiltonian, in a (two-dimensional) annular trap, with $N = 17$ atoms, $\omega/\omega_0 = 1$, $R_0/a_0 = 4$, and $g/g_0 = 30$. The (red) crosses show the single-component case, with $N_A = 17$ and $N_B = 0$ ($x_A = 1$), while the (black) squares correspond to a two-component system, with $N_A = 15$ and $N_B = 2$ ($x_A \approx 0.88$). While in the first case there are two local minima in $E(L)$ at $L/N = 1$ and $L/N = 2$, in the second case there is only one minimum, around $L/N = 1$.

diagonalization of the Hamiltonian are shown in Fig. 4 and are consistent with those of the mean-field approximation. Figure 4 shows the dispersion relation, i.e., the lowest eigenvalue of the Hamiltonian for each value of L, with $x_A = 1$ (red crosses) and $x_A \approx 0.88$ (black squares). More specifically, we have considered $N = 17$ atoms, with $N_A = 17$ and $N_B = 0$ in the one case, and $N_A = 15$ and $N_B = 2$ atoms in the other one. In this calculation the single-particle basis includes all the states with $|m| \leq 5$, and we consider the same parameters as before, with $R_0/a_0 = 4$, $\omega/\omega_0 = 1$, and $g/g_0 = 30$.

The dispersion relation $E(L)$ in the case of a single component has two local minima, at $L/N = 1$ and $L/N = 2$, showing that persistent currents are stable in both cases. This is consistent with the result obtained from the mean-field calculations shown in Fig. 3. On the other hand, in the case of two components, $E(L)$ has only one local minimum around $L/N = 1$. Again, this is consistent with the result of Fig. 3, since for the corresponding values of x_A and $g_{\min}/g_0$, only the state with $L/N = 1$ is metastable.

V. SUMMARY AND CONCLUSIONS

According to the present study, Bose-Einstein condensed atoms which move in annular traps offer a very interesting system for the study of persistent currents.

In principle one could investigate a surface in a three-

dimensional phase diagram showing the minimum interaction strength for stable currents as function of both the confinement strength ω, which determines the deviation from purely-one-dimensional motion, and the population imbalance x_A. Given the size and the numerical effort of such a calculation, we have instead restricted ourselves to certain parts of this diagram, and in particular we have investigated two different questions.

Firstly, we considered the case of one component, and showed that as the width of the annulus becomes larger, and thus there are deviations from the strictly one-dimensional motion, the critical coupling $g_{\min}$ that is necessary to achieve metastability increases. This result is consistent with the fact that a necessary condition for metastability is that the confining potential – and as a result the particle density – does not decrease monotonically from the center of the trap; in the picture of vortex dynamics, there is a force acting on the vortex that is associated with the rotational motion of the superfluid, which is in the opposite direction of the gradient of the single-particle density distribution.

Secondly, we investigated the case of a mixture of two components, in a fixed annular potential. We found that $g_{\min}$ which corresponds to persistent currents with one unit of circulation increases with the addition of the second component. Also, we found a similar behavior as in the one-dimensional problem, where $g_{\min}$ diverges at $x_A = 3/4$. Thus, the main effect of the finite width of the annulus is a relative increase in the critical coupling.

On the other hand, a novel result that is absent in the one-dimensional case is the stability of persistent currents with circulation for the larger component which is higher than one unit. When the motion is one-dimensional, $g_{\min}$ is infinite, and there are no stable currents. As the width of the annulus increases, these currents become stable for interaction strengths which are sufficiently strong, but finite. When the annulus becomes even wider, $g_{\min}$ becomes infinite again. Therefore, the effect of the deviation from the one-dimensional motion is rather dramatic in this case.

From the above results it is clear that the phase diagram $g_{\min} = g_{\min}(\omega, x_A)$ has an interesting structure and that the physics of persistent currents of Bose-Einstein condensed gases trapped in annular potentials is very rich. With the recent progress in building such trapping potentials in the laboratory, it will be of much interest to investigate all these effects also experimentally.

VI. ACKNOWLEDGEMENTS

[1] A. J. Leggett, Rev. Mod. Phys. **73**, 307 (2001).

[2] S. Gupta, K. W. Murch, K. L. Moore, T. P. Purdy, and D. M. Stamper-Kurn, Phys. Rev. Lett. **95**, 143201 (2005).

[3] S. E. Olson, M. L. Terraciano, M. Bashkansky, and F. K. Fatemi, Phys. Rev. A **76**, 061404(R) (2007).

[4] C. Ryu, M. F. Andersen, P. Clade, V. Natarajan, K. Helmerson, and W. D. Phillips, Phys. Rev. Lett. **99**, 260401 (2007).

[5] K. Henderson, C. Ryu, C. MacCormick, and M. G. Boshier, New J. Phys. **11**, 043030 (2009).

[6] I. Lesanovsky and W. von Klitzing, Phys. Rev. Lett. 99, 083001 (2007).

[7] K. Kärkkäinen, J. Christensson, G. Reinisch, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A **76**, 043627 (2007).

[8] J. Smyrnakis, S. Bargi, G. M. Kavoulakis, M. Magiropoulos, K. Kärkkäinen, and S. M. Reimann, Phys. Rev. Lett. **103**, 100404 (2009).

[9] P. Mason and N. G. Berloff, Phys. Rev. A **79**, 043620 (2009).

[10] F. Malet, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A **81**, 013630 (2010).

[11] I. Lesanovsky and W. von Klitzing, Phys. Rev. Lett. **98**, 050401 (2007).

[12] J. Brand, T. J. Haigh, and U. Zülicke, Phys. Rev. A **80**, 011602 (2009).

[13] C. Lanczos, J. Res. Natl. Bur. Stand. **45**, 255 (1950).

[14] S. A. Chin and E. Krotscheck, Phys. Rev. E **72**, 036705 (2005).

[15] B. H. Bransden and C. J. Joachain, *Introduction to quantum mechanics*, Longman (1989).

[16] A. D. Jackson, G. M. Kavoulakis, and C. J. Pethick, Phys. Rev. A **58**, 2417 (1998).

[17] R. Kanamoto, H. Saito, and M. Ueda, Phys. Rev. A **68**, 043619 (2003).

[18] G. M. Kavoulakis, Phys. Rev. A **69**, 023613 (2004).

[19] K. Kärkkäinen, J. Christensson, G. Reinisch, G. M. Kavoulakis, and S. M. Reimann, Phys. Rev. A **76**, 043627 (2007).