Classification of Basis States for $(p - f)-$Nuclei $(41 \leq A \leq 80)$ with Minimal Configuration Energy

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We give a complete classification of basis states with unitary $(U(A-1), U(3))$ and permutational$(S(A))$ symmetries. These states are suitable as basis functions for $(p-f)-$nuclei $(41 \leq A \leq 80)$ with minimal configuration energy. We also give a brief survey of the way in which they are obtained.

1 Introduction

In the tradicional nonrelativistic treatment, the nucleus is considered as a system of A fermions, the nucleons, with spin and isospin 1/2, and three spatial degrees of freedom interacting through one- and two-body forces. The bound states of such a system are described by totally antisymmetric wave functions.

The introduction of Jacobi vectors

$$\vec{\rho}_i = \frac{1}{\sqrt{i(i+1)}}\left(\sum_{j=1}^{i}\vec{r}_j - i\,\vec{r}_{i+1}\right)\ ;\ i = 1, 2, ..., A-1\ , \tag{1}$$

$$\vec{\rho}_A = \frac{1}{\sqrt{A}}\sum_{j=1}^{A}\vec{r}_j \tag{2}$$

allows us to remove the center of mass and pay attention only to the relative motion described by the translationally invariant Jacoby vectors $\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_{A-1}$.

To describe the bound states of such system, we will use as basis the basis functions of irrep $[1^{\,7(A-1)}]$ of $U(7(A-1)) \supset U^{(r)}(3(A-1)) \times U^{(s)}(4(A-1))$. The spin-isospin part is described using the chain

$$\begin{array}{lllll} U(4(A-1)) & \supset & U(4) & \times & U(A-1) \\ & & \cup & & \cup \\ & & U^{(S)}(2) \times U^{(T)}(2) & & O^{(s)}(A-1) \\ & & & & \cup \\ & & & & \mathrm{S}^{(s)}(A) \end{array} \tag{3}$$

while the space part is described by

$$\begin{array}{lllll} \mathrm{U(3(A\text{-}1)} & \supset & \mathrm{U(3)} & \times & \mathrm{U}^{(r)}(A-1) \\ & & \cup & & \cup \\ & & \mathrm{O}^{+}(3) & & \mathrm{O}^{(r)}(A-1) \\ & & & & \cup \\ & & & & \mathrm{S}^{(r)}(A) \quad . \end{array} \tag{4}$$

The labelling of basis functions in the spin-isospin chain of subgroups, Eq.(3), is given by the Wigner Supermultiplet Theory.

We will focus our attention on the problem of labelling the basis functions for the space chain of subgroups, Eq(4).

Since the basis functions of irrep $\{\lambda\}$ of $U(3(A-1))$ are functions only of the coordinates of the first $A-1$ Jacobi vectors, they have to be symmetric. We then write $\{\lambda\} = \{E\}$. Since the wave functions of the $p-$dimensional harmonic oscillator carry the irrep $\{E\}$ of $U(p)$, it is usual to associate E with the configuration energy of the nuclear states whose space part is described by wave functions labelled by the chain (4).

This association allows us to stablish a link with the harmonic oscillator shell model. The basis functions of the irrep $\{E\}$ could alternatively be labelled by the chain of subgroups

$$U(3(A-1)) \supset U^{(1)}(3) \times U^{(2)}(3) \times ... \times U^{(A-1)}(3) \tag{5}$$

in which each link $U^{(i)}(3)$ acts only in the 3 coordinates of the Jacoby vector $\overrightarrow{\rho}_i$. In this case the irreps associated to these $U^{(i)}(3)$ would be all symmetric $[E^{(i)}]$ and their basis functions would be eigenstates of harmonic oscillators with energy $\mathcal{E}^{(i)} = (E^{(i)} + 3/2)\hbar\omega$ and it would result

$$E = \sum_{i=1}^{A-1} E^{(i)}. \tag{6}$$

The number of linearly independent wave functions of the 3-dimensional harmonic oscillator with energy $\mathcal{E} = (E + 3/2)\hbar\omega$ is equal to the dimension of the irrep $\{E\}$ of $U(3)$ given by

$$\dim_{\{E\}} = \frac{1}{2}(E+1)(E+2)\ . \tag{7}$$

In this way, by the Pauli principle, in the E shell one can put at most $4\dim_{\{E\}} = 2(E+1)(E+2)$ nucleons. The minimal configuration energy is obtained by filling the shells $E = 0(s), E = 1(p), E = 2(s-d), ..., E_0 - 1$ and putting the remaining nucleons in the first partially filled shell E_0. In this way, it follows that

$$E_{\min} = \sum_{E=0}^{E_0-1} 4E\dim_{\{E\}} + E_0 n_0$$

$$= E_0 A - \frac{1}{6}E_0(E_0+1)(E_0+2)(E_0+3)\ , \tag{8}$$

where n_0 is the number of nucleons in the partially filled shell E_0.

Our aim is to label the states of a system of A nucleons with minimal configuration energy with the labels given by the unitary chain (4).

In [1], Elliott gives the labelling for $p-$ and $(s-d)-$nuclei in a different, but equivalent, organization than the one used here. In his paper, Elliott only mentions that the classification was obtained by the plethysm technique.

In a recent paper [2], which we will refer to as (I), we review the plethysm technique, propose a general algorithm to compute all plethysms of two Schur functions of degrees n and m using as input the plethysm $\{n\} \otimes \{m\}$ of symmetric Schur functions [3] and show how the plethysm technique can be applied to our problem. Ultimately, one has to find the reduction $U(A-1) \supset O(A-1) \supset S(A)$. [We refer the readers to (I) for definitions and notations.] An alternative method for obtaining the reduction $O(A-1) \supset S(A)$, exploiting the complementarity between $O(A-1)$ and $Sp(3,R)$ was proposed in [7].

According to the plethysm technique exposed in (I), the groups $U(3)$ and $U^{(r)}(A-1)$ in Eq.(4) must share the same irrep

$$\{E_1, E_2, E_3\} \text{ with } E_1 + E_2 + E_3 = E \tag{9}$$

and the irreps $[\lambda]$ of $S^{(r)}(A)$ and $S^{(s)}(A)$ must be conjugate to each other.

The branching rules for irreps in the restriction $U(n) \to O(n)$ have definite rules [4, 5, 6, 2]. According to them, for states of minimal configuration energy, the irrep of $O(n)$ in this restriction is the same as the one of $U(n)$. This then fixes the $O(A-1)$ irrep. Then we must concern ourselves only with the restriction $U^{(r)}(A-1) \supset S(A)^{(r)}$.

Besides, the Pauli principle imposes an additional restriction. The treatment of the spin-isospin part by the Wigner supermultiplet model implies that the $S^{(r)}(A)$ irrep $[\widetilde{\lambda}]$ must have at most 4 lines, that is,

$$[\tilde{\lambda}] = [\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4] \quad . \tag{10}$$

Therefore $\{\lambda\}$, being its conjugate, must have at most 4 columns.

2 Reduction $U(A-1) \supset S(A)$

The reduction $U(A-1) \supset S(A)$ is given by the inner plethysm $\{A-1, 1\} \odot \{\lambda'\}$ of $U(A-1)$ irreps expanded in terms of $S(A)$ irreps

$$\{A-1,1\} \odot \{\lambda'\} = \sum_{\lambda''} V_{\lambda'\lambda''}[A - r'', \lambda''_1, \lambda''_2, ..., \lambda''_{A-1}] \tag{11}$$

where $V_{\lambda'\lambda''}$ are numerical coefficients, $\{\lambda''_1, \lambda''_2, ..., \lambda''_{A-1}\}$ are $U(A-1)$ irreps and $r'' = \sum_{i=1}^{A-1} \lambda''_i$ are their degrees.

The numerical coefficients and the $U(A-1)$ irreps are obtained by the following procedure.

One first defines the operator $\widehat{\mathcal{D}}(\{\lambda\})$ by its action on an $U(A-1)$ irrep $\{\mu\}$:

$$\widehat{\mathcal{D}}(\{\lambda\})\{\mu\} = \sum_{\mu'} \alpha(\{\lambda\}\{\mu\} \to \{\mu'\})\{\mu'\} \qquad (12)$$

where $\alpha(\{\lambda\}\{\mu\} \to \{\mu'\})$ is the multiplicity of irrep $\{\mu'\}$ in the outer product $\{\lambda\}\{\mu\}$.

From the properties of the outer product of Schur functions, it follows that the operators $\widehat{\mathcal{D}}$ satisfy the relations:

$$\widehat{\mathcal{D}}(\{\lambda'\})\widehat{\mathcal{D}}(\{\lambda''\}) = \widehat{\mathcal{D}}(\{\lambda'\}\{\lambda''\}) \, , \qquad (13)$$

$$\widehat{\mathcal{D}}(\{\lambda'\}) + \widehat{\mathcal{D}}(\{\lambda''\}) = \widehat{\mathcal{D}}(\{\lambda'\} + \{\lambda''\}) \, . \qquad (14)$$

Next one defines an operator $\widehat{D}$ by

$$\widehat{D} = \sum_{t_2=0}^{\infty} \sum_{\lambda_{t_2}} \sum_{t_3=0}^{\infty} \sum_{\lambda_{t_3}} ... \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} ...(\{\lambda\}_{t_2}\{\lambda\}_{t_3}...)(\widehat{\mathcal{D}}(\{2\} \otimes \{\lambda\}_{t_2}) \cdot \cdot(\widehat{\mathcal{D}}(\{3\} \otimes \{\lambda_{t_3}\})...(\widehat{\mathcal{D}}(\{2\} \otimes \{j_2\})(\widehat{\mathcal{D}}(\{3\} \otimes \{j_3\})... \quad . \qquad (15)$$

where $\{j_2\}, \{j_3\}, ...$ are symmetric Schur functions and the $\{\lambda\}_{t_2}, \{\lambda\}_{t_3}, ...$ are general Schur functions of degrees $t_2, t_3, ...$. [Note that the only plethysms needed are those with a symmetric Schur function in the left. An algorithm to compute them is presented in (I).]

The action of $\widehat{D}$ over an $U(A-1)$ irrep $\{\lambda'\}$, by use of (13) and (14) is transformed in a sum of irreps $\{\lambda''\}$ with multiplicities $V_{\lambda' \ \lambda''}$:

$$\widehat{D}\{\lambda'\} = \sum_{\lambda''} V_{\lambda' \ \lambda''}\{\lambda''\} \, . \qquad (16)$$

This expression provides the numerical coefficients $V_{\lambda' \ \lambda''}$ and the $U(A-1)$ irreps $\{\lambda''\}$ that appear in (11). To each $U(A-1)$ irrep $\{\lambda''\}$ corresponds one $S(A)$ irrep $[\lambda] = [A - r'', \lambda_1'', \lambda_2'', \lambda_{A-1}'']$.

This is the mathematical framework. When applied to the classification of nuclear states new ingredients appear. First, the $U(A-1)$ irrep $\{\lambda'\}$ in which $\widehat{\mathcal{D}}$ acts has, by (9), at most 3 rows. Second, the $S(A)$ irreps $[\lambda] = [A - r'', {\lambda_1'}', \lambda_2'', ..., \lambda_{A-1}'']$ with physical meaning, by (10), are only the ones with at most 4 columns, that is,

$$[\lambda] = [4^{k_4}, 3^{k_3}, 2^{k_2}, 1^{k_1}], \text{ with } 4k_4 + 3k_3 + 2k_2 + k_1 = A. \qquad (17)$$

The k_i are interpreted[8] as the number of space levels occupyied by 1,1,3,4 nucleons, respectively.

These conditions restrict the t_k and j_k in (15) that may give meaningful $S(A)$ irreps $[\lambda]$ when $\widehat{D}$ is applied to a given $U(A-1)$ irrep $\{\lambda'\}$ representing a nuclear state with configuration energy $E \geq E_{\text{min}}$. These t_k and j_k are obtained following 2 steps:

1) take a nonnegative integer i in the range

$$\left[\frac{E+1}{2}\right] - 6 \leq i \leq E - A + 4; \qquad (18)$$

2) for each i in this range, find the nonnegative integers j_k and t_k that satisfy

$$t_2 + \sum_{k=2} k(t_{k+1} + j_k) = i \, , \qquad \sum_{k=2} k(t_k + j_k) = r_{\lambda''} \, , \qquad (19)$$

where $r_{\lambda''}$ must be in the range

$$E - 12 \leq r_{\lambda''} \leq 2i \qquad (20)$$

Once these t_k 's and j_k 's are obtained, one replaces them in (15) applied to $\{\lambda'\}$, computes the resulting plethysms and outer products, linearizes the resulting expression with respect to $\widehat{\mathcal{D}}$ using (13) and (14) ending with an expression of type (16). From the $S(A)$ irreps $[\lambda] = [A - r'', \lambda_1'', \lambda_2'', ..., \lambda_{A-1}'']$ produced by each $\{\lambda''\}$ one keeps only the ones that satisfy (17).

For $E = E_{\text{min}}$, which we are interested in, the solutionis of steps 1) and 2) for nuclei with $A \leq 80$ are given below.

For $p-$nuclei $(5 \leq A \leq 16)$, $E_{\text{min}} = A - 4$,

$$t_2 = t_3 = ... = 0; \; j_2 = j_3 = ... = 0; \;\; \widehat{D}\{\lambda\}_{A-4}^{(A)} \doteq \{\lambda\}_{A-4}^{(A)} \qquad (21)$$

and the reduction $U(A-1) \supset S(A)$ is

$$\{\lambda\}_{A-4}^{(A)} \doteq [4, \lambda_1, \lambda_2, \lambda_3] \text{ with } \lambda_1 + \lambda_2 + \lambda_3 = A - 4 \text{ and } \lambda_i \leq 4. \qquad (22)$$

[here and in the following the symbol $\doteq$ means that on the RHS only the terms which may produce physically acceptable $S(A)$ irreps are considered.]

For $(s-d)$-nuclei $(17 \leq A \leq 40)$, $E_{\text{min}} = 2A - 20$,

$$t_2 = A-16;\ t_3=t_4=...=0;\ j_2=j_3=....=0, \tag{23}$$

$$\widehat{D}\{\lambda\}^{(A)}_{2A-20} \doteq \sum_{\{\lambda\}_{A-16}} \{4^3,\{\lambda\}_{A-16}\}\alpha(\{2\}\dot{\otimes}\{\lambda\}_{A-16}\} \to \{\lambda\}_{2(A-16)} = \{\lambda\}-\{4^3\}). \tag{24}$$

[The symbol $\dot{\otimes}$ denotes a reduced plethysm, that is, a plethysm expansion in which only the terms with up to 3 rows are considered.]

The Schur function $\{4^3,\{\lambda\}_{A-16}\}$ will produce, by Eq.(11), $S(A)$ irreps $[4^4,\{\lambda\}_{A-16}]$.

For $(p-f)-$nuclei $(41 \le A \le 80)$, $E_{\text{min}} = 3A-60$,

$$t_2 = 24; \quad t_3 = A-40; \quad t_4=t_5=0; \quad j_2=j_3=...=0\,, \tag{25}$$

$$\widehat{D}\{\lambda\}^{(A)}_{3A-60} \doteq \sum_{\{\lambda\}_{A-40}} \{4^9,\{\lambda\}_{A-40}\}\alpha(\{3\}\dot{\otimes}\{\lambda\}_{A-40} \to \{\lambda\}_{3(A-40)} = \{\lambda\}_{3A-60}-\{20^3\})\,. \tag{26}$$

The Schur functions $\{4^9,\{\lambda\}_{A-40}\}$ will produce, by Eq.(11), $S(A)$ irreps $[4^{10},\{\lambda\}_{A-40}]$.

Analogous to the case of $p-$ and $(s-d)-$nuclei, Eq.(26) allows us to read the reduction $U(A-1) \supset S(A)$ for nuclei in the ground configuration of this shell directly from the table of multiplicities of Schur functions $\{\lambda\}_{3(A-40)}$ in the reduced plethysms $\{3\}\dot{\otimes}\{\lambda\}_{A-40}$. The column associated to a given Schur function $\{\lambda\}_{3(A-40)}$ corresponds to the $U(A-1)$ irrep $\{20^3\}+\{\lambda\}_{3(A-40)}$. Its entries, in each line labelled by $\{\lambda\}_{A-40}$ give the multiplicity of $S(A)$ irrep $[4^{10},\{\lambda\}_{A-40}]$ in the reduction.

The $U(A-1) \supset S(A)$ reductions for $p-$ and $(s-d)-$nuclei in minimal energy configuration are given in (I) and in [1] in a different organization.

For $(p-f)-$nuclei the reductions are given in the tables below.

3 Explanation of tables

The equations in the tables give the reduction of $U(A-1)$ irrep $\{E_1,E_2,E_3\}$ into $S(A)$ irreps $[\lambda]$. On their RHS are listed only the $S(A)$ irreps physically acceptable, that is, those satisfying (17) as the symbol $\doteq$ indicates.

For a given A and minimal configuration energy, only the $U(A-1)$ irreps that have at least one physically acceptable irrep in its reduction to $S(A)$ are listed.

Only the first half of the shell $(41 \le A \le 60)$ is listed in the tables. The second half, $(61 \le A \le 79)$, is obtained by the use of particle-hole symmetry in the open shell. For this shell this symmetry reads as

$$\begin{aligned} A &\leftrightarrow \bar{A}; \\ \lambda &\equiv [4^{10},\lambda^{(0)}_1,\lambda^{(0)}_2,...,\lambda^{(0)}_{10}] \leftrightarrow [4^{10},4-\lambda^{(0)}_{10},4-\lambda^{(0)}_9,...,4-\lambda^{(0)}_1]. \end{aligned} \tag{27}$$

For $A=80$ one has

$$\{60^3\} \doteq [4^{20}]. \tag{28}$$

In the tables are listed only the first 5 irreps $\{E_1,E_2,E_3\}$ more symmetric in $U(3)$ labels, *i.e.*, those with greatest values of $U(3)$ Casimir invariant.

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TABLES

$A = 41$

$\{23, 20^2\} \doteq [4^{10}, 1]$

$A = 42$

$\{26, 20^2\} \doteq [4^{10}, 2]$; $\{25, 21, 20\} \doteq [4^{10}, 1^2]$; $\{24, 22, 20\} \doteq [4^{10}, 2]$; $\{23^2, 20\} \doteq [4^{10}, 1^2]$

$A = 43$

$\{29, 20^2\} \doteq [4^{10}, 3]$; $\{28, 21, 20\} \doteq [4^{10}, 2, 1]$; $\{27, 22, 20\} \doteq [4^{10}, 3] + [4^{10}, 2, 1]$;
$\{27, 21^2\} \doteq [4^{10}, 1^3]$; $\{26, 23, 20\} \doteq [4^{10}, 3] + [4^{10}, 2, 1] + [4^{10}, 1^3]$; ...

$A = 44$

$\{32, 20^2\} \doteq [4^{11}]$; $\{31, 21, 20\} \doteq [4^{10}, 3, 1]$; $\{30, 22, 20\} \doteq [4^{11}] + [4^{10}, 3, 1] + [4^{10}, 2^2]$;
$\{30, 21^2\} \doteq [4^{10}, 2, 1^2]$; $\{29, 23, 20\} \doteq [4^{11}] + 2[4^{10}, 3, 1] + [4^{10}, 2, 1^2]$; ...

$A = 45$

$\{34, 21, 20\} \doteq [4^{11}, 1]$; $\{33, 22, 20\} \doteq [4^{11}, 1] + [4^{10}, 3, 2]$; $\{33, 21^2\} \doteq [4^{10}, 3, 1^2]$;
$\{32, 23, 20\} \doteq 2[4^{11}, 1] + [4^{10}, 3, 2] + [4^{10}, 3, 1^2]$; $\{32, 22, 21\} \doteq [4^{11}, 1] + [4^{10}, 3, 2] + [4^{10}, 3, 1^2] + [4^{10}, 2^2, 1]$; ...

$A = 46$

$\{36, 22, 20\} \doteq [4^{11}, 2]$; $\{36, 21^2\} \doteq [4^{11}, 1^2]$; $\{35, 23, 20\} \doteq [4^{11}, 2] + [4^{11}, 1^2] + [4^{10}, 3^2]$;
$\{35, 22, 21\} \doteq [4^{11}, 2] + [4^{11}, 1^2] + [4^{10}, 3, 2, 1]$; $\{34, 24, 20\} \doteq 3[4^{11}, 2] + [4^{11}, 1^2] + [4^{10}, 3, 2, 1]$; ...

$A = 47$

$\{38, 23, 20\} \doteq [4^{11}, 3]$; $\{38, 22, 21\} \doteq [4^{11}, 2, 1]$; $\{37, 24, 20\} \doteq [4^{11}, 3] + [4^{11}, 2, 1]$;
$\{37, 23, 21\} \doteq [4^{11}, 3] + 2[4^{11}, 2, 1] + [4^{11}, 1^3] + [4^{10}, 3^2, 1]$; $\{37, 22^2\} \doteq [4^{11}, 3] + [4^{11}, 2, 1] + [4^{10}, 3, 2^2]$; ...

$A = 48$

$\{40, 24, 20\} \doteq [4^{12}]$; $\{40, 23, 21\} \doteq [4^{11}, 3, 1]$; $\{40, 22^2\} \doteq [4^{11}, 2^2]$; $\{39, 25, 20\} \doteq [4^{11}, 3, 1]$;
$\{39, 24, 21\} \doteq [4^{12}] + 2[4^{11}, 3, 1] + [4^{11}, 2^2] + [4^{11}, 2, 1^2]$; ...

$A = 49$

$\{42, 24, 21\} \doteq [4^{12}, 1]$; $\{42, 23, 22\} \doteq [4^{11}, 3, 2]$; $\{41, 26, 20\} \doteq [4^{12}, 1]$;
$\{41, 25, 21\} \doteq [4^{12}, 1] + [4^{11}, 3, 2] + [4^{11}, 3, 1^2]$;
$\{41, 24, 22\} \doteq 2[4^{12}, 1] + 2[4^{11}, 3, 2] + [4^{11}, 3, 1^2] + [4^{11}, 2^2, 1]$; ...

$A = 50$

$\{44, 24, 22\} \doteq [4^{12}, 2]$; $\{44, 23^2\} \doteq [4^{11}, 3^2]$; $\{43, 26, 21\} \doteq [4^{12}, 2] + [4^{12}, 1^2]$;
$\{43, 25, 22\} \doteq [4^{12}, 2] + [4^{12}, 1^2] + [4^{11}, 3^2] + [4^{11}, 3, 2, 1]$;
$\{43, 24, 23\} \doteq 2[4^{12}, 2] + [4^{12}, 1^2] + [4^{11}, 3^2] + [4^{11}, 3, 2, 1]$; ...

$A = 51$

$\{46, 24, 23\} \doteq [4^{12}, 3]$; $\{45, 26, 22\} \doteq [4^{12}, 3] + [4^{12}, 2, 1]$; $\{45, 25, 23\} \doteq [4^{12}, 3] + [4^{12}, 2, 1] + [4^{11}, 3^2, 1]$;
$\{45, 24^2\} \doteq [4^{12}, 3] + [4^{12}, 2, 1]$; $\{44, 28, 21\} \doteq [4^{12}, 3] + [4^{12}, 2, 1]$; ...

$A = 52$

$\{48, 24^2\} \doteq [4^{13}]$; $\{47, 26, 23\} \doteq [4^{13}] + [4^{12}, 3, 1]$; $\{47, 25, 24\} \doteq [4^{12}, 3, 1]$;
$\{46, 28, 22\} \doteq [4^{13}] + [4^{12}, 3, 1] + [4^{12}, 2^2]$; $\{46, 27, 23\} \doteq [4^{13}] + 3[4^{12}, 3, 1] + [4^{12}, 2^2] + [4^{12}, 2, 1^2] + [4^{11}, 3^2, 2]$; ...

$A = 53$

$\{49, 26, 24\} \doteq [4^{13}, 1]$; $\{48, 28, 23\} \doteq [4^{13}, 1] + [4^{12}, 3, 2]$; $\{48, 27, 24\} \doteq 2[4^{13}, 1] + [4^{12}, 3, 2] + [4^{12}, 3, 1^2]$;
$\{48, 26, 25\} \doteq [4^{13}, 1] + [4^{12}, 3, 2] + [4^{12}, 3, 1^2]$; $\{47, 30, 22\} \doteq [4^{13}, 1] + [4^{12}, 3, 2]$; ...

$A = 54$

$\{50, 28, 24\} \doteq [4^{13}, 2]$; $\{50, 27, 25\} \doteq [4^{13}, 1^2]$; $\{50, 26^2\} \doteq [4^{13}, 2]$;
$\{49, 30, 23\} \doteq [4^{13}, 2] + [4^{12}, 3^2]$; $\{49, 29, 24\} \doteq 2[4^{13}, 2] + 2[4^{13}, 1^2] + [4^{12}, 3^2] + [4^{12}, 3, 2, 1]$; ...

$A = 55$

$\{51, 30, 24\} \doteq [4^{13}, 3]$; $\{51, 29, 25\} \doteq [4^{13}, 2, 1]$; $\{51, 28, 26\} \doteq [4^{13}, 3] + [4^{13}, 2, 1]$;
$\{51, 27^2\} \doteq [4^{13}, 1^3]$; $\{50, 32, 23\} \doteq [4^{13}, 3]$; ...

$A = 56$

$\{52, 32, 24\} \doteq [4^{14}]$; $\{52, 31, 25\} \doteq [4^{13}, 3, 1]$; $\{52, 30, 26\} \doteq [4^{14}] + [4^{13}, 3, 1] + [4^{13}, 2^2]$;
$\{52, 29, 27\} \doteq [4^{13}, 3, 1] + [4^{13}, 2, 1^2]$; $\{52, 28^2\} \doteq [4^{14}] + [4^{13}, 2^2]$; ...

$A = 57$

$\{53, 33, 25\} \doteq [4^{14}, 1]$; $\{53, 32, 26\} \doteq [4^{14}, 1] + [4^{13}, 3, 2]$; $\{52, 35, 24\} \doteq [4^{14}, 1]$;
$\{53, 31, 27\} \doteq [4^{14}, 1] + [4^{13}, 3, 2] + [4^{13}, 3, 1^2]$; $\{53, 30, 28\} \doteq [4^{14}, 1] + [4^{13}, 3, 2] + [4^{13}, 2^2, 1]$; ...

$A = 58$

$\{54, 34, 26\} \doteq [4^{14}, 2]$; $\{54, 33, 27\} \doteq [4^{14}, 2] + [4^{14}, 1^2] + [4^{13}, 3^2]$; $\{53, 36, 25\} \doteq [4^{14}, 2] + [4^{14}, 1^2]$;
$\{52, 38, 24\} \doteq [4^{14}, 2]$; $\{54, 32, 28\} \doteq 2[4^{14}, 2] + [4^{13}, 3, 2, 1]$; ...

$A = 59$

$\{55, 35, 27\} \doteq [4^{14}, 3]$; $\{55, 34, 28\} \doteq [4^{14}, 3] + [4^{14}, 2, 1]$; $\{50, 44, 23\} \doteq [4^{14}, 3]$;
$\{54, 37, 26\} \doteq [4^{14}, 3] + [4^{14}, 2, 1]$; $\{52, 41, 24\} \doteq [4^{14}, 3]$; ...

$A = 60$

$\{56, 36, 28\} \doteq [4^{15}]$; $\{52, 44, 24\} \doteq [4^{15}]$; $\{56, 35, 29\} \doteq [4^{14}, 3, 1]$;
$\{51, 45, 24\} \doteq [4^{14}, 3, 1]$; $\{55, 38, 27\} \doteq [4^{15}] + [4^{14}, 3, 1]$; ...