# Thermodynamics of Abelian Forms in Real Compact Hyperbolic Spaces

A. A. Bytsenko,
*Departamento de Física, Universidade Estadual de Londrina, Londrina, Brazil*

V. S. Mendes,
*Departamento de Física, Universidade Estadual de Londrina, Londrina, Brazil*

and A. C. Tort
*Departamento de Física Teórica, Instituto de Física,*
*Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil*

Received on 4 December, 2003

We analyze gauge theories based on abelian $p-$forms in real compact hyperbolic manifolds. The explicit thermodynamic functions associated with skew–symmetric tensor fields are obtained via zeta–function regularization and the trace tensor kernel formula. Thermodynamic quantities in the high–temperature expansions are calculated and the entropy/energy ratios are established.

## 1 Introduction

It is known that the thermodynamics of quantum fields in an Einstein universe for some radius is equivalent to that of an instantaneously static closed Friedmann–Robertson–Walker universe. The field thermodynamics of positive curvature Einstein spaces was discussed by several authors before. In particular, the so-called entropy bounds or entropy to thermal energy ratios were calculated and compared with known bounds such as the Bekenstein bound or the Cardy-Verlinde bound. For example, for a massless scalar field in $\mathbf{S}^3$ space this was done in [1] and for a massive scalar field in [2]. Here we wish to extend the evaluation of those type of bounds to the case of skew symmetric tensor fields in real hyperbolic spaces.

We shall work with a $D-$dimensional compact hyperbolic space $X$ with universal covering $M$ and fundamental group $\Gamma$. We can represent $M$ as the symmetric space $G/K$, where $G = SO_1(D,1)$ and $K = SO(D)$ is a maximal compact subgroup of $G$. Then we regard $\Gamma$ as a discrete subgroup of $G$ acting isometrically on $M$, and we take $X$ to be the quotient space by that action: $X = \Gamma\backslash M = \Gamma\backslash G/K$. Let $\tau$ be an irreducible representation of $K$ on a complex vector space $V_\tau$, and form the induced homogeneous vector bundle $G\times_K V_\tau$ (the fiber product of $G$ with $V_\tau$ over $K$) $\rightarrow M$ over $M$. Restricting the $G$ action to $\Gamma$ we obtain the quotient bundle $E_\tau = \Gamma\backslash(G\times_K V_\tau) \rightarrow X = \Gamma\backslash M$ over $X$. The natural Riemannian structure on $M$ (therefore on $X$) induced by the Killing form $(\ ,\ )$ of $G$ gives rise to a connection Laplacian $\mathcal{L}$ on $E_\tau$. If $\Omega_K$ denotes the Casimir operator of $K$ – that is $\Omega_K = -\sum y_j^2$, for a basis $\{y_j\}$ of the Lie algebra $k_0$ of $K$, where $(y_j\ , y_\ell) = -\delta_{j\ell}$, then $\tau(\Omega_K) = \lambda_\tau$ for a suitable scalar $\lambda_\tau$. Moreover for the Casimir operator $\Omega$ of $G$, with $\Omega$ operating on smooth sections $\Gamma^\infty E_\tau$ of $E_\tau$ one has $\mathcal{L} = \Omega - \lambda_\tau\mathbf{1}$ ; see Lemma 3.1 of [3]. For $\lambda\geq 0$ let

$$\Gamma^\infty\left(X\ , E_\tau\right)_\lambda = \{s\in\Gamma^\infty E_\tau\,|-\mathcal{L}s = \lambda s\} \qquad (1)$$

be the space of eigensections of $\mathcal{L}$ corresponding to $\lambda$. Here we note that since $X$ is compact we can order the spectrum of $-\mathcal{L}$ by taking $0=\lambda_0<\lambda_1<\lambda_2<\cdots$; $\lim_{j\rightarrow\infty}\lambda_j = \infty$. It will be convenient moreover to work with the normalized Laplacian $\mathcal{L}_p = -c(D)\mathcal{L}$ where $c(D) = 2(D-1) = 2(2N-1)$. $\mathcal{L}_p$ has spectrum $\{c(D)\lambda_j\ , m_j\}_{j=0}^\infty$ where the multiplicity $m_j$ of the eigenvalue $c(D)\lambda_j$ is given by $m_j = \dim\Gamma^\infty\left(X\ , E_{\tau^{(p)}}\right)_{\lambda_j}$.

It is easy to prove the following properties for operators and forms: $dd=\delta\delta=0$, $\delta = (-1)^{Dp+D+1} * d*$, $**\omega_p = (-1)^{p(D-p)}\omega_p$. Let $\alpha_p$, $\beta_p$ be $p-$forms; then the invariant inner product is defined by $(\alpha_p,\beta_p) := \int_M \alpha_p\wedge *\beta_p$. The operators $d$ and $\delta$ are adjoint to each other with respect to this inner product for $p-$forms: $(\delta\alpha_p,\beta_p) = (\alpha_p, d\beta_p)$. In quantum field theory the Lagrangian associated with $\omega_p$ takes the form: $L = d\omega_p\wedge *d\omega_p$ (gauge field); $L = \delta\omega_p\wedge *\delta\omega_p$ (co–gauge field). The Euler–Lagrange equations supplied with the gauge give $\mathcal{L}_p\omega_p = 0\,,\ \delta\omega_p = 0$ (Lorentz gauge); $\mathcal{L}_p\omega_p = 0\,,\ d\omega_p = 0$ (co–Lorentz gauge). These Lagrangians give possible representation of tensor fields or generalized Abelian gauge fields. The two representations of tensor fields are not completely independent. Indeed, there is a duality property in the exterior calculus which gives a connection between star–conjugated gauge tensor fields and co–gauge fields.

## 2 The trace formula applied to the tensor Kernel

We can apply the version of the trace formula developed by Fried in [4]. First we define additional notation. For $\sigma_p$ the natural representation of $SO(2N-1)$ on $\Lambda^p\mathbf{C}^{2N-1}$, one has the corresponding Harish–Chandra–Plancherel density, given for a suitable normalization of Haar measure $dx$ on $G$ by

$$\mu_{\sigma_p(r)} = \frac{\pi}{2^{4k-4}[\Gamma(N)]^2}\left(\begin{array}{c} 2N-1 \\ p \end{array}\right) P_{\sigma_p}(r) r\tanh(\pi r)\ ,$$

for $0 \leq p \leq N-1$, where

$$\begin{aligned} P_{\sigma_p}(r) &= \prod_{\ell=2}^{p+1}\left[r^2+\left(N-\ell+\frac{3}{2}\right)^2\right] \\ &\times\prod_{\ell=p+2}^{N}\left[r^2+\left(N-\ell+\frac{1}{2}\right)^2\right] \end{aligned}$$

is an even polynomial of degree $2N-2$. One has that $P_{\sigma_p}(r) = P_{\sigma_{2N-1-p}}(r)$ and $\mu_{\sigma_p}(r) = \mu_{\sigma_{2N-1-p}}(r)$ for $N \leq p \leq 2N-1$. Now define the Miatello coefficients (see the ref. [5]) $a_{2\ell}^{(p)}$ for $G = SO_1(2N+1,1)$ by $P_{\sigma_p}(r) = \sum_{\ell=0}^{N-1} a_{2\ell}^{(p)} r^{2\ell}$ , $0 \leq p \leq 2N-1$ . Let $\mathrm{Vol}(\Gamma\backslash G)$ denote the integral of the constant function $\mathbf{1}$ on $\Gamma\backslash G$ with respect to the $G$ – invariant measure on $\Gamma\backslash G$, induced by $dx$. For $0 \leq p \leq D-1$, the Fried trace formula [4] applied to the tensor kernel associated to the Laplace operator on co-exact forms $\mathcal{L}_p^{CE}$ is [6, 7]:

$$\begin{aligned} &\mathrm{Tr}\left(e^{-t\mathcal{L}_p^{(CE)}}\right) = \\ &\sum_{j=1}^{p}(-1)^j\left[I_\Gamma^{(p-j)}(\mathcal{K}_t)+I_\Gamma^{(p-1-j)}(\mathcal{K}_t)\right. \\ &\left.+H_\Gamma^{(p-j)}(\mathcal{K}_t)+H_\Gamma^{(p-1-j)}(\mathcal{K}_t)-b_{p-j}\right] \qquad (2) \end{aligned}$$

where $b_p$ are the Betti numbers. In the above formula $I_\Gamma^{(p)}(\mathcal{K}_t)$ and $H_\Gamma^{(p)}(\mathcal{K}_t)$ are the identity and hyperbolic orbital integrals, respectively.

## 3 The spectral functions of exterior forms

The spectral zeta function related to the Laplace operator $\mathcal{L}_j$ can be represented by the inverse Mellin transform of the heat kernel $\mathcal{K}_t = \mathrm{Tr}\,\exp\left(-t\mathcal{L}_j\right)$. Using the Fried formula, we can write the zeta function as a sum of two contributions:

$$\begin{aligned} \zeta(s|\mathcal{L}_j) &= \frac{1}{\Gamma(s)}\int_0^\infty dt t^{s-1}\left(I_\Gamma^{(j)}(\mathcal{K}_t)+I_\Gamma^{(j-1)}(\mathcal{K}_t)\right. \\ &\quad\left.+H_\Gamma^{(j)}(\mathcal{K}_t)+H_\Gamma^{(j-1)}(\mathcal{K}_t)\right) \\ &\equiv \zeta_I^{(D)}(s,j)+\zeta_H^{(D)}(s,j). \qquad (3) \end{aligned}$$

For the identity component we have

$$\zeta_I^{(D)}(s,j) = \frac{V_\Gamma}{\Gamma(s)}\int_0^\infty dt t^{s-1}\int_{\mathbf{R}} dr\,\mu_{\sigma_j} e^{-t(r^2+\alpha_j^2)}, \quad (4)$$

where $V_\Gamma = \chi(1)\mathrm{Vol}\left(\Gamma\backslash G\right)/4\pi$, and we define $\alpha_j^2 = b^{(j)}+(\rho_0-j)^2$, $\rho_0 = (D-1)/2$ and $b^{(j)}$ are constants. Replacing the Harish–Chandra–Plancherel measure, we obtain two representations for $\zeta_I^{(D)}(s,j)$ which hold for odd and even dimension:

$$\begin{aligned} \zeta_I^{(2N)}(s,j) &= \frac{V_\Gamma C_{2N}^{(j)}}{\Gamma(s)}\sum_{\ell=0}^{N-1} a_{2\ell,2N}^{(j)}\int_0^\infty dt\,t^{s-1} \\ &\times\int_{\mathbf{R}} dr r^{2\ell+1}\tanh(\pi r)\,e^{-t(r^2+\alpha_j^2)} \\ &= \frac{V_\Gamma C_{2N}^{(j)}}{\Gamma(s)}\sum_{\ell=0}^{N-1} a_{2\ell,2N}^{(j)}\left[\frac{\Gamma(\ell+1)\Gamma(s-\ell-1)}{\alpha_j^{2s-2\ell-2}}\right. \\ &\left.+\sum_{n=0}^{\infty}\xi_{n\ell}\frac{\Gamma(s+n)}{\alpha_j^{2s+2n}}\right], \qquad (5) \end{aligned}$$

$$\begin{aligned} \zeta_I^{(2N+1)}(s,j) &= \frac{V_\Gamma C_{2N+1}^{(j)}}{\Gamma(s)}\sum_{\ell=0}^{N} a_{2\ell,2N+1}^{(j)}\int_0^\infty dt\,t^{s-1} \\ &\times\int_{\mathbf{R}} dr r^{2\ell} e^{-t(r^2+\alpha_j^2)} \\ &= \frac{V_\Gamma C_{2N+1}^{(j)}}{\Gamma(s)}\sum_{\ell=0}^{N}\Gamma\left(\ell+\frac{1}{2}\right) \\ &\times\Gamma\left(s-\ell-\frac{1}{2}\right)\frac{a_{2\ell,2N+1}^{(j)}}{\alpha_j^{-2s+2\ell+1}}, \qquad (6) \end{aligned}$$

where $B_n$ is the $n$-th Bernoulli number. Moreover, we define

$$\xi_{n\ell} := \frac{(-1)^{\ell+1}\left(1-2^{-2\ell-2n-1}\right)}{n!\,(2\ell+2n+2)}B_{2\ell+2n+2}. \qquad (7)$$

In fact, we do not need the hyperbolic component $\zeta_H^{(D)}(s,j)$ since in the high temperature limit (see next section), only the function $\zeta_I^{(D)}(s,j)$ will present contribution.

## 4 The high temperature limit

Using the Mellin representation for the zeta function one can obtain useful formulae for the non–trivial temperature dependent part of the identity and hyperbolic orbital components of the free energy (for details see the ref. [8, 9, 10])

$$\begin{aligned} F_{I,H}^{(D)}(\beta,j) &= -\frac{1}{2\pi i}\int_{\Re z=c} dz\,\zeta_R(z) \\ &\times\Gamma(z-1)\zeta_{I,H}^{(D)}\left(\frac{z-1}{2},j\right)\beta^{-z}. \qquad (8) \end{aligned}$$

A tedious calculation gives the following result:

$$F_I^{(2N)}(\beta,j) = -\frac{V_\Gamma C_{2N}^{(j)} a_{2N-2,2N}^{(j)}}{\sqrt{4\pi}}\Gamma(N)\zeta(2N+1) \times\Gamma\left(N+\frac{1}{2}\right)\beta^{-2N-1} -\frac{V_\Gamma C_{2N}^{(j)}}{\sqrt{4\pi}}\zeta(2N-1) \times\Gamma\left(N-\frac{1}{2}\right)\left[a_{2N-4,2N}^{(j)}\Gamma(N-1) - a_{2N-2,2N}^{(j)}\Gamma(N)\right]\beta^{-2N+1} +\mathcal{O}(\beta^{-2N+3}), \quad (9)$$

$$F_I^{(2N+1)}(\beta,j) = -\frac{V_\Gamma C_{2N+1}^{(j)} a_{2N,2N+1}^{(j)}}{\sqrt{4\pi}}\Gamma\left(N+\frac{1}{2}\right) \times\zeta(2N+2)\Gamma(N+1)\beta^{-2N-2} -\frac{V_\Gamma C_{2N+1}^{(j)}}{\sqrt{4\pi}}\zeta(2N)\Gamma(N) \times\left[a_{2N-2,2N+1}^{(j)}\Gamma\left(N-\frac{1}{2}\right) - a_{2N,2N+1}^{(j)}\Gamma\left(N+\frac{1}{2}\right)\alpha_j^2\right]\beta^{-2N} +\mathcal{O}\left(\beta^{-2N+2}\right). \quad (10)$$

The contribution associated to the hyperbolic orbital component is negligible small.

### 4.1 The thermodynamic functions and the entropy bound

In the context of the Hodge theory, the physical degrees of freedom are represented by the co–exact forms. Thus the free energy becomes

$$\mathcal{F}^{(D)}(\beta) = \sum_{j=0}^{p}(-1)^j\left(F_I^{(D}(\beta,p-j) + F_I^{(D)}(\beta,p-j-1)\right).$$

However, if we perform the sum explicitly, we see that

$$\mathcal{F}^{(D)}(\beta) = F_I^{(D)}(\beta,p).$$

In the high temperature limit ($\beta\rightarrow 0$) we have

$$\mathcal{F}^{(D)}(\beta) = -A_1(D;\Gamma)\beta^{-D-1} - A_2(D;\Gamma)\beta^{-D+1} +\mathcal{O}(\beta^{-D+3}), \quad (11)$$

where, for the even dimensional case

$$A_1(2N;\Gamma) = \frac{V_\Gamma C_{2N}^{(p)}}{\sqrt{4\pi}}\zeta(2N+1)\Gamma(N) \times\Gamma\left(N+\frac{1}{2}\right)a_{2N-2,2N}^{(p)}. \quad (12)$$

$$A_2(2N;\Gamma) = \frac{V_\Gamma C_{2N}^{(p)}}{\sqrt{4\pi}}\zeta(2N-1)\Gamma\left(N-\frac{1}{2}\right) \times\left[\Gamma(N-1)a_{2N-4,2N}^{(p)} + \Gamma(N)a_{2N-2,2N}^{(p)}\alpha_{p-j}^2\right], \quad (13)$$

and, for the odd dimensional case

$$A_1(2N+1;\Gamma) = \frac{V_\Gamma C_{2N+1}^{(p)}}{\sqrt{4\pi}}\zeta(2N+2) \times\Gamma\left(N+\frac{1}{2}\right)\Gamma(N+1)a_{2N,2N+1}^{(p)}, \quad (14)$$

$$A_2(2N+1;\Gamma) = \frac{V_\Gamma C_{2N+1}^{(p)}}{\sqrt{4\pi}}\zeta(2N)\Gamma(N) \times\left[\Gamma\left(N-\frac{1}{2}\right)a_{2N-2,2N+1}^{(p)} - \Gamma\left(N+\frac{1}{2}\right)a_{2N,2N+1}^{(p)}\alpha_p^2\right]. \quad (15)$$

In fact, in the sums (12) - (15) only terms containing the Miatello coefficients $a_{2\ell,D}^{(p)}$ survive and define the coefficients $A_1$ and $A_2$. The entropy and the total energy can be obtained with the help of the following thermodynamic relations: $S^{(D)}(\beta) = \beta^2\partial\mathcal{F}^{(D)}(\beta)/\partial\beta$, $E^{(D)}(\beta) = \partial(\beta\mathcal{F}^{(D)}(\beta))/\partial\beta$. Therefore,

$$S^{(D)}(\beta) = (D+1)A_1(D;\Gamma)\beta^{-D} +(D-1)A_2(D;\Gamma)\beta^{-D+2}\mathcal{O}\left(\beta^{-D+4}\right), \quad (16)$$

$$E^{(D)}(\beta) = -DA_1(D;\Gamma)\beta^{-D-1} -(D-2)A_2(D;\Gamma)\beta^{-D+1} + \mathcal{O}\left(\beta^{-D+3}\right). \quad (17)$$

The entropy/energy ratio becomes

$$\frac{S^{(D)}(\beta)}{E^{(D)}(\beta)} = \frac{D+1}{D}\beta + \frac{2}{D^2}\frac{A_2(D;\Gamma)}{A_1(D;\Gamma)}\beta^3 + \mathcal{O}\left(\beta^5\right). \quad (18)$$

## 5 Concluding remarks

We have obtained the high–temperature expansion for the entropy/energy ratios of abelian gauge fields in real compact hyperbolic spaces. The dependence on the Miatello coefficients related to the structure of the Harish–Chandra–Plancherel measure starts from the second term of the expansion. In the case of scalar fields ($p = 0$) we have eq. (18) with

$$\frac{A_2(2N;\Gamma)}{A_1(2N;\Gamma)} = \frac{2}{2N-1}\frac{\zeta(2N-1)}{\zeta(2N+1)} \times\left(\frac{1}{N-1}\frac{a_{2N-4,2N}^{(0)}}{a_{2N-2,2N}^{(0)}} - \alpha_0^2\right),$$

$$\frac{A_2(2N+1;\Gamma)}{A_1(2N+1;\Gamma)} = \frac{1}{N}\frac{\zeta(2N)}{\zeta(2N+2)} \times\left(\frac{2}{2N-1}\frac{a^{(0)}_{2N-2,2N+1}}{a^{(0)}_{2N,2N+1}}-\alpha_0^2\right),$$

where $\alpha_0^2 = \rho_0^2 + m^2$ ($\alpha_0^2 = \rho_0^2$ for the massless case). For three–dimensional hyperbolic manifolds the Miatello coefficients reads [11]: $a_0^{(0)} = a_2^{(0)} = 1$ and therefore $S^{(3)}(\beta)/E^{(3)}(\beta) = (4/3)\beta+(10/3\pi^2)(2-\alpha_0^2)\beta^3+\mathcal{O}(\beta^5)$. This formula is in agreement with result obtained in [2] where entropy bounds have been calculated for spherical geometry and where the dependence on the geometry of the background also starts from the second term of the expansion.

**Acknowledgements**

A.A.B. would like to thank Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP/Brazil) and the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq/Brazil) for partial financial support, and the Instituto de Física Teórica (IFT/UNESP) for kind hospitality.

# References

[1] I. Brevik, K. A. Milton, and S. D. Odintsov, Ann. of Phys. **302**, 120 (2002).

[2] E. Elizalde and A. C. Tort, Phys. Rev. D **67**, 124014 (2003).

[3] N. Wallach, J. Diff. Geom. **11**, 91 (1976).

[4] D. Fried, Invent. Math. **84**, 523 (1986).

[5] R. Miatello, Trans. Am. Math. Soc. **260**, 1 (1980).

[6] A. A. Bytsenko, L. Vanzo and S. Zerbini, Nucl. Phys. B **505**, 641 (1997).

[7] A. A. Bytsenko, A. E. Gonçalves and F. L. Williams, Int. J. Mod. Phys. A **18**, 2041 (2003).

[8] E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).

[9] A. A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini, Phys. Rep. **266**, 1 (1996).

[10] A. A. Bytsenko, G. Cognola, E. Elizalde, V. Moretti and S. Zerbini, *Analytic Aspects of Quantum Fields* (World Scientific, Singapore, 2003).

[11] A. A. Bytsenko, E. Elizalde and M. E. X. Guimarães, Int. J. Mod. Phys. A **18**, 2179 (2003).