Brownian markets

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Financial market dynamics is rigorously studied via the exact generalized Langevin equation. Assuming Brownian market self-similarity, the market return memory and autocorrelation functions are derived, which exhibit an oscillatory-decaying behavior and a long-time tail similar to the empirical observations.

In 1900 Bachelier, at that time a student of Poincaré, has published his doctoral thesis entitled "Théorie de la speculation" [1], where he has developed the mathematical theory of the Brownian motion five years before the famous Einstein paper [2] has came out to explain its physics. Bachelier introduced also the geometric Brownian motion, which is the background of the well-known Black-Scholes option pricing model [3] and the most powerful tool for qualitative description of financial market fluctuations [4-6]. According to the geometric Brownian motion, the market fluctuations obey a stochastic differential equation

$$dM = \mu M dt + \sigma M dW \tag{1}$$

where M is the market prize, μ is the market mean rate of return, t is time, σ is the market volatility and W is a random Wiener process. As is seen, the noise in Eq. (1) is multiplicative. In finances, the stochastic products MdW is traditionally treated via the Ito rules [7] but there are also other definitions proposed in the literature for handling this peculiarity [8, 9].

Equation (1) describes geometric Brownian motion without memory, while the financial markets are driven by people, who possess ability to remember. Hence, the model in Eq. (1) is oversimplified and requires a generalization, which is the scope of the present paper. Thus, an explicit expression for the market memory function is derived based on the Brownian dynamic self-similarity concept [10]. The latter was already applied to hydrodynamics memory [11] and corresponds to the simplest Hermitian dynamics, governed by an infinite-dimensional hyper-spherical Hilbert space [12].

In the frames of the classical mechanics the evolution of an observable $R(t)$, being a function of momentums and coordinates of some particles of the Universe, is governed by the following dynamic equation

$$dR(t) = i\hat{L}R(t)dt \tag{2}$$

where $i\hat{L}$ is the global Liouville operator. The latter takes into account all the interactions in the Universe, including the human activities as well. Equation (2) is an alternative presentation of the Newton laws from the classical mechanics. The formal solution of Eq. (2) can be written in the form

$$R(t) = \exp(i\hat{L}t)R \tag{3}$$

where $R \equiv R(0)$ is the initial value of the observable. This exact solution is, however, useless since no one is able to define precisely the Universe Liouville operator and even its approximations will not make the problem easier since Eq. (3) involves infinite number of differentiations.

Obviously, we are not able to describe the evolution of the whole Universe but our interest is concentrated solely on the description of a very small part of it, particularly, the prize M of a market. Of course, the latter is influenced by processes in the whole Universe but some of them are important, while others are meaningless. Hence, the basic idea in statistical physics is to introduce a projection operator $\hat{P}$, which focuses the observation on the variable R. Evidently the projector satisfies idempotence ($\hat{P}^2 = \hat{P}$) and a possible definition of the projection operator reads

$$\hat{P}X \equiv R < RX > / < R^2 > \tag{4}$$

where $< \cdot >$ denotes a statistical average. As is seen, the operator $\hat{P}$ from Eq. (4) projects the effect of X on R via the correlation $< RX >$ between these two quantities. If they are statistically independent and zero centered than $<RX>=<R><X>=0$, and the evolution of R will not be affected by X in an average sense. On the other hand the projector (4) preserves completely the information about R since $\hat{P}R = R$.

In the physical literature a general integral representation for the exponential operator form Eq. (3) is proposed [13, 14], which is the base of the Mori-Zwanzig formalism

$$\exp(i\hat{L}t) = \int_0^t \exp(i\hat{L}s)]\hat{P}i\hat{L}\exp[(1-\hat{P})i\hat{L}(t-s)]ds + \exp[(1-\hat{P})i\hat{L}t] \tag{5}$$

Applying this integral identity on the initial velocity $i\hat{L}R$ and using Eq. (4) leads to the following dynamic equation equivalent to Eq. (2)

$$\frac{dR(t)}{dt} = -\int_0^t \frac{<F(t)F(s)>}{<R^2>} R(s)ds + F(t) \tag{6}$$

where the fluctuation force is introduced via $F(t) \equiv \exp[(1-\hat{P})i\hat{L}t]i\hat{L}R$. The benefit of the exact Mori-Zwanzig presentation (5) and the generalized Langevin equation (6) is the separation of the whole interaction into two general forces, dissipation and fluctuation ones, governing the evolution on a macroscopic level. The integral in Eq. (6) represents the dissipation force. The fluctuation-dissipation theorem is also emphasized in Eq. (6) by the fact that the memory kernel in the integral is proportional to the autocorrelation function of the fluctuation force. In addition, the rigorous definition of the fluctuation force above proves the relations $<F(t)>=0$ and $<F(t)R>=0$, where the latter means that there is no correlation between the Langevin force at a given moment and the observable at the beginning. Using these relations one can derive, via multiplying Eq. (6) by R and taking an average value, an integro-differential equation

$$\frac{dC_{RR}(t)}{dt} = -\int_0^t \frac{C_{FF}(t-s)}{C_{RR}(0)} C_{RR}(s)ds \tag{7}$$

for the observable autocorrelation function $C_{RR}(t) \equiv < R(t)R >$ as related to the Langevin force autocorrelation function $C_{FF}(t) \equiv < F(t)F >$. Applying the standard Laplace transformation to Eq. (7) results in the following image expression

$$\tilde{C}_{RR}(p) = C_{RR}(0) / [p + \tilde{C}_{FF}(p) / C_{RR}(0)] \tag{8}$$

where p is the transformation variable. As is seen the autocorrelation function of the Langevin force C_{FF} determines uniquely the autocorrelation function of the observable $R(t)$.

The derivation of the equations above is general and can be applied to arbitrary observable, which is stationary and zero centered. A very popular model for the fluctuation Langevin force is the white noise with a constant spectral density $\tilde{C}_{FF}(p) = C_{RR}(0) / \tau_R$, where τ_R is the relaxation time of the observable $R(t)$. In this case the inverse image of Eq. (8) represents an exponentially decaying autocorrelation function

$$C_{RR}(\tau) = C_{RR}(0)\exp(-\tau / \tau_R) \tag{9}$$

which according to the Doob theorem [15] is typical for stationary Gaussian Markov processes. The stochastic differential equation corresponding to the white noise fluctuation force reads

$$dR(t) = -R(t)dt / \tau_R + \sqrt{< R^2 > / \tau_R}\, dW \tag{10}$$

To apply the general results obtained above to stock markets one should define first a proper observable $R(t)$. Since we are looking for a zero centered ($< R >= 0$) stationary variable with a constant dispersion $< R^2 >$, a reasonable candidate is the market return fluctuation

$$R(t) = d \ln M \ / \ dt - \mu \tag{11}$$

where M is the market prize and μ is its mean return rate. In physics, stationary processes are usually the rates of change of some quantities, e.g. the velocity of a molecule, etc. For this reason the variable in Eq. (11) is not simply proportional to the market prize M but to its relative rate of change. If the time t is much larger than the relaxation time τ_R one can neglect the left-hand-side of Eq. (10) and hence it simplifies to $R(t)dt = \sqrt{< R^2 > \tau_R} dW$. Introducing here Eq. (11) results in a stochastic differential equation for the market prize

$$dM = \mu M dt + \sqrt{< R^2 > \tau_R} M dW \tag{12}$$

Comparing now this equation with Eq. (1) unveils an expression relating the dispersion $< R^2 >$ of the relative rate of market fluctuations and the volatility of the market prize σ

$$\sigma^2 = < R^2 > \tau_R \tag{13}$$

If the return fluctuations obey the Poisson law than $< R^2 > = \mu^2$ and the correlation time from Eq. (13) acquires the form $\tau_R = \sigma^2 / \mu^2$. However, since according to Eq. (12) the return fluctuations $R(t)$ are proportional to the white noise, which is only Gaussian [16], it is necessary to accept that the return rate fluctuations are Gaussian as well.

The analysis above shows that Eq. (1) is valid only for large times $t > \tau_R$ and the lack of memory effects. A general way to determine the return rate autocorrelation and memory functions is to assume dynamic Brownian self-similarity [10] of the market. According to this model the autocorrelation functions of the observable and its conjugated Langevin force are the same

$$C_{RR}(\tau) / C_{RR}(0) = C_{FF}(\tau) / C_{FF}(0) \tag{14}$$

which can be translated in common words as "a market is driven by the market itself". Strictly streaking the assumption (14) implies a Hermitian dynamics in an infinite-dimensional hyper-spherical Hilbert space [12]. Combining Eq. (14) with Eq. (8) results in the following expression for the Laplace image of the return rate autocorrelation function

$$\tilde{C}_{RR}(p) = < R^2 > \tau_R[\sqrt{1+(\tau_R p/2)^2} - \tau_R p/2] \quad (15)$$

where the relaxation time equals to $\tau_R \equiv \sqrt{< R^2 > / < F^2 >}$. The inverse Laplace transformation of Eq. (15) leads straightforward to return rate autocorrelation function

$$C_{RR}(\tau)/ < R^2 >= J_1(2\tau/\tau_R)\tau_R/\tau \quad (16)$$

This is a universal oscillatory-decaying function, whose amplitude exhibits a long-time tail falling asymptotically as $(\tau_R/\tau)^{3/2}$. The plot of the autocorrelation from Eq. (16) in Fig. 1 shows the existence of a sequence of correlations and anti-correlations of the market return. Similar auto-correlation functions are empirically detected, for instance, in the Dow Jones Industrial Average index [17].

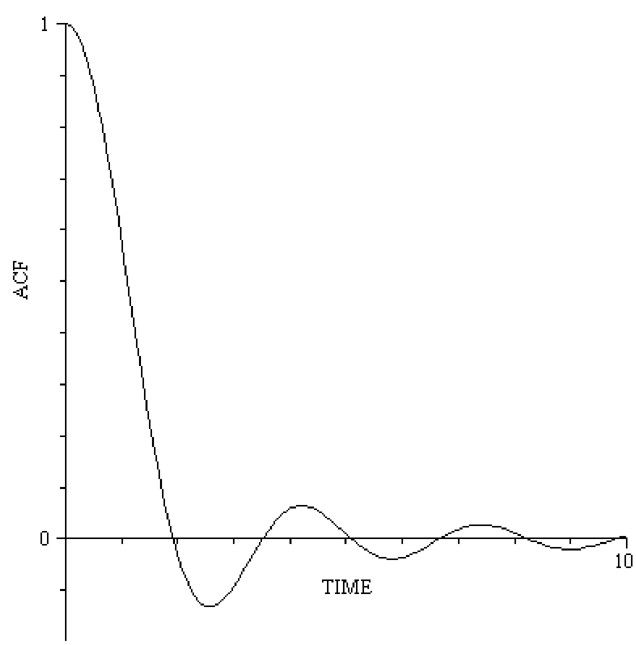

Figure 1. The dependence of the return rate autocorrelation function $C_{RR}/ < R^2 >$ from Eq. (16) as a function of the dimensionless time τ/τ_R

According to Eqs. (14) and (16) the memory function of return rate fluctuations equals to $C_{FF}(\tau)/<R^2>=J_1(2\tau/\tau_R)/\tau_R\tau$ and thus Eq. (6) becomes fully specified. Since the Langevin force F is not a white noise anymore, the return rate fluctuations $R(t)$ are not restricted to be Gaussian in contrast to Eq. (1). Hence, Eq. (6) describes completely the market stochastic dynamics by accounting for the red color of the noise and the corresponding memory effects.

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