A TIME BEFORE WHICH INSIDERS WOULD NOT UNDERTAKE RISK

CONSTANTINOS KARDARAS

Abstract. In a continuous-path semimartingale market model, we perform an initial enlargement of the filtration by including the overall minimum of the numéraire portfolio. We establish that all discounted asset-price processes, when stopped at the time of the overall minimum of the numéraire portfolio, become local martingales under the enlarged filtration. This implies that risk-averse insider traders would refrain from investing in the risky assets before that time. A partial converse to the previous result is also established, showing that the time of the overall minimum of the numéraire portfolio is in a certain sense unique in rendering undesirable the act of undertaking risky positions before it. Our results shed light to the importance of the numéraire portfolio as an indicator of overall market performance.

0. Introduction

When modeling insider trading, one usually enlarges the "public" information flow by including knowledge of a non-trivial random variable, which represents the extra information of the insider, from the very beginning. (This method called *initial filtration enlargement*, as opposed to *progressive filtration enlargement* — for more details, see [11, Chapter VI].) It is then of interest to explore the effect that the extra information has on the trading behavior of the insider — for an example, see [1]. Under this light, the topic of the present paper may be considered slightly unorthodox, as we identify an initial filtration enlargement and a stopping time of the enlarged filtration (which is *not* a stopping time of the original filtration) with the property that risk-averse insider traders would refrain from taking risky positions before that time. As will be revealed, this apparently "negative" result, though not helpful in the theory of insider trading, sheds more light to the importance of the numéraire portfolio as an indicator of overall market performance.

Our setting is a continuous-path semimartingale market model with d asset-price processes $S^1, \ldots, S^d$. All wealth is discounted with respect to some riskless asset, or money market account. Natural structural assumptions are imposed — in particular, we only enforce a mild market viability condition, and allow for the existence of some discounted wealth process that will grow unconditionally as time goes to infinity. Such assumptions are satisfied in every reasonable infinite time-horizon model. In such an environment, the numéraire portfolio — an appellation coined in [9] — is the unique nonnegative wealth process $\widehat{X}$ with unit initial capital such that all processes

Date: October 12, 2010.

2000 *Mathematics Subject Classification.* 60G07, 60G44.

$S^i/\widehat{X}$, $i \in \{1, \ldots, d\}$, become local martingales. The numéraire portfolio has several interesting optimality properties. For instance, it maximizes expected logarithmic utility for all time-horizons and achieves maximal long-term growth — for more information, check [6]. The goal of the present paper add yet one more to the remarkable list of properties of the numéraire portfolio.

The original filtration $\mathbf{F}$ is enlarged to $\mathbf{G}$, which further contains information on the overall minimum level $\min_{t \in \mathbb{R}_+} \widehat{X}(t)$ of the numéraire portfolio. In particular, the time ρ that this overall minimum is achieved (which can be shown to be almost surely unique) becomes a stopping time with respect to $\mathbf{G}$. Our first main result states that all S^i, $i \in \{1, \ldots, d\}$, become local martingales up to time ρ under the *enlarged filtration* $\mathbf{G}$ and *original probability* $\mathbb{P}$. Note that the asset-price processes are discounted by the money market account, and not by the numéraire portfolio, which makes asset price processes local martingales under $\mathbf{F}$. In essence, $\mathbb{P}$ becomes a risk-neutral measure for the model with enlarged filtration up to time ρ. An immediate consequence of this fact is that a risk-averse investor would refrain from taking risky positions up to time ρ, since they would result in no compensation (in terms of excess return relative to the money market account) for the risk that is being undertaken. (Note, however, that an insider can arbitrage unconditionally after time ρ with *no* downside risk whatsoever involved, simply by taking immediately after ρ arbitrarily large long positions in the the numéraire.) In effect, trading in the market occurs simply because traders do not have information about the time of the overall minimum of the numéraire. In fact, until time ρ, not only the numéraire, but the whole market performs badly, since the expected outcome of any portfolio at time ρ is necessarily less or equal than the initial capital used to set it up.

A partial converse to the previous result is also presented. Under an extra completeness condition on the market, we show that if a random time ϕ, coming from the specific class of honest times that avoid all stopping times, is such that $\mathbb{E}[X(\phi)] \leq X(0)$ holds for any nonnegative wealth process X formed by trading with information $\mathbf{F}$, then ϕ is necessarily equal to the time of the overall minimum of the numéraire. Combined with our first main result, this clarifies the unique role of the numéraire as an indicator of market performance.

The structure of the remainder of the paper is simple. In Section 1 the results are presented, while Section 2 contains the proofs.

1. Results

1.1. **The set-up.** Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space — here, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\mathbf{F} = (\mathcal{F}(t))_{t \in \mathbb{R}_+}$ is a right-continuous filtration such that, for each $t \in \mathbb{R}_+$, $\mathcal{F}(t) \subseteq \mathcal{F}$ and $\mathcal{F}(t)$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ — in other words, $\mathbf{F}$ satisfies the *usual conditions*. Without affecting in any way the generality of our discussion, we shall be assuming that $\mathcal{F}(0)$ is trivial modulo $\mathbb{P}$. Relationships involving random variables are to be understood in the $\mathbb{P}$-a.s. sense; relationships involving processes hold modulo evanescence.

On $(\Omega, \mathbf{F}, \mathbb{P})$, let $S = (S^i)_{i=1,\dots,d}$ be a vector-valued semimartingale with continuous paths. For each $i \in \{1, \dots, d\}$, S^i represents the discounted, with respect to some baseline security, price of a liquid asset in the market. This baseline security should be thought as a locally riskless money market account, whereas the other assets represent riskier investments. We also set $S^0 := 1$ to denote the wealth accumulated by the baseline security, discounted by itself.

Starting with capital $x \in \mathbb{R}_+$, and investing according to some d-dimensional, $\mathbf{F}$-predictable and S-integrable strategy ϑ (modeling the number of liquid assets held in the portfolio), an economic agent's discounted wealth is given by $X^{x,\vartheta} = x + \int_0^\cdot \vartheta^\top(t) \mathrm{d}S(t)$. We define $\mathcal{X}_\mathbf{F}(x)$ as the set of all processes $X^{x,\vartheta}$ in the previous notation that remain nonnegative at all times. We also set $\mathcal{X}_\mathbf{F} := \bigcup_{x \in \mathbb{R}_+} \mathcal{X}_\mathbf{F}(x)$.

Below, we gather some definitions and results that have appeared previously in the literature; more information can be found in [6] and, for the special case of continuous-path semimartingales that is considered here, in [7, Section 4].

Definition 1.1. We shall say that the market allows for *arbitrage of the first kind* if there exists $T \in \mathbb{R}_+$ and an $\mathcal{F}(T)$-measurable random variable ξ with $\mathbb{P}[\xi \geq 0] = 1$, $\mathbb{P}[\xi > 0] > 0$, such that for all $x > 0$ there exists $X \in \mathcal{X}(x)$ satisfying $\mathbb{P}[X(T) \geq \xi] = 1$. If the market does not allow for any arbitrage of the first kind, we say that condition NA_1 holds.

Condition NA_1 is weaker than the "No Free Lunch with Vanishing Risk" market viability condition of [2], and is actually equivalent to the requirement that $\lim_{\ell \to \infty} \sup_{X \in \mathcal{X}^\mathbf{F}(x)} \mathbb{P}\left[X(T) > \ell\right] = 0$ holds for all $x \in \mathbb{R}_+$ and $T \in \mathbb{R}_+$ — see [7, Proposition 1]. The latter boundedness-in-probability requirement is coined condition BK in [5] and condition "No Unbounded Profit with Bounded Risk" (NUPBR) in [6].

Definition 1.2. A strictly positive local martingale deflator is a strictly positive process Y with $Y(0) = 1$ such that YS^i is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $i \in \{0, \dots, d\}$. (The last requirement is equivalent to asking that YX is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $X \in \mathcal{X}_\mathbf{F}$.) A strictly positive process $\widehat{X} \in \mathcal{X}_\mathbf{F}(1)$ will be called *the numéraire in the market* if $\widehat{Y} := 1/\widehat{X}$ is a (necessarily, strictly positive) local martingale deflator.

By Jensen's inequality, it is straightforward to see that if the numéraire $\widehat{X}$ exists in the market, then it is unique. Obviously, if the numéraire exists in the market then there exists at least one strictly positive local martingale deflator. Interestingly, the converse also holds, i.e., existence of the numéraire in the market is equivalent to existence of at least one strictly positive local martingale deflator. Furthermore, the previous are also equivalent to condition NA_1 holding in the market.

Condition NA_1 can also be described in terms of the asset-prices process drifts and volatilities. More precisely, consider the Doob-Meyer decomposition $S = A + M$ of the continuous-path semimartingale S, where $A = (A^1, \dots, A^d)$ has continuous paths and is of finite variation, and

$M = (M^1, \ldots, M^d)$ is a continuous-path local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. For $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, d\}$, denote by $[S^i, S^k]$ the quadratic (co)variation of S^i and S^k — of course, $[S^i, S^k] = [M^i, M^k]$. Also, let $[S, S]$ be the $d \times d$ nonnegative-definite symmetric matrix-valued process whose (i, k)-component is $[S^i, S^k]$ for $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, d\}$. Call now $G := \mathsf{trace}[S, S]$, where trace is the operator returning the trace of a matrix. Observe that G is an increasing, adapted, continuous process, and that there exists a $d \times d$ nonnegative-definite symmetric matrix-valued process c such that $[S^i, S^k] = \int_0^\cdot c^{i,k}(t) \mathrm{d}G(t)$ for $i \in \{1, \ldots, d\}$ and $k \in \{1, \ldots, d\}$; $[S, S] = \int_0^\cdot c(t) \mathrm{d}G(t)$ in short. Then, condition NA_1 is equivalent to the existence of a d-dimensional, predictable process ξ such that $A = \int_0^\cdot (c(t)\xi(t)) \mathrm{d}G(t)$, satisfying $\int_0^T \left(\xi^\top(t) c(t) \xi(t)\right) \mathrm{d}G(t) < \infty$ for all $T \in \mathbb{R}_+$. In fact, with the previous notation, it is straightforward to check that the numéraire in the market is given by $\widehat{X} = \mathcal{E}\left(\int_0^\cdot \xi^\top(t) \mathrm{d}S(t)\right)$, where "$\mathcal{E}$" denotes the stochastic exponential operator.

Definition 1.3. We shall say that *the discounting process is asymptotically suboptimal* if there exists $X \in \mathcal{X}_{\mathbf{F}}$ such that $\mathbb{P}\left[\lim_{t\to\infty} X(t) = \infty\right] = 1$.

The previous definition is self-explanatory — the discounting process is asymptotically suboptimal if it can be beaten unconditionally in the long run by some other wealth process in the market. As a simple example where the discounting process is asymptotically suboptimal, we mention any multi-dimensional Black-Scholes model such that the original probability is not a risk-neutral one.

Given condition NA_1, or equivalently the existence of the numéraire $\widehat{X}$, the condition that the discounting process is asymptotically suboptimal is equivalent to $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}(t) = \infty\big] = 1$; indeed, if there exists some $X \in \mathcal{X}_{\mathbf{F}}$ such that $\mathbb{P}\left[\lim_{t\to\infty} X(t) = \infty\right] = 1$, the supermartingale property of $X/\widehat{X}$ and Doob's nonnegative supermartingale convergence theorem give $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}(t) = \infty\big] = 1$. Furthermore, under condition NA_1, and with the notation used in the paragraph right before Definition 1.3, it is not hard to see that the discounting process is asymptotically suboptimal if and only if $\int_0^\infty \left(\xi^\top(t) c(t) \xi(t)\right) \mathrm{d}G(t) = \infty$.

1.2. **The first result.** For the purposes of §1.2, assume that condition NA_1 holds in the market and the the discounting process is asymptotically suboptimal. Recall that this is equivalent to existence of the numéraire $\widehat{X}$ in the market, which satisfies $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}(t) = \infty\big] = 1$.

Define the nonincreasing process $I := \inf_{t\in[0,\cdot]} \widehat{X}(t)$; then, $I(\infty) = \inf_{t\in\mathbb{R}_+} \widehat{X}(t)$ is the overall minimum of $\widehat{X}$. Let $\mathbf{G} = (\mathcal{G}(t))_{t\in\mathbb{R}_+}$ be the smallest filtration satisfying the usual hypotheses, containing $\mathbf{F}$, and making $I(\infty)$ a $\mathcal{G}(0)$-measurable random variable. Consider any random time ρ such that $\widehat{X}(\rho) = \inf_{t\in\mathbb{R}_+} \widehat{X}(t) = I(\infty)$ — in other words, $\widehat{X}$ achieves at ρ its overall minimum. Since $\mathbb{P}\big[\lim_{t\to\infty} \widehat{X}(t) = \infty\big] = 1$, such a time is $\mathbb{P}$-a.s. finite — in fact, it is also $\mathbb{P}$-a.s. unique, as will be revealed in Theorem 1.4 below. Therefore, $\mathbb{P}$-a.s., $\rho = \inf\big\{t \in \mathbb{R}_+ \mid \widehat{X}(t) = I(\infty)\big\}$, the latter being a stopping time on $(\Omega, \mathbf{G})$; since $\mathcal{G}(0)$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$, it follows that ρ is a stopping time on $(\Omega, \mathbf{G})$. Therefore, $\mathbf{G}$ is strictly larger than the smallest filtration that satisfies the usual hypotheses, contains $\mathbf{F}$ and makes ρ a stopping time.

What follows is the first result of the paper — its proof is given in Section 2.

Theorem 1.4. *Assume that condition* NA_1 *holds and that the discounting process is asymptotically suboptimal. Then, the time of minimum of* $\widehat{X}$ *is* $\mathbb{P}$*-a.s. unique. With* ρ *denoting such a time, the process* $S^\rho = (S(\rho \wedge t))_{t \in \mathbb{R}_+}$ *is a local martingale on* $(\Omega, \mathbf{G}, \mathbb{P})$.

Remark 1.5. The result of Theorem 1.4 does not appear to follow directly from known results in the theory of filtration enlargements. In particular:

- In order to use the theory of initial enlargement of filtrations, the random variable $I(\infty)$ must satisfy the so-called *Jacod's criterion* [3], which states that the conditional law of $I(\infty)$ given $\mathcal{F}(t)$ is absolutely continuous with respect to its unconditional law for all $t \in \mathbb{R}_+$. However, the conditional law of $I(\infty)$ given $\mathcal{F}(t)$ has a Dirac component of mass $1 - I(t)\widehat{Y}(t)$ at the point $I(t)$ as follows from Doob's maximal identity [10, Lemma 2.1], while the unconditional law of $I(\infty)$ is standard uniform. Therefore, Jacod's criterion fails.
- The Jeulin-Yor semimartingale decomposition result (see [4]) cannot be utilized, because this is not a case of progressive filtration enlargement. Furthermore, as already noted, the filtration $\mathbf{G}$ is strictly larger than the smallest filtration that satisfies the usual hypotheses, contains $\mathbf{F}$, and makes ρ a stopping time.

One could use the general results of [10] in order to establish the validity of Theorem 1.4; however, we provide here a self-contained simple proof that sheds light in a financial context, as it involves the concepts of local martingale deflators and martingale measures.

Remark 1.6. Theorem 1.4 justifies the title of the paper. With the insider information flow $\mathbf{G}$, investing in the risky assets before time ρ gives the same instantaneous return as the locally riskless asset, but entails (locally) higher risk; therefore, before ρ an insider would not be willing to take any position on the risky assets. Let us make the point more precise. Let $\mathcal{X}^\rho_{\mathbf{G}}$ be the class of nonnegative processes of the form $x + \int_0^\cdot \vartheta^\top(t) \mathrm{d}S^\rho(t)$, where now x is $\mathcal{G}(0)$-measurable and ϑ is $\mathbf{G}$-predictable and S^ρ-integrable. By Theorem 1.4, all processes in $\mathcal{X}^\rho_{\mathbf{G}}$ are nonnegative local martingales on $(\Omega, \mathbf{G}, \mathbb{P})$, which implies that they are nonnegative supermartingales on $(\Omega, \mathbf{G}, \mathbb{P})$. Therefore, $\mathbb{E}[X(\rho) \mid I(\infty)] \le X(0)$ holds for all $X \in \mathcal{X}^\rho_{\mathbf{G}}$. (In particular, $\mathbb{E}[X(\rho)] \le X(0)$ holds for all $X \in \mathcal{X}_{\mathbf{F}}$, which sharpens the conclusion of [8, Theorem 2.15] for continuous-path semimartingale models.) Jensen's inequality then implies that any expected utility maximizer having an increasing and concave utility function, information flow $\mathbf{G}$, and time-horizon before ρ, would not invest at all in the risky assets.

Remark 1.7. At first sight, Theorem 1.4 appears counterintuitive. If the overall minimum of $\widehat{X}$ is known from the outset exactly, and especially if it is going to to be extremely low, taking an opposite (short) position in it should ensure particularly good performance at the time of the overall minimum of $\widehat{X}$. Of course, admissibility constraints prevent one from taking an *absolute* short position on the numéraire; still, one can imagine that a *relative* short position on the numéraire should result in something substantial. To understand better why this intuition fails, remember that $\widehat{X} = \mathcal{E}\left(\int_0^\cdot \xi^\top(t)\mathrm{d}S(t)\right)$ in the notation of §1.1. A relative short position would result in the wealth $X = \mathcal{E}\left(-\int_0^\cdot \xi^\top(t)\mathrm{d}S(t)\right)$. Straightforward computations show that

$$X(\rho) = \frac{1}{\widehat{X}(\rho)} \exp\left(-\int_0^\rho \left(\xi^\top(t)c(t)\xi(t)\right)\mathrm{d}G(t)\right)$$

Even though $\widehat{X}(\rho)$ can be very close to zero, the term $\exp\left(-\int_0^\rho \left(\xi^\top(t)c(t)\xi(t)\right)\mathrm{d}G(t)\right)$ will compensate for the small values of $\widehat{X}(\rho)$. In effect, the cumulative volatility of the numéraire up to the time of its overall minimum will eliminate any chance of profit by taking short positions on it.

1.3. **A partial converse to Theorem 1.4.** In Remark 1.6, we argued that $\mathbb{E}[X(\rho)] \leq X(0)$ holds for all $X \in \mathcal{X}_{\mathbf{F}}$. We shall now present a partial converse of the previous result in a special case. Before stating the result, some definitions are needed.

Definition 1.8. Consider a market as described in §1.1, satisfying condition NA_1. The market will be called *complete* if for any stopping time τ and any $\mathcal{F}_\tau$-measurable nonnegative random variable H_τ with $\mathbb{E}\big[\widehat{Y}_\tau H_\tau\big] < \infty$, there exists $X \in \mathcal{X}_{\mathbf{F}}$ such that $X_\tau = H_\tau$.

Remark 1.9. A market as described in §1.1 satisfies condition NA_1 *if and only if* there exists at least one strictly positive supermartingale deflator. It can be actually shown that the market is further complete in the sense of Definition 1.8 *if and only if* there exists a unique strictly positive supermartingale deflator — the proof is similar to the one for the case where an equivalent martingale measure exists in the market. In fact, it can be shown that in a complete market, for any stopping time τ and $\mathcal{F}_\tau$-measurable nonnegative random variable H_τ, we have

$$\mathbb{E}\big[\widehat{Y}_\tau H_\tau\big] = \min\left\{x \in \mathbb{R}_+ \mid \text{ there exists } X \in \mathcal{X}_{\mathbf{F}}(x) \text{ with } X_\tau = H_\tau\right\},$$

which gives a formula for the minimal hedging price of the payoff H_τ delivered at time τ.

Definition 1.10. Let ϕ be a random time on $(\Omega, \mathbf{F}, \mathbb{P})$. If $\mathbb{P}[\phi = \tau] = 0$ holds for all stopping times τ on $(\Omega, \mathbf{F})$, we shall say that ϕ *avoids all stopping times on* $(\Omega, \mathbf{F}, \mathbb{P})$. Furthermore, ϕ will be called an honest time on $(\Omega, \mathbf{F})$ if for all $t \in \mathbb{R}_+$ there exists an $\mathcal{F}_t$-measurable random variable ϕ_t such that $\phi = \phi_t$ holds on $\{\phi \leq t\}$.

It is not hard to see that ρ, as defined in §1.2, is an honest time that avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$. The next result shows that, if the market is viable and complete, ρ is the *unique* honest time that avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$, and with the property that a wealth processes sampled at the random time has expectation dominated by its initial capital.

Theorem 1.11. *Assume that condition* NA_1 *holds and that the market is complete. Let* ϕ *be an honest time that avoids all stopping times on* $(\Omega, \mathbf{F}, \mathbb{P})$*, such that* $\mathbb{E}[X(\phi)] \leq X(0)$ *holds for all* $X \in \mathcal{X}_{\mathbf{F}}$*. Then, the discounting process is asymptotically suboptimal and* $\phi = \rho$*.*

2. Proofs

2.1. **Proof of Theorem 1.4.** We first show that ρ is $\mathbb{P}$-a.s. unique. Define the random times $\rho' := \inf\{t \in \mathbb{R}_+ \mid \widehat{X}(t) = I(\infty)\}$ and $\rho'' := \sup\{t \in \mathbb{R}_+ \mid \widehat{X}(t) = I(\infty)\}$. Since $\widehat{Y} := 1/\widehat{X}$ a nonnegative local martingale that vanishes at infinity on $(\Omega, \mathbf{F}, \mathbb{P})$, one can show [8, proof of Theorem 2.14] that $\mathbb{P}[\rho' > t \mid \mathcal{F}(t)] = \mathbb{P}[\rho'' > t \mid \mathcal{F}(t)] = I(t)\widehat{Y}(t)$ for all $t \in \mathbb{R}_+$. The previous imply that ρ' and ρ'' have the same law under $\mathbb{P}$. Since $\rho' \leq \rho''$, it follows that $\mathbb{P}[\rho' = \rho''] = 1$. Furthermore, since for any time ρ of minimum of $\widehat{X}$ we have $\rho' \leq \rho \leq \rho''$, it follows that the time of minimum of $\widehat{X}$ is $\mathbb{P}$-a.s. unique.

Continuing, recall that $\widehat{Y}$ is a nonnegative local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ such that $\widehat{Y}(0) = 1$ and $\mathbb{P}\big[\lim_{t\to\infty} \widehat{Y}(t) = 0\big] = 1$. Observe that ρ is a time of overall maximum of $\widehat{Y}$ and that $1/I = \sup_{t\in[0,\cdot]} \widehat{Y}(t)$. Clearly, we have $I(\rho) = I(\infty)$. Define also the nonnegative nondecreasing process $U = 1 - I$. Both $I(\infty)$ and $U(\infty) = U(\rho)$ have the standard uniform law under $\mathbb{P}$.

For all $u \in [0, 1)$ define $\eta_u := \inf\{t \in \mathbb{R}_+ \mid \widehat{Y}(t) = 1/(1-u)\}$; then, $(\eta_u)_{u\in[0,1)}$ is a nondecreasing collection of stopping times on $(\Omega, \mathbf{F})$. Observe that $\mathbb{P}\big[\sup_{t\in\mathbb{R}_+} \widehat{Y}^{\eta_u}(t) \leq 1/(1-u)\big] = 1$ holds for all $u \in [0, 1)$. Also, $\mathbb{P}[\eta_u < \infty] = 1 - u$ for all $u \in [0, 1)$, as follows from Doob's maximal identity. For $u \in [0, 1)$, let $\mathbb{P}_u$ be the probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$ conditioned on $\{\eta_u < \infty\}$; as $\mathbb{P}\big[\widehat{Y}(\eta_u) = (1/(1-u))\,\mathbb{I}_{\{\eta_u<\infty\}}\big] = 1$, it follows that $\mathbb{P}_u$ is absolutely continuous with respect to $\mathbb{P}$, and that $\mathrm{d}\mathbb{P}_u/\mathrm{d}\mathbb{P} = \widehat{Y}(\eta_u) = (1/(1-u))\,\mathbb{I}_{\{\eta_u<\infty\}}$, for all $u \in [0, 1)$. We use "$\mathbb{E}_u$" to denote expectation under $\mathbb{P}_u$ for $u \in [0, 1)$ and "$\mathbb{E}$" to denote expectation under $\mathbb{P} = \mathbb{P}_0$.

Remark 2.1. Since all $\widehat{Y}S^i$, $i \in \{1, \ldots, d\}$, are local martingales on $(\Omega, \mathbf{F}, \mathbb{P})$, it follows that S^{η_u} is a local martingale in $(\Omega, \mathbf{F}, \mathbb{P}_u)$ for all $u \in [0, 1)$. In other words, $\mathbb{P}_u$ is an absolutely continuous local martingale measure for S^{η_u} for all $u \in [0, 1)$.

In order to prove Theorem 1.4, we shall use the following auxiliary result.

Lemma 2.2. *For all $u \in [0, 1)$, $\mathbb{P}_u[\eta_u < \infty] = 1$ holds; in particular, $\mathbb{P}_u[U(\eta_u) = u] = 1$. Furthermore, for any bounded and d-dimensional process V that is optional on $(\Omega, \mathbf{F})$, we have*

$$\mathbb{E}[V(\rho)] = \mathbb{E}\left[\int_{\mathbb{R}_+} V(t)\widehat{Y}(t)\mathrm{d}U(t)\right] = \int_{[0,1)} \mathbb{E}_u[V(\eta_u)]\,\mathrm{d}u. \tag{2.1}$$

Proof. First of all, we have $\mathbb{P}_u[\eta_u < \infty] = \mathbb{E}[(1/(1-u))\mathbb{I}_{\{\eta_u<\infty\}}] = (1/(1-u))\mathbb{P}[\eta_u < \infty] = 1$.

In order to establish (2.1), start by observing that $\mathbb{P}[\rho > t \mid \mathcal{F}(t)] = I(t)\widehat{Y}(t)$ holds for all $t \in \mathbb{R}_+$, in view of Doob's maximal identity. Fix $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ with $s \leq t$. The definition of I and a use the integration-by-parts formula give $I(s)\widehat{Y}(s) - I(t)\widehat{Y}(t) = \log(I(s)) - \log(I(t)) - \int_s^t I(v)\mathrm{d}\widehat{Y}(v)$. From the latter equality, and upon using the bounds $0 \leq I\widehat{Y} \leq 1$, it easily follows that

$$\mathbb{P}[s < \rho \leq t \mid \mathcal{F}(s)] = \mathbb{E}\big[I(s)\widehat{Y}(s) - I(t)\widehat{Y}(t) \mid \mathcal{F}(s)\big] = \mathbb{E}\left[\log(I(s)) - \log(I(t)) \mid \mathcal{F}(s)\right].$$

As $-\log(I)$ is non-decreasing and adapted, we conclude that it coincides with the optional compensator (dual optional projection) of $\mathbb{I}_{[\![\rho,\infty[\![}$ on $(\Omega,\, \mathbf{F},\, \mathbb{P})$. In other words, we have

$$\begin{aligned}
\mathbb{E}[V(\rho)] &= \mathbb{E}\left[-\int_{\mathbb{R}_+} V(t)\frac{\mathrm{d}I(t)}{I(t)}\right] \\
&= \mathbb{E}\left[-\int_{\mathbb{R}_+} V(t)\widehat{Y}(t)\mathrm{d}I(t)\right] \\
&= \mathbb{E}\left[\int_{\mathbb{R}_+} V(t)\widehat{Y}(t)\mathrm{d}U(t)\right] \\
&= \mathbb{E}\left[\int_{[0,1)} V(\eta_u)\widehat{Y}(\eta_u)\mathbb{I}_{\{\eta_u<\infty\}}\mathrm{d}u\right] \\
&= \int_{[0,1)} \mathbb{E}\left[\widehat{Y}(\eta_u)V(\eta_u)\right]\mathrm{d}u \ = \ \int_{[0,1)} \mathbb{E}_u\left[V(\eta_u)\right]\mathrm{d}u,
\end{aligned}$$

the second equality following from the fact that $\int_{\mathbb{R}_+} \mathbb{I}_{\left\{\widehat{Y}(t)\neq 1/I(t)\right\}}\mathrm{d}I(t) = 0$ and the fourth by a simple time-change. The above establishes (2.1) and completes the proof of Lemma 2.2. □

Continuing with the proof of Theorem 1.4, note that we may assume that S is actually bounded via a simple localization argument. In all that follows, fix arbitrary $s \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$ with $s \leq t$, $B \in \mathcal{F}_s$, as well as a bounded deterministic function $f : [0,1) \mapsto \mathbb{R}_+$. A use of the π-λ theorem implies that in order for the result to hold, we only need to show that $\mathbb{E}\left[S^\rho(t)f(U(\infty))\mathbb{I}_B\right] = \mathbb{E}\left[S^\rho(s)f(U(\infty))\mathbb{I}_B\right]$. Further noticing that $\mathbb{P}[U(\infty) = U(\rho)] = 1$, and using the obvious equality $S^\rho(t)f(U(\rho))\mathbb{I}_B = S^\rho(s)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho\leq s\}} + S^\rho(t)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}$, one only needs to establish

$$\mathbb{E}\left[S^\rho(t)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}\right] = \mathbb{E}\left[S^\rho(s)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}\right] \tag{2.2}$$

Since S is assumed bounded, Remark 2.1 implies that S^{η_u} is a martingale on $(\Omega,\, \mathbf{F},\, \mathbb{P}_u)$ for all $u \in [0,1)$. Observe that the process $V := S^t f(U)\mathbb{I}_B\mathbb{I}_{]\!]s,\infty[\![}$ is optional on $(\Omega,\, \mathbf{F})$; furthermore, $V(\rho) = S^\rho(t)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}$. Therefore, from Lemma 2.2, recalling that $\mathbb{P}_u[U(\eta_u) = u]$ for all $u \in [0,1)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[S^\rho(t)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}\right] &= \int_{[0,1)} f(u)\mathbb{E}_u\left[S^{\eta_u}(t)\mathbb{I}_B\mathbb{I}_{\{\eta_u>s\}}\right]\mathrm{d}u \\
&= \int_{[0,1)} f(u)\mathbb{E}_u\left[S^{\eta_u}(s)\mathbb{I}_B\mathbb{I}_{\{\eta_u>s\}}\right]\mathrm{d}u \ = \ \mathbb{E}\left[S^\rho(s)f(U(\rho))\mathbb{I}_B\mathbb{I}_{\{\rho>s\}}\right],
\end{aligned}$$

which is exactly (2.2) and completes the proof of Theorem 1.4.

2.2. **Proof of Theorem 1.11.** To begin with, note that $(\Omega,\, \mathbf{F},\, \mathbb{P})$ supports only continuous local martingales. Indeed, otherwise there would exist a nontrivial strictly positive process N with $N(0) = 1$, such that N is a purely discontinuous local martingale on $(\Omega,\, \mathbf{F},\, \mathbb{P})$; but then, $N\widehat{Y}$ would be a strictly positive local martingale deflator in the market, which contradicts the uniqueness of the strictly positive local martingale deflator $\widehat{Y}$.

Since all local martingales on $(\Omega, \mathbf{F})$ are continuous and ϕ is an honest time that avoids all stopping times on $(\Omega, \mathbf{F}, \mathbb{P})$, [10, Theorem 4.1] implies that ϕ is the time of overall maximum of a nonnegative continuous local martingale L on $(\Omega, \mathbf{F}, \mathbb{P})$ with $L(0) = 1$ and $\mathbb{P}\left[\lim_{t\to\infty} L(t) = 0\right] = 1$. We shall show below that $L = \widehat{Y}$; this shows at the same time that $\phi = \rho$ and that the discounting process is asymptotically suboptimal, the latter following from $\mathbb{P}\left[\lim_{t\to\infty} L(t) = 0\right] = 1$.

As in the proof of Theorem 1.4, with L replacing $\widehat{Y}$ and ϕ replacing ρ, for all $u \in [0, 1)$ define $\eta_u := \inf\{t \in \mathbb{R}_+ \mid L(t) = 1/(1-u)\}$ and $\mathbb{P}_u$ via $\mathrm{d}\mathbb{P}_u = L(\eta_u)\mathrm{d}\mathbb{P} = (1/(1-u))\,\mathbb{I}_{\{\eta_u<\infty\}}$. Define the nondecreasing processes $L^* := \sup_{t\in[0,\cdot]L(t)}$ and $K := 1 - 1/L^*$. Following the reasoning of Lemma 2.2 (replacing $\widehat{Y}$ and U there by L and K respectively — note that in the proof of Lemma 2.2, we only use the facts that $\widehat{Y}$ is a nonnegative continuous local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ with $\widehat{Y}(0) = 1$ and $\mathbb{P}\big[\lim_{t\to\infty} \widehat{Y}(t) = 0\big] = 1$, properties that $\widehat{Y}$ shares with L), we obtain

$$\mathbb{E}\left[V(\phi)\right] = \mathbb{E}\left[\int_{\mathbb{R}_+} V(t)L(t)\mathrm{d}K(t)\right], \tag{2.3}$$

holding for all nonnegative optional process V on $(\Omega, \mathbf{F})$.

Lemma 2.3. *For a uniformly bounded $X \in \mathcal{X}_{\mathbf{F}}$, we have*

$$\mathbb{E}_u\left[\int_0^{\eta_u} (1 - K(t))\mathrm{d}X(t)\right] \leq 0, \text{ for all } u \in [0, 1). \tag{2.4}$$

Proof. Let $B := \int_{[0,\cdot]} X(t)\mathrm{d}K(t)$; clearly, B is a uniformly bounded nondecreasing continuous and adapted process on $(\Omega, \mathbf{F})$. Fix $u \in [0, 1)$. Using integration-by-parts, write

$$\begin{aligned}\int_{\mathbb{R}_+} X^{\eta_u}(t)L(t)\mathrm{d}K(t) &= \int_0^{\eta_u} L(t)\mathrm{d}B^{\eta_u}(t) + X(\eta_u)\int_{\eta_u}^{\infty} L(t)\mathrm{d}K(t)\\ &= L(\eta_u)B(\eta_u) - \int_0^{\eta_u} B(t)\mathrm{d}L(t) + X(\eta_u)\left(\log(L^*(\infty)) + \log(1-u)\right)\mathbb{I}_{\{\eta_u<\infty\}}.\end{aligned}$$

Now, observe that $\mathbb{E}\left[L(\eta_u)B(\eta_u)\right] = \mathbb{E}_u[B(\eta_u)] = \mathbb{E}_u[\int_0^{\eta_u} X(t)\mathrm{d}K(t)]$ and $\mathbb{E}\left[\int_0^{\eta_u} B(t)\mathrm{d}L(t)\right] = 0$, the latter following from the facts that B is uniformly bounded and L^{η_u} is a uniformly bounded martingale on $(\Omega, \mathbf{F}, \mathbb{P})$. Furthermore, using Doob's maximal identity we obtain that

$$\mathbb{E}\left[\log(L^*(\infty)) + \log(1-u) \mid \mathcal{F}(\eta_u)\right] = 1 \text{ holds on } \{\eta_u < \infty\}.$$

Therefore, $\mathbb{E}\left[X(\eta_u)\left(\log(L^*(\infty)) + \log(1-u)\right)\mathbb{I}_{\{\eta_u<\infty\}}\right] = \mathbb{E}\left[X(\eta_u)\mathbb{I}_{\{\eta_u<\infty\}}\right] = (1-u)\mathbb{E}_u[X(\eta_u)]$. In view of the fact that $\mathbb{E}\left[\int_{\mathbb{R}_+} X^{\eta_u}(t)L(t)\mathrm{d}K(t)\right] = \mathbb{E}[X^{\eta_u}(\phi)] \leq X(0)$, as follows from (2.3) and the assumptions of Theorem 1.11, all the previous give

$$\mathbb{E}_u\left[\int_0^{\eta_u} X(t)\mathrm{d}K(t) + (1-u)X(\eta_u)\right] \leq X(0).$$

Since $\int_0^{\eta_u} X(t)\mathrm{d}K(t) = K(\eta_u)X(\eta_u) - \int_0^{\eta_u} K(t)\mathrm{d}X(t) = uX(\eta_u) - \int_0^{\eta_u} K(t)\mathrm{d}X(t)$ holds on $\{\eta_u < \infty\}$ and $\mathbb{P}_u[\eta_u < \infty] = 1$, we furthermore obtain

$$\mathbb{E}_u\left[X(\eta_u) - \int_0^{\eta_u} K(t)\mathrm{d}X(t)\right] \leq X(0),$$

which is the same as (2.4) and proves Lemma 2.3. □

Continuing, for each $i \in \{1, \dots, d\}$ and $n \in \mathbb{N}$, define $\tau_n^i := \inf \left\{ t \in \mathbb{R}_+ \mid |S^i(t) - S^i(0)| \geq n \right\}$, which is a stopping time on $(\Omega, \mathbf{F})$. Furthermore, define $X_n^i := (1 - n^{-1}) + n^{-1}(S^i - S^i(0))^{\tau_n^i}$ — it is clear that $X_n^i \in \mathcal{X}_{\mathbf{F}}(1)$ and that $0 \leq X_n^i \leq 2$. For an arbitrary stopping time τ on $(\Omega, \mathbf{F})$, apply (2.4) with $(X_n^i)^\tau$ replacing X; one then obtains $\mathbb{E}_u \left[\int_0^{\eta_u \wedge \tau_n^i \wedge \tau} (1 - K(t)) \mathrm{d}S^i(t) \right] \leq 0$. Performing exactly the previous work by redefining $X_n^i := (1 - n^{-1}) - n^{-1}(S^i - S^i(0))^{\tau_n^i}$, one obtains $\mathbb{E}_u \left[\int_0^{\eta_u \wedge \tau_n^i \wedge \tau} (1 - K(t)) \mathrm{d}S^i(t) \right] \geq 0$. In other words, $\mathbb{E}_u \left[\int_0^{\eta_u \wedge \tau_n^i \wedge \tau} (1 - K(t)) \mathrm{d}S^i(t) \right] = 0$ holds for all $i \in \{1, \dots, d\}$, $n \in \mathbb{N}$, and any stopping time τ on $(\Omega, \mathbf{F})$. This implies that each process $\int_0^{\eta_u \wedge \cdot} (1 - K(t)) \mathrm{d}S^i(t)$ is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P}_u)$. Since $1 - K > 0$, we further obtain that each process $(S^i)^{\eta_u}$ is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P}_u)$. By the definition of the collection $(\mathbb{P}_u)_{u \in [0,1)}$, we conclude that LS^i is a local martingale on $(\Omega, \mathbf{F}, \mathbb{P})$ for all $i \in \{1, \dots, d\}$. This would imply that L is a local martingale deflator. Since $1/\widehat{X}$ is the unique local martingale deflator, we finally conclude that $L = 1/\widehat{X}$, which proves Theorem 1.11.

References

[1] S. Ankirchner, S. Dereich, and P. Imkeller, *The Shannon information of filtrations and the additional logarithmic utility of insiders*, Ann. Probab., 34 (2006), pp. 743–778.

[2] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann., 300 (1994), pp. 463–520.

[3] J. Jacod, *Grossissement initial, hypothèse (H) et théorème de Girsanov*, vol. 1118 of Lecture Notes in Mathematics (T. Jeulin and M. Yor, Grossissements de filtrations: exemples et applications), Springer, Berlin, 1985.

[4] T. Jeulin and M. Yor, *Grossissement d'une filtration et semi-martingales: formules explicites*, in Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977), vol. 649 of Lecture Notes in Math., Springer, Berlin, 1978, pp. 78–97.

[5] Y. M. Kabanov, *On the FTAP of Kreps-Delbaen-Schachermayer*, in Statistics and control of stochastic processes (Moscow, 1995/1996), World Sci. Publ., River Edge, NJ, 1997, pp. 191–203.

[6] I. Karatzas and C. Kardaras, *The numéraire portfolio in semimartingale financial models*, Finance Stoch., 11 (2007), pp. 447–493.

[7] C. Kardaras, *Finitely additive probabilities and the fundamental theorem of asset pricing*, in Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen, Springer, Berlin Heidelberg, 2010, pp. 19–34.

[8] ———, *Numéraire-invariant preferences in financial modeling*, Ann. Appl. Probab., 20 (2010), pp. 1697–1728.

[9] J. B. Long, Jr., *The numéraire portfolio*, Journal of Financial Economics, 26 (1990), pp. 29–69.

[10] A. Nikeghbali and M. Yor, *Doob's maximal identity, multiplicative decompositions and enlargements of filtrations*, Illinois J. Math., 50 (2006), pp. 791–814 (electronic).

[11] P. Protter, *Stochastic integration and differential equations*, vol. 2.1 of Applications of Mathematics (New York), Springer-Verlag, Berlin, 1990. A new approach.

Constantinos Kardaras, Mathematics and Statistics Department, Boston University, 111 Cummington Street, Boston, MA 02215, USA.

E-mail address: `kardaras@bu.edu`