Optimal consumption and investment in incomplete markets with general constraints*

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Abstract. We study an optimal consumption and investment problem in a possibly incomplete market with general, not necessarily convex, stochastic constraints. We give explicit solutions for investors with exponential, logarithmic and power utility. Our approach is based on martingale methods which rely on recent results on the existence and uniqueness of solutions to BSDEs with drivers of quadratic growth.

1 Introduction

We consider an investor receiving stochastic income who can invest in a financial market. The question is how to optimally consume and invest if utility is derived from intermediate consumption and the level of remaining wealth at some final time T. More specifically, we assume our investor receives income at rate e_t and a lump sum payment E at the final time. The investor chooses a rate of consumption c_t and an investment policy so as to maximize the expectation

$$\mathbb{E}\left[\int_0^T \alpha e^{-\beta t} u(c_t)dt + e^{-\beta T} u(X_T + E)\right],$$

where α and β are constants, $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a concave utility function and X_T is his/her wealth immediately before s/he receives the lump sum payment E. There exists an extensive literature on problems of this form; see for instance, Karatzas and Shreve [7], Schachermayer [11] or Morlais [10] for an overview.

The novelty of this paper is that we put general, not necessarily convex, stochastic constraints on consumption and investment. We provide explicit solutions for investors with exponential, logarithmic and power utility. Our approach is based on the same idea as Hu et al. [6], which studies constraint investment problems without intermediate consumption. To every admissible strategy we associate a utility process, which we show to always be a supermartingale and a martingale if and only if the strategy is optimal. This method relies on results from Kobylanski [9] and Morlais [10] on the existence and properties of solutions to BSDEs with drivers of quadratic growth (for extensions to unbounded terminal conditions, see Briand and Hu [2, 3], Ankirchner et al. [1] as well as Delbaen et al. [5]).

The structure of the paper is as follows: Section 2 introduces the model. In Section 3 we discuss the case of constant absolute risk aversion, which corresponds to exponential utility functions.

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Section 4 treats the case of constant relative risk aversion, which is covered by logarithmic and power utility functions. The specification of the constraints and the definition of admissible strategies will be slightly different from case to case.

2 The model

Let $T \in \mathbb{R}_+$ be a finite time horizon and $(W_t)_{0 \le t \le T}$ an n-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $(\mathcal{F}_t)$ the augmented filtration generated by (W_t). We consider a financial market consisting of a money market and $m \le n$ stocks. Money can be lent to and borrowed from the money market at a constant interest rate $r \ge 0$ and the stock prices follow the dynamics

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + \sigma_t^i dW_t, \quad S_0^i > 0, \quad i = 1, \dots, m,$$

for bounded predictable processes μ_t^i and σ_t^i taking values in $\mathbb{R}$ and $\mathbb{R}^{1 \times n}$, respectively. If $m < n$, the stocks do not span all uncertainty and the market is incomplete even if there are no constraints.

Consider an investor with initial wealth $x \in \mathbb{R}$ receiving income at a predictable rate e_t and an $\mathcal{F}_T$-measurable lump sum payment E at time T who can consume at intermediate times and invest in the financial market. If the investor consumes at a predictable rate c_t and invests according to a predictable trading strategy π_t taking values in $\mathbb{R}^{1 \times m}$, where π_t^i is the amount of money invested in stock i at time t, his/her wealth evolves like

$$X_t = x + \int_0^t \left(X_s - \sum_{i=1}^m \pi_s^i \right) r ds + \sum_{i=1}^m \int_0^t \frac{\pi_s^i}{S_s^i} dS_s^i + \int_0^t (e_s - c_s) ds.$$

Denote by σ_t the matrix with rows σ_t^i, $i = 1, \dots, m$. Assume that $\sigma\sigma^T$ is invertible $\nu \otimes \mathbb{P}$-almost everywhere, where ν is the Lebesgue measure on $[0, T]$, and the process

$$\theta = \sigma^T (\sigma\sigma^T)^{-1} (\mu - r1)$$

is bounded. Then for $p = \pi\sigma$, one can write

$$X_t^{(c,p)} = x + \int_0^t X_s^{(c,p)} r ds + \int_0^t p_t [dW_t + \theta_t dt] + \int_0^t (e_s - c_s) ds.$$

We assume our agent chooses c and π so as to maximize

$$\mathbb{E}\left[\int_0^T \alpha e^{-\beta t} u(c_t) dt + e^{-\beta T} u(X_T^{(c,p)} + E) \right] \tag{2.1}$$

for constants $\alpha > 0$, $\beta \in \mathbb{R}$ and a concave function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. The specific cases we will discuss are:

- $u(x) = -\exp(-\gamma x)$ for $\gamma > 0$
- $u(x) = \log(x)$
- $u(x) = x^\gamma / \gamma$ for $\gamma \in (-\infty, 0) \cup (0, 1)$.

As usual, for $\gamma > 0$, we understand x^γ/γ to be $-\infty$ on $(-\infty, 0)$ while $\log(x)$ and x^γ/γ for $\gamma < 0$ are meant to be $-\infty$ on $(-\infty, 0]$.

To formulate consumption and investment constraints we introduce non-empty subsets $C \subset \mathcal{P}$ and $Q \subset \mathcal{P}^{1 \times m}$, where $\mathcal{P}$ denotes the set of all real-valued predictable processes $(c_t)_{0 \le t \le T}$ and $\mathcal{P}^{1 \times m}$ the set of all predictable processes $(\pi_t)_{0 \le t \le T}$ with values in $\mathbb{R}^{1 \times m}$. In Section 3 we do not put restrictions on consumption and just require π to be in Q. In Section 4 consumption and investment will be of the form $c = \tilde{c}X$ and $\pi = \tilde{\pi}X$, respectively, and we will require $\tilde{c}$ to be in C and $\tilde{\pi}$ in Q.

Note that the expected value (2.1) does not change if (c, p) is replaced by a pair (c', p') which is equal $\nu \otimes \mathbb{P}$-almost everywhere. So we identify predictable processes that agree $\nu \otimes \mathbb{P}$-almost everywhere and use the following concepts from Cheridito et al. [4]: We call a subset A of $\mathcal{P}^{1\times k}$ **sequentially closed** if it contains every process a that is the $\nu \otimes \mathbb{P}$-almost everywhere limit of a sequence $(a^n)_{n\geq 1}$ of processes in A. We call it $\mathcal{P}$**-stable** if it contains $1_B a + 1_{B^c} a'$ for all $a, a' \in A$ and every predictable set $B \subset [0,T] \times \Omega$. We say A is $\mathcal{P}$**-convex** if it contains $\lambda a + (1-\lambda)a'$ for all $a, a' \in A$ and every process $\lambda \in \mathcal{P}$ with values in $[0,1]$.

We always assume that C and Q are sequentially closed and $\mathcal{P}$-stable. This will allow us to show existence of optimal strategies. If, in addition, C and Q are $\mathcal{P}$-convex, the optimal strategies will be unique. Since we assumed $\sigma\sigma^T$ to be bounded and invertible for $\nu \otimes \mathbb{P}$-almost all (t,ω), $P = Q\sigma$ is a sequentially closed, $\mathcal{P}$-stable subset of $\mathcal{P}^{1\times n}$, which is $\mathcal{P}$-convex if and only if Q is so.

For a process q in $\mathcal{P}^{1\times n}$, we denote by $\text{dist}(q,P)$ the predictable process

$$\text{dist}(q,P) := \operatorname*{ess\,inf}_{p\in P} |q-p|,$$

where $|.|$ is the Euclidean norm on $\mathbb{R}^{1\times n}$ and ess inf denotes the greatest lower bound with respect to the $\nu\otimes\mathbb{P}$-almost everywhere order. It is shown in Cheridito et al. [4] that there exists a process $p \in \mathcal{P}^{1\times n}$ satisfying $|q-p| = \text{dist}(q,P)$ and that it is unique if P is $\mathcal{P}$-convex. We denote the set of all these processes by $\Pi_P(q)$.

By $\mathcal{P}^{1\times n}_{\text{BMO}}$ we denote the processes $Z \in \mathcal{P}^{1\times n}$ for which there exists a constant $D \geq 0$ such that

$$\mathbb{E}\left[\int_\tau^T |Z_t|^2 dt \mid \mathcal{F}_\tau\right] \leq D \quad \text{for all stopping times } \tau \leq T.$$

For every $Z \in \mathcal{P}^{1\times n}_{\text{BMO}}$, $\int_0^\cdot Z_s dW_s$ is a BMO-martingale and $\mathcal{E}(Z\cdot W)_t$, $0 \leq t \leq T$, a positive martingale. Moreover, if $Z, V \in \mathcal{P}^{1\times n}_{\text{BMO}}$, then Z is also in $\mathcal{P}^{1\times n}_{\text{BMO}}$ with respect to the Girsanov transformed measure

$$\mathbb{Q} = \mathcal{E}(V\cdot W)_T \cdot \mathbb{P};$$

see for instance, Kazamaki [8].

3 CARA or exponential utility

We first assume that our investor has constant absolute risk aversion $-u''(x)/u'(x) = \gamma > 0$. Then, up to affine transformations, the utility function u is of the exponential form

$$u(x) = -\exp(-\gamma x).$$

Here we do not constrain consumption, that is, $C = \mathcal{P}$, and we assume that the set P of possible investment strategies contains at least one bounded process $\bar{p}$.

Introduce the bounded positive function h on $[0,T]$ by

$$h(t) = 1/(T-t) \quad \text{if} \quad r = 0$$

and

$$h(t) = \frac{r}{1-(1-r)\exp(-r(T-t))} \quad \text{if} \quad r > 0.$$

Note that in both cases it solves the quadratic ODE

$$h'(t) = h(t)(h(t)-r), \quad h(T) = 1.$$

Definition 3.1 *If $u(x) = -\exp(-\gamma x)$, an admissible strategy consists of a pair $(c, p) \in \mathcal{P} \times P$ such that $\int_0^T (|c_t| + |p_t|^2)dt < \infty$ $\mathbb{P}$-almost surely,*

$$\exp\left(-\gamma h(t) X_t^{(c,p)}\right)_{0 \le t \le T} \text{ is of class (D)} \quad \text{and} \quad \int_0^T \mathbb{E}\left[e^{-\gamma c_t}\right] dt < \infty.$$

Consider the BSDE

$$Y_t = E + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T Z_s dW_s \tag{3.1}$$

with driver

$$f(t, y, z) = -\frac{\gamma}{2}\text{dist}_t^2\left(z + \frac{1}{\gamma}\theta, hP\right) + z\theta_t + \frac{1}{2\gamma}|\theta_t|^2 + h(t)(e_t - y) + \frac{h(t)}{\gamma}\left(\log\frac{h(t)}{\alpha} - 1\right) + \frac{\beta}{\gamma}.$$

Since θ, e, E and h are bounded and the set P contains a bounded process $\bar{p}$, there exists a constant $K \in \mathbb{R}_+$ such that

$$|f(t, y, z)| \le K(1 + |y| + |z|^2)$$

and

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \le K(|y_1 - y_2| + (1 + |z_1| + |z_2|)|z_1 - z_2|).$$

So it follows from [9] and [10] that equation (3.1) has a unique solution (Y, Z) such that Y is bounded and Z belongs to $\mathcal{P}_{\text{BMO}}^{1\times n}$.

Theorem 3.2 *The optimal value of the optimization problem (2.1) for $u(x) = -\exp(-\gamma x)$ over all admissible strategies is*

$$-\exp\left[-\gamma(h(0)x + Y_0)\right], \tag{3.2}$$

and (c^*, p^*) is an optimal admissible strategy if and only if

$$c^* = hX^{(c^*,p^*)} + Y - \frac{1}{\gamma}\log\frac{h}{\alpha} \quad \text{and} \quad p^* \in \Pi_P\left(\frac{Z + \theta/\gamma}{h}\right). \tag{3.3}$$

Moreover, if P is $\mathcal{P}$-convex, there is only one optimal strategy (c^*, p^*).

Proof. For every admissible strategy (c, p) define the process

$$R_t^{(c,p)} = -e^{-\beta t}e^{-\gamma\left(h(t)X_t^{(c,p)} + Y_t\right)} - \int_0^t \alpha e^{-\beta s}e^{-\gamma c_s}ds.$$

Then

$$R_0^{(c,p)} = -e^{-\gamma(h(0)x + Y_0)}, \quad R_T^{(c,p)} = -e^{-\beta T}e^{-\gamma\left(X_T^{(c,p)} + E\right)} - \int_0^T \alpha e^{-\beta s}e^{-\gamma c_s}ds$$

and

$$dR_t^{(c,p)} = \gamma e^{-\beta t}e^{-\gamma\left(h(t)X_t^{(c,p)} + Y_t\right)}\left[(h(t)p_t - Z_t)dW_t + A_t^{(c,p)}dt\right],$$

where

$$\begin{aligned} A_t^{(c,p)} = \quad & h(t)p_t\theta_t - \frac{\gamma}{2}|h(t)p_t - Z_t|^2 - f(t, Y_t, Z_t) \\ & + h(t)(e_t - c_t) - \frac{\alpha}{\gamma}e^{\gamma\left(h(t)X_t^{(c,p)} + Y_t\right)}e^{-\gamma c_t} + h'(t)X_t^{(c,p)} + h(t)rX_t^{(c,p)} + \frac{\beta}{\gamma}. \end{aligned}$$

First note that

$$h(t)p_t\theta_t - \frac{\gamma}{2}|h(t)p_t - Z_t|^2 = -\frac{\gamma}{2}\left|h(t)p_t - \left(Z_t + \frac{1}{\gamma}\theta_t\right)\right|^2 + Z_t\theta_t + \frac{1}{2\gamma}|\theta_t|^2$$
$$\leq \; -\frac{\gamma}{2}\operatorname{dist}_t^2\left(Z + \frac{1}{\gamma}\theta, hP\right) + Z_t\theta_t + \frac{1}{2\gamma}|\theta_t|^2,$$

and the inequality becomes an $\nu \otimes \mathbb{P}$-almost everywhere equality if and only if

$$p \in \Pi_P\left(\frac{Z + \theta/\gamma}{h}\right).$$

Furthermore,

$$h(t)(e_t - c_t) - \frac{\alpha}{\gamma}e^{\gamma\left(h(t)X_t^{(c,p)} + Y_t\right)}e^{-\gamma c_t} + h'(t)X_t^{(c,p)} + h(t)rX_t^{(c,p)} + \frac{\beta}{\gamma}$$
$$\leq \; h(t)e_t + \frac{h(t)}{\gamma}\log\frac{h(t)}{\alpha} - h^2(t)X_t^{(c,p)} - h(t)Y_t - \frac{h(t)}{\gamma} + h'(t)X_t^{(c,p)} + h(t)rX_t^{(c,p)} + \frac{\beta}{\gamma}$$
$$= \; h(t)e_t + \frac{h(t)}{\gamma}\log\frac{h(t)}{\alpha} - h(t)Y_t - \frac{h(t)}{\gamma} + \frac{\beta}{\gamma}, \tag{3.4}$$

where the inequality is attained if and only if

$$\alpha e^{\gamma\left(hX^{(c,p)} + Y\right)}e^{-\gamma c} = h \quad \Leftrightarrow \quad c = hX^{(c,p)} + Y - \frac{1}{\gamma}\log\frac{h}{\alpha}$$

(note that in (3.4) the X-terms disappear due to our choice of the function h). It follows that for every admissible pair (c,p), $R^{(c,p)}$ is a local supermartingale, which by our definition of admissible strategies, is of class (D). Therefore, it is a supermartingale, and one obtains

$$R_0^{(c,p)} \geq \mathbb{E}\left[R_T^{(c,p)}\right].$$

If we can show that each pair (c^*, p^*) satisfying (3.3) is admissible and $R^* = R^{(c^*,p^*)}$ is a martingale, we can conclude that

$$R_0^* = \mathbb{E}\left[R_T^*\right],$$

and it follows that (c^*, p^*) is optimal.

But if (c^*, p^*) satisfies (3.3), c^* is continuous. Therefore, it belongs to $\mathcal{P}$ and $\int_0^T c_t^* dt < \infty$ $\mathbb{P}$-almost surely. Moreover, since θ as well as h are bounded and P contains a bounded process $\bar{p}$, there exists a constant L such that $|p^*| \leq L(1+|Z|)$. It follows that $p^* \in \mathcal{P}^{1\times n}_{\mathrm{BMO}}$, and in particular, $\int_0^T |p_t^*|^2 dt < \infty$ $\mathbb{P}$-almost surely. Since $A^* = A^{(c^*,p^*)} = 0$, $-R^*$ is a positive local martingale, and one obtains

$$\mathbb{E}\left[e^{-\gamma X_T^*}\right] + \mathbb{E}\left[\int_0^T e^{-\gamma c_t^*}dt\right] \leq M\mathbb{E}\left[-R_T^*\right] < \infty,$$

where M is a suitable constant and the inequality $\mathbb{E}\left[-R_T^*\right] < \infty$ follows from Fatou's lemma. By Girsanov's theorem,

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s ds$$

is an n-dimensional Brownian motion under the measure

$$\mathbb{Q} = \mathcal{E}(-\theta \cdot W)_T \cdot \mathbb{P},$$

and one has

$$
\begin{aligned}
& d(h(t)X_t^*) = h'(t)X_t^*dt + h(t)p_t^*dW_t + h(t)[X_t^*r + p_t^*\theta_t + e_t - c_t^*]dt \\
= \;& h'(t)X_t^*dt + h(t)p_t^*dW_t + h(t)\left[X_t^*r + p_t^*\theta_t + e_t - h(t)X_t^* - Y_t + \frac{1}{\gamma}\log\left(\frac{h(t)}{\alpha}\right)\right]dt \\
= \;& h(t)p_t^*dW_t + h(t)\left[p_t^*\theta_t + e_t - Y_t + \frac{1}{\gamma}\log\left(\frac{h(t)}{\alpha}\right)\right]dt \\
= \;& h(t)p_t^*dW_t^{\mathbb{Q}} + h(t)\left[e_t - Y_t + \frac{1}{\gamma}\log\left(\frac{h(t)}{\alpha}\right)\right]dt. \qquad (3.5)
\end{aligned}
$$

Since p^* belongs to $\mathcal{P}_{\text{BMO}}^{1\times n}$, the process $V_t = \int_0^t h(s)p_s^*dW_u^{\mathbb{Q}}$ is a BMO-martingale under $\mathbb{Q}$, and it can be seen from (3.5) that there exist constants c_1, c_2 such that

$$
e^{-\gamma h(t)X_t^*} \le c_1 e^{-\gamma V_t} \quad \text{and} \quad e^{-\gamma V_t} \le c_2 e^{-\gamma h(t)X_t^*} \quad \text{for all } t \in [0,T].
$$

Hence, one obtains for every stopping time $\tau \le T$,

$$
\begin{aligned}
e^{-\gamma h(\tau)X_\tau^*} \;&\le\; c_1 e^{-\gamma V_\tau} \le c_1\left(\mathbb{E}_{\mathbb{Q}}\left[e^{-\frac{\gamma}{2}V_T} \mid \mathcal{F}_\tau\right]\right)^2 \\
&=\; c_1\left(\mathbb{E}\left[e^{-\frac{\gamma}{2}V_T}\mathcal{E}(-\theta\cdot W)_T \mid \mathcal{F}_\tau\right]\right)^2 \mathcal{E}(-\theta\cdot W)_\tau^{-2} \\
&\le\; c_1\mathbb{E}\left[e^{-\gamma V_T} \mid \mathcal{F}_\tau\right]\mathbb{E}\left[\mathcal{E}(-\theta\cdot W)_T^2 \mid \mathcal{F}_\tau\right]\mathcal{E}(-\theta\cdot W)_\tau^{-2} \\
&\le\; c_1c_2\mathbb{E}\left[e^{-\gamma X_T^*} \mid \mathcal{F}_\tau\right]\mathbb{E}\left[\mathcal{E}(-\theta\cdot W)_T^2 \mid \mathcal{F}_\tau\right]\mathcal{E}(-\theta\cdot W)_\tau^{-2}.
\end{aligned}
$$

But since θ is bounded, there exists a constant c_3 such that

$$
\begin{aligned}
&\mathbb{E}\left[\mathcal{E}(-\theta\cdot W)_T^2 \mid \mathcal{F}_\tau\right]\mathcal{E}(-\theta\cdot W)_\tau^{-2} \\
=\;& \mathbb{E}\left[\frac{\mathcal{E}(-2\theta\cdot W)_T}{\mathcal{E}(-2\theta\cdot W)_\tau}\exp\left(\int_\tau^T |\theta_s|^2 ds\right) \mid \mathcal{F}_\tau\right] \\
\le\;& c_3 \quad \text{for every stopping time } \tau \le T.
\end{aligned}
$$

So one has

$$
e^{-\gamma h(\tau)X_\tau^*} \le c_1c_2c_3\mathbb{E}\left[e^{-\gamma X_T^*} \mid \mathcal{F}_\tau\right] \quad \text{for every stopping time } \tau \le T.
$$

This shows that $\exp(-\gamma h(t)X_t^*)_{0\le t\le T}$ is of class (D). Therefore, (c^*, p^*) is admissible and R^* a martingale.

If P is $\mathcal{P}$-convex, $\Pi_P\left(\frac{Z+\theta/\gamma}{h}\right)$ contains only one process. So (c^*, p^*) is unique. □

4 CRRA utility

We now assume that the investor has constant relative risk aversion $-xu''(x)/u'(x) = \delta > 0$. For $\delta = 1$, this corresponds to $u(x) = \log(x)$, and for $\delta \neq 1$ to $u(x) = x^\gamma/\gamma$, where $\gamma = 1 - \delta$. We discuss the cases $\delta = 1$ and $\delta \neq 1$ separately. In both of them we assume $E = 0$.

We here suppose that the initial wealth is strictly positive: $x > 0$. To avoid $-\infty$ utility, the agent must keep the wealth process positive. Therefore, we can parameterize e, c and π by $\tilde{e} = e/X$, $\tilde{c} = c/X$ and $\tilde{\pi} = \pi/X$, respectively. If one denotes $\tilde{p} = \tilde{\pi}\sigma$, the corresponding wealth evolves according to

$$
\frac{dX_t^{(c,p)}}{X_t^{(c,p)}} = \tilde{p}_t(dW_t + \theta_t dt) + (r + \tilde{e}_t - \tilde{c}_t)dt, \quad X_0^{(c,p)} = x,
$$

and one can write

$$X_t^{(c,p)} = x\,\mathcal{E}\left(\tilde{p}\cdot W^{\mathbb{Q}}\right)_t \exp\left(\int_0^t (r+\tilde{e}_s-\tilde{c}_s)ds\right) > 0, \tag{4.1}$$

where $W_t^{\mathbb{Q}} = W_t + \int_0^t \theta_s ds$.

The constraints are now of the following form: $\tilde{c}$ must be in the set C and $\tilde{p}$ in $P = Q\sigma$. Additionally, $\tilde{c}$ will be required to be positive or non-negative depending on the specific utility function being used. Moreover, $\tilde{c}$ and $\tilde{p}$ will have to satisfy suitable integrability conditions. For all CRRA utility functions u we make the following assumption:

$$\text{there exists a pair } (\bar{c},\bar{p}) \in C\times P \text{ such that } u(\bar{c})-\bar{c} \text{ and } \bar{p} \text{ are bounded.} \tag{4.2}$$

This implies that $u(\bar{c})$ and $\bar{c}$ are both bounded.

4.1 Logarithmic utility

If the utility function is logarithmic, we introduce the positive function

$$h(t) = \begin{cases} 1+\alpha(T-t) & \text{if } \beta = 0 \\ \alpha/\beta + (1-\alpha/\beta)e^{-\beta(T-t)} & \text{if } \beta > 0 \end{cases},$$

and note that

$$h'(t) = \beta h(t) - \alpha \quad \text{with} \quad h(T) = 1.$$

Definition 4.1 *For $u(x) = \log(x)$, an admissible strategy is a pair $(\tilde{c},\tilde{p}) \in C\times P$ satisfying*

$$\mathbb{E}\left[\int_0^T |\log(\tilde{c}_t)|dt + \int_0^T \tilde{c}_t dt + \int_0^T |\tilde{p}_t|^2 dt\right] < \infty. \tag{4.3}$$

Notice that (4.3) implies $\tilde{c} > 0$.

Let us set

$$\max_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right) := \operatorname*{ess\,sup}_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right), \tag{4.4}$$

where ess sup is the smallest upper bound with respect to $\nu\otimes\mathbb{P}$-almost everywhere inequality. Due to assumption (4.2), (4.4) defines a bounded predictable process. By

$$\arg\max_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right) \tag{4.5}$$

we denote the set of all process in C which attain the ess sup. It follows from Cheridito et al. [4] that (4.5) is not empty and contains exactly one process if C is $\mathcal{P}$-convex.

Consider the BSDE

$$Y_t = \int_t^T f(s,Y_s)ds + \int_t^T Z_s dW_s \tag{4.6}$$

with driver

$$f(t,y) = \frac{1}{2}\operatorname{dist}_t^2(\theta,P) - \frac{1}{2}|\theta_t|^2 - \frac{\alpha y}{h(t)} - \max_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right)_t - r - \tilde{e}_t. \tag{4.7}$$

$f(t,y)$ is of linear growth in y, and all the other terms are bounded. So by Kobylanski [9] and Morlais [10], equation (4.6) has a unique solution (Y,Z) such that Y is bounded and $Z \in \mathcal{P}^{1\times n}_{\text{BMO}}$.

Theorem 4.2 *For $u(x) = \log(x)$, the optimal value of the optimization problem* (2.1) *over all admissible strategies is*

$$h(0)(\log(x) - Y_0), \tag{4.8}$$

and $(\tilde{c}^*, \tilde{p}^*)$ is an optimal admissible strategy if and only if

$$\tilde{c}^* \in \arg\max_{\tilde{c}\in C} \left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right) \quad \textit{and} \quad \tilde{p}^* \in \Pi_P(\theta). \tag{4.9}$$

Moreover, if C and P are $\mathcal{P}$-convex, there is just one optimal strategy $(\tilde{c}^*, \tilde{p}^*)$.

Proof. For every admissible pair $(\tilde{c}, \tilde{p})$ define the process

$$R_t^{(c,p)} = h(t)e^{-\beta t}\left(\log\left(X_t^{(c,p)}\right) - Y_t\right) + \int_0^t \alpha e^{-\beta s}\log(c_s)ds.$$

One has

$$R_0^{(c,p)} = h(0)(\log(x) - Y_0), \quad R_T^{(c,p)} = e^{-\beta T}\log\left(X_T^{(c,p)}\right) + \int_0^T \alpha e^{-\beta s}\log(c_s)ds$$

and

$$dR_t^{(c,p)} = h(t)e^{-\beta t}\left[(\tilde{p}_t + Z_t)dW_t + A_t^{(c,p)}dt\right], \tag{4.10}$$

where

$$A_t^{(c,p)} = \quad \tilde{p}_t\theta_t - \frac{1}{2}|\tilde{p}_t|^2 + \frac{\alpha Y_t}{h(t)} + f(t, Y_t) + \frac{\alpha}{h(t)}\log(\tilde{c}_t) + r + \tilde{e}_t - \tilde{c}_t.$$

First note that

$$\tilde{p}_t\theta_t - \frac{1}{2}|\tilde{p}_t|^2 + \frac{\alpha Y_t}{h(t)} = -\frac{1}{2}|\tilde{p}_t - \theta_t|^2 + \frac{1}{2}|\theta_t|^2 + \frac{\alpha Y_t}{h(t)} \le -\frac{1}{2}\operatorname{dist}_t^2(\theta, P) + \frac{1}{2}|\theta_t|^2 + \frac{\alpha Y_t}{h(t)},$$

and the inequality becomes an equality if and only if

$$\tilde{p} \in \Pi_P(\theta).$$

Furthermore,

$$\frac{\alpha}{h}\log(\tilde{c}) + r + \tilde{e} - \tilde{c} \le \max_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right) + r + \tilde{e},$$

where equality is attained if and only if

$$\tilde{c} \in \arg\max_{\tilde{c}\in C}\left(\frac{\alpha}{h}\log(\tilde{c}) - \tilde{c}\right).$$

It follows that for every admissible pair $(\tilde{c}, \tilde{p})$, the process $R^{(c,p)}$ is a local supermartingale. But it can be seen from (4.10) that the local martingale part of $R^{(c,p)}$ is a true martingale and its finite variation part is of integrable total variation. So $R^{(c,p)}$ is a supermartingale and one obtains

$$R_0^{(c,p)} \ge \mathbb{E}\left[R_T^{(c,p)}\right].$$

If $(\tilde{c}^*, \tilde{p}^*)$ satisfies (4.9), then the pair is in $C \times P$ and $\log(\tilde{c}^*)$ as well as $\tilde{p}^*$ are bounded. It follows that $(\tilde{c}^*, \tilde{p}^*)$ is admissible and the corresponding process R^* is a martingale. We conclude that

$$R_0^* = \mathbb{E}\left[R_T^*\right],$$

which shows that $(\tilde{c}^*, \tilde{p}^*)$ is optimal. Finally, if C and P are $\mathcal{P}$-convex, there exists just one pair $(\tilde{c}^*, \tilde{p}^*)$ satisfying condition (4.9), and the proof is complete. □

Example 4.3 *If consumption is unconstrained, that is $C = \mathcal{P}$, then*

$$\tilde{c}^* = \frac{\alpha}{h}, \quad \max_{\tilde{c} \in C} \left(\frac{\alpha}{h} \log(\tilde{c}) - \tilde{c} \right) = \frac{\alpha}{h} \left(\log \left(\frac{\alpha}{h} \right) - 1 \right),$$

and the driver (4.7) *becomes*

$$f(t, y) = \frac{1}{2} \text{dist}_t^2 (\theta, P) - \frac{1}{2} |\theta_t|^2 - \frac{\alpha y}{h(t)} - \frac{\alpha}{h(t)} \left(\log \left(\frac{\alpha}{h(t)} \right) - 1 \right) - r - \tilde{e}_t.$$

4.2 Power utility

Let us now turn to the case $u(x) = x^\gamma / \gamma$ for $\gamma \in (-\infty, 0) \cup (0, 1)$. The definition of admissible strategies is slightly different for $\gamma > 0$ and $\gamma < 0$. But the optimal value of the optimization problem (2.1) and the optimal strategies will in both cases be of the same form.

Definition 4.4 *In the case $\gamma > 0$, an admissible strategy is a pair $(\tilde{c}, \tilde{p}) \in C \times P$ such that*

$$\tilde{c} \geq 0 \quad \textit{and} \quad \int_0^T \tilde{c}_t dt + \int_0^T |\tilde{p}_t|^2 dt < \infty \quad \mathbb{P}\textit{-almost surely.}$$

For $\gamma < 0$, we additionally require the process $(X^{(c,p)})^\gamma$ to be of class (D) *and $\mathbb{E} \left[\int_0^T c_t^\gamma dt \right] < \infty$.*

Note that for $\gamma < 0$, the condition $\mathbb{E} \left[\int_0^T c_t^\gamma dt \right] < \infty$ implies $\tilde{c} > 0$.

For every continuous bounded process Y we define

$$\max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right) := \operatorname*{ess\,sup}_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right), \tag{4.11}$$

where ess sup denotes the smallest upper bound with respect to $\nu \otimes \mathbb{P}$-almost everywhere ordering. By our assumption (4.2), (4.11) defines a bounded predictable process. We denote the set of all processes in C which attain the ess sup by

$$\arg\max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right). \tag{4.12}$$

It follows from Cheridito et al. [4] that (4.12) is not empty and contains exactly one process if C is $\mathcal{P}$-convex.

Consider the BSDE

$$Y_t = \int_t^T f(s, Y_s, Z_s) ds + \int_t^T Z_s dW_s \tag{4.13}$$

with driver

$$f(t, y, z) = \gamma \left(\frac{1 - \gamma}{2} \text{dist}_t^2 \left(\frac{z + \theta}{1 - \gamma}, P \right) - \frac{|z + \theta_t|^2}{2(1 - \gamma)} - \frac{1}{2\gamma} |z|^2 - \max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^y - \tilde{c} \right)_t - r - \tilde{e}_t + \frac{\beta}{\gamma} \right). \tag{4.14}$$

Note that $f(t, y, z)$ grows exponentially in y. But with a truncation argument it can be deduced from the results in Kobylanski [9] and Morlais [10] that equation (4.13) has a unique solution (Y, Z) such that Y is bounded and Z is in $\mathcal{P}_{\text{BMO}}^{1 \times n}$.

Theorem 4.5 *If $u(x) = x^\gamma/\gamma$ for $\gamma \in (-\infty, 0) \cup (0, 1)$, the optimal value of the optimization problem (2.1) over all admissible strategies is*

$$\frac{1}{\gamma} x^\gamma e^{-Y_0}, \tag{4.15}$$

and $(\tilde{c}^*, \tilde{p}^*)$ is an optimal admissible strategy if and only if

$$\tilde{c}^* \in \arg\max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right) \quad \text{and} \quad \tilde{p}^* \in \Pi_P \left(\frac{Z + \theta}{1 - \gamma} \right). \tag{4.16}$$

If C and P are $\mathcal{P}$-convex, then the optimal strategy $(\tilde{c}^*, \tilde{p}^*)$ is unique.

Proof. For every admissible strategy $(\tilde{c}, \tilde{p})$ define the process

$$R_t^{(c,p)} = e^{-\beta t} \frac{1}{\gamma} \left(X_t^{(c,p)} \right)^\gamma e^{-Y_t} + \int_0^t \alpha e^{-\beta s} \frac{1}{\gamma} c_s^\gamma ds.$$

Then

$$R_0^{(c,p)} = \frac{1}{\gamma} x^\gamma e^{-Y_0}, \quad R_T^{(c,p)} = e^{-\beta T} \frac{1}{\gamma} \left(X_T^{(c,p)} \right)^\gamma + \int_0^T \alpha e^{-\beta s} \frac{1}{\gamma} c_s^\gamma ds$$

and

$$dR_t^{(c,p)} = e^{-\beta t} \left(X_t^{(c,p)} \right)^\gamma e^{-Y_t} \left[\left(\tilde{p}_t + \frac{1}{\gamma} Z_t \right) dW_t + A_t^{(c,p)} dt \right],$$

where

$$\begin{aligned} A_t^{(c,p)} = \quad & \tilde{p}_t (Z_t + \theta_t) + \frac{1}{2} (\gamma - 1) |\tilde{p}_t|^2 + \frac{1}{2\gamma} |Z_t|^2 + \frac{1}{\gamma} f(t, Y_t, Z_t) \\ & + \frac{\alpha}{\gamma} \tilde{c}_t^\gamma e^{Y_t} + \tilde{e}_t - \tilde{c}_t + r - \frac{\beta}{\gamma}. \end{aligned}$$

First note that

$$\begin{aligned} & \tilde{p}_t (Z_t + \theta_t) + \frac{1}{2} (\gamma - 1) |\tilde{p}_t|^2 + \frac{1}{2\gamma} |Z_t|^2 \\ = \quad & \frac{\gamma - 1}{2} \left| \tilde{p}_t - \frac{Z_t + \theta_t}{1 - \gamma} \right|^2 + \frac{1}{2(1 - \gamma)} |Z_t + \theta_t|^2 + \frac{1}{2\gamma} |Z_t|^2 \\ \leq \quad & \frac{\gamma - 1}{2} \mathrm{dist}_t^2 \left(\frac{Z + \theta}{1 - \gamma}, P \right) + \frac{1}{2(1 - \gamma)} |Z_t + \theta_t|^2 + \frac{1}{2\gamma} |Z_t|^2, \end{aligned}$$

and the inequality becomes an equality if and only if

$$\tilde{p} \in \Pi_P \left(\frac{Z + \theta}{1 - \gamma} \right).$$

Furthermore,

$$\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y + \tilde{e} - \tilde{c} + r - \frac{\beta}{\gamma} \leq \max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right) + \tilde{e} + r - \frac{\beta}{\gamma},$$

where equality is attained if and only if

$$\tilde{c} \in \arg\max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^\gamma e^Y - \tilde{c} \right).$$

If $\gamma > 0$, the process $R^{(c,p)}$ is for every admissible pair $(\tilde{c}, \tilde{p})$, a positive local supermartingale, and therefore a supermartingale. In particular,

$$R_0^{(c,p)} \geq \mathbb{E}\left[R_T^{(c,p)}\right]. \tag{4.17}$$

Now let $(\tilde{c}^*, \tilde{p}^*)$ be a strategy satisfying (4.16). Then $\tilde{c}^*$ is non-negative and bounded. Moreover, since Z is in $\mathcal{P}^{1\times n}_{\mathrm{BMO}}$, $\tilde{p}^*$ is again in $\mathcal{P}^{1\times n}_{\mathrm{BMO}}$. It follows that the pair $(\tilde{c}^*, \tilde{p}^*)$ is admissible. Furthermore,

$$X_t^{(c^*,p^*)} = x\,\mathcal{E}\left(\tilde{p}^* \cdot W^{\mathbb{Q}}\right)_t \exp\left(\int_0^t (r + \tilde{e}_s - \tilde{c}^*_s) ds\right) \leq M\mathcal{E}\left(\tilde{p}^* \cdot W^{\mathbb{Q}}\right)_t \tag{4.18}$$

for some constant $M \in \mathbb{R}_+$. Choose $\gamma < \gamma' < 1$ and let $\varepsilon = 1 - \gamma'$. Since θ is bounded, one has $\mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(\theta \cdot W^{\mathbb{Q}}\right)_T^{1/\varepsilon}\right] < \infty$, and by Hölder's inequality, one obtains for every stopping time $\tau \leq T$,

$$\begin{aligned}
\mathbb{E}\left[(X_\tau^{(c^*,p^*)})^{\gamma'}\right] &= \mathbb{E}_{\mathbb{Q}}\left[(X_\tau^{(c^*,p^*)})^{\gamma'}\mathcal{E}(\theta \cdot W^{\mathbb{Q}})_T\right] \\
\leq \; \mathbb{E}_{\mathbb{Q}}\left[(X_\tau^{(c^*,p^*)})\right]^{\gamma'} & \mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(\theta \cdot W^{\mathbb{Q}}\right)_T^{1/\varepsilon}\right]^{\varepsilon} \leq M^{\gamma'}\mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}\left(\theta \cdot W^{\mathbb{Q}}\right)_T^{1/\varepsilon}\right]^{\varepsilon}.
\end{aligned}$$

It follows that $(X^{(c^*,p^*)})^\gamma$ is of class (D) and R^* a martingale. In particular,

$$R_0^* = \mathbb{E}\left[R_T^*\right].$$

This shows that $(\tilde{c}^*, \tilde{p}^*)$ is optimal.

If $\gamma < 0$, $R^{(c,p)}$ is for every admissible pair $(\tilde{c}, \tilde{p})$ a supermartingale due to our assumption that $(X^{(c,p)})^\gamma$ is of class (D) and $\mathbb{E}\left[\int_0^T c_t^\gamma dt\right] < \infty$. So again,

$$R_0^{(c,p)} \geq \mathbb{E}\left[R_T^{(c,p)}\right].$$

If $(\tilde{c}^*, \tilde{p}^*)$ satisfies (4.16), $u(\tilde{c}^*) - \tilde{c}^*$ is bounded and $\tilde{p}^*$ belongs to $\mathcal{P}^{1\times n}_{\mathrm{BMO}}$. Moreover, $-R^*$ is a positive local martingale. So $-R^*$ is a supermartingale and $\mathbb{E}[-R_T^*] < \infty$. It follows that

$$\mathbb{E}\left[(X_T^*)^\gamma + \int_0^T (c_t^*)^\gamma dt\right] < \infty,$$

which by (4.1), implies

$$\mathbb{E}\left[\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_T^\gamma\right] < \infty.$$

From Jensen's inequality, one obtains for every stopping time $\tau \leq T$,

$$\begin{aligned}
\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_\tau^\gamma &\leq \left(\mathbb{E}_{\mathbb{Q}}\left[\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_T^{\gamma/2} | \mathcal{F}_\tau\right]\right)^2 \\
&= \left(\mathbb{E}\left[\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_T^{\gamma/2}\frac{\mathcal{E}(-\theta \cdot W)_T}{\mathcal{E}(-\theta \cdot W)_\tau} | \mathcal{F}_\tau\right]\right)^2 \\
&\leq \mathbb{E}\left[\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_T^\gamma | \mathcal{F}_\tau\right]\mathbb{E}\left[\frac{\mathcal{E}(-\theta \cdot W)_T^2}{\mathcal{E}(-\theta \cdot W)_\tau^2} | \mathcal{F}_\tau\right] \\
&\leq N\mathbb{E}\left[\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})_T^\gamma | \mathcal{F}_\tau\right]
\end{aligned}$$

for some constant $N \in \mathbb{R}_+$. This shows that $\mathcal{E}(\tilde{p}^* \cdot W^{\mathbb{Q}})^\gamma$ and $(X^*)^\gamma$ are of class (D). Hence, R^* is a martingale and $R_0^* = \mathbb{E}\left[R_T^*\right]$, which shows that $(\tilde{c}^*, \tilde{p}^*)$ is optimal. If C and P are $\mathcal{P}$-convex, then there exists only one pair $(\tilde{c}^*, \tilde{p}^*)$ satisfying condition (4.16). □

Example 4.6 *If consumption is unconstrained, that is $C = \mathcal{P}$, then*

$$c^* = \alpha^{1/(1-\gamma)} e^{Y_t/(1-\gamma)}, \quad \max_{\tilde{c} \in C} \left(\frac{\alpha}{\gamma} \tilde{c}^{\gamma} e^{y} - \tilde{c} \right) = \frac{1-\gamma}{\gamma} \alpha^{1/(1-\gamma)} e^{y/(1-\gamma)},$$

and the driver (4.14) *becomes*

$$f(t,y,z) = \gamma \left(\frac{1-\gamma}{2} \text{dist}_t^2 \left(\frac{z+\theta}{1-\gamma}, P \right) - \frac{|z+\theta_t|^2}{2(1-\gamma)} - \frac{1}{2\gamma}|z|^2 - \frac{1-\gamma}{\gamma} \alpha^{1/(1-\gamma)} e^{y/(1-\gamma)} - r - \tilde{e}_t + \frac{\beta}{\gamma} \right).$$

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