Internal Space-time Symmetries of Particles derivable from Periodic Systems in Optics

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Abstract

While modern optics is largely a physics of harmonic oscillators and two-by-two matrices, it is possible to learn about some hidden properties of the two-by-two matrix from optical systems. Since two-by-two matrices can be divided into three conjugate classes depending on their traces, optical systems force us to establish continuity from one class to another. It is noted that those three classes are equivalent to three different branches of Wigner's little groups dictating the internal space-time symmetries massive, massless, and imaginary-mass particles. It is shown that the periodic systems in optics can also be described by have the same class-based matrix algebra. The optical system allow us to make continuous, but not analytic, transitions from massiv to massless, and massless to imaginary-mass cases.

1   Introduction

Two-by-two matrices with real elements have three independent parameters if their determinants are constrained to be one. They constitute building blocks for many branches of physics, including beam-transfer matrices in optics [1] and Wigner's little groups for internal space-time symmetry of particles [2, 3, 4].

For the two-by-two matrix, we are accustomed to solve a quadratic equation to get the eigenvalues and construct a rotation matrix to get the eigenvalues. This procedure does not always lead to correct answers, because squeeze matrices should also be considered [5]. We are quite familiar with

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rotations, but the concept of squeeze started getting our attention only after squeezed states of light appeared in the physics literature [6].

In particle physics, Lorentz boosts are squeeze transformations, and this aspect was addressed by Paul A. M. Dirac in his 1949 and 1963 papers [7, 8]. It is possible to apply this concept to high-speed hadrons which are bound state of the quarks which are thought to be more fundamental particles [3]. Thus, high-energy hadronic physics and modern optics share the same mathematical base, and it is profitable to trade physics between these two branches of physics using the common mathematics language.

In this report, we note first that optical beam transfer matrix, often called the $ABCD$ matrix, has three independent parameters and its determinant is one. We study then how this matrix can be decomposed into one-parameter matrices. The Bargmann decomposition and Iwasawa decomposition are already familiar to us, and are used often in the literature [9, 10].

We show that, in addition, there is a decomposition based on the concept of conjugate classes [11]. There are three conjugate classes. The first class consists of those matrices with their traces smaller than two, the second class consists of those with the traces equal to two, and the third consisting of those matrices with their traces greater than two. It is remarkable that this purely mathematical theorem corresponds to Wigner's construction of his little groups. which dictate the internal space-time symmetries of massive, massless, and imaginary-mass particles respectively [2].

In Sec. 2, we introduce the Wigner decomposition based on the conjugate classes of the two-by-two matrices. It is then shown that the Wigner decomposition can be translated into the Bargmann and Iwasawa decompositions. In Sec. 3, we discuss how we can formulate those three decompositions while studying optical multilayer systems. It is shown that the Wigner decomposition is needed for repeated application of the $ABCD$ matrix for periodic system. In Sec. 4, we study how those three decompositions can serve useful purposes in studying Wigner's little groups and thus the internal space-time symmetries of elementary particles.

2   Decompositions of the ABCD Matrix

The two-by-two matrix with real elements and unit determinant can be written as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1}$$

with $AD - BC = 1$ has three independent elements. This matrix is commonly used as the beam transfer matrices in optics [1]. The complete set of these matrices is like the group $SU(1,1)$ which serves as the fundamental language for squeezed states of light [6, 12]

All the matrices in this set can be divided into three classes depending on their traces [11]. Without changing its trace, we can bring every matrix to an equi-diagonal form by a rotation [15].

We shall use the notation $[ABCA]$ for the equi-diagonal $ABCD$ matrix. This equi-diagonal matrix now has two-independent parameters. If its trace is smaller than two, the matrix can be written as

$$[ABCA] = \begin{pmatrix} \cos\phi & -e^{\eta}\sin\phi \\ e^{-\eta}\sin\phi & \cos\phi \end{pmatrix}. \tag{2}$$

If the trace is greater than two, it can take the form

$$[ABCA] = \begin{pmatrix} \cosh\chi & -e^{\eta}\sinh\chi \\ -e^{-\eta}\sinh\chi & \cosh\chi \end{pmatrix}. \tag{3}$$

If the trace is equal to two, it can be brought to the form

$$[ABCA] = \begin{pmatrix} 1 & -\gamma e^{\eta} \\ 0 & 1 \end{pmatrix}. \tag{4}$$

. We choose to use the notation $W(\tau)$ collectively for the following three matrices.

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix}, \quad \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}. \tag{5}$$

The parameter τ could be θ, χ, or γ. If we define the matrix $B(\eta)$ as

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{6}$$

Then every equi-diagonal $ABCA$ matrix can be written as

$$[ABCA] = B(\eta)W(\tau)B(-\eta). \tag{7}$$

Indeed, the matrix $W(\tau)$ constitutes a set of three matrices which play an important role in the theory of two-by-two matrices.

It is important to note that these three matrices constitute the basic elements for Wigner's little group which dictates the internal space-time symmetries of elementary particles [2, 3]. We thus choose to call $W(\tau)$ the "Wigner matrix" [13] and call the expression of Eq.(7) "Wigner decomposition." The Wigner decomposition leads to

$$[ABCA]^N = B(\eta)[W(\tau)]^N B(-\eta) = B(\eta)W(N\tau)B(-\eta), \qquad (8)$$

convenient in dealing with repeated applications of the $ABCA$ matrix in periodic systems.

It is known that the $ABCA$ matrix can be written as the product of three one-parameter matrices

$$\begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \cosh\lambda & -\sinh\lambda \\ -\sinh\lambda & \cosh\lambda \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$
$$(9)$$

This form is called the Bargmann decomposition [9, 14]. This expression can be compressed to

$$[ABCA] = \begin{pmatrix} \cosh\lambda\cos\theta & -\sinh\lambda - \cosh\lambda\sin\theta \\ -\sinh\lambda + \cosh\lambda\sin\theta & \cosh\lambda\cos\theta \end{pmatrix}, \qquad (10)$$

with two independent parameters. We shall call this expression the "Bargmann matrix."

If $\sin\theta = \tanh\lambda$, the $ABCA$ matrix takes the triangular form given in Eq.(5). Then this special case of the Bargmann decomposition is called the "Iwasawa" decomposition. This form is also known to correspond to gauge transformations of massless particles [3, 14].

We shall see in this report that both Wigner and Bargmann decompositions play essential roles in optics and space-time symmetries. The question then is whether one transformation can be translated into the other. If $\cosh\lambda\cos\theta$ is smaller than one, the off-diagonal elements have opposite signs. The Bargmann matrix of Eq.(10) will be translated into Eq.(2) with

$$\cos\phi = \cosh\lambda\cos\theta, \qquad e^\eta = \sqrt{\frac{\sin\theta - \tanh\lambda}{\sin\theta + \tanh\lambda}} \qquad (11)$$

If the diagonal element is greater than one, the matrix will be translated into Eq.(3) with

$$\cosh\chi = \cosh\lambda\cos\theta, \qquad e^\eta = \sqrt{\frac{\tanh\lambda - \sin\theta}{\tanh\lambda + \sin\theta}} \qquad (12)$$
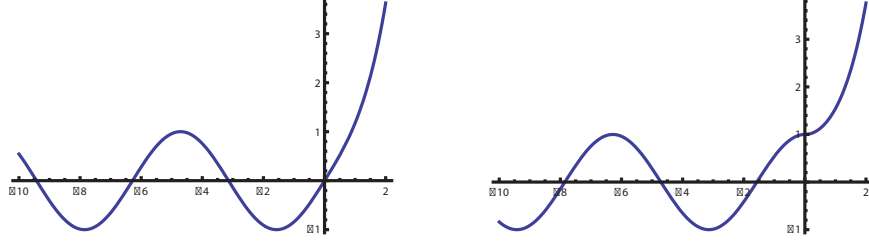
Figure 1: Transitions from sin to sinh, and from cos to cosh. They are continuous transitions. Their first derivatives are also continuous, but the second derivatives are not. Thus, they are not analytic continuations.

These variable transformations are simple enough, but the real issue is what happens when the diagonal element makes a transition from less-than-one to greater-than-one. If it becomes one, the lower left element of the Bargmann matrix becomes zero, and the matrix becomes triangular.

In order to study this transition, we introduce a small number

$$\epsilon = \cosh\lambda\sin\theta - \sinh\lambda, \tag{13}$$

and see what happens when it changes its sign. When ϵ is positive, the Bargmann matrix can be written as

$$\begin{pmatrix} 1 - \epsilon\sinh\eta\cosh\eta & -2\sinh\eta \\ \epsilon\cosh\eta & 1 - \epsilon\sinh\eta\cosh\eta \end{pmatrix}. \tag{14}$$

If we let

$$\alpha = \sqrt{2\epsilon\sinh\eta\cosh\eta}, \qquad \beta = \sqrt{\frac{\epsilon\cosh\eta}{2\sinh\eta}}, \tag{15}$$

the matrix becomes

$$\begin{pmatrix} 1 - \alpha^2/2 & -\alpha/\beta \\ \alpha\beta & 1 - \alpha^2/2 \end{pmatrix}, \tag{16}$$

which can be decomposed into

$$\begin{pmatrix} 1/\sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} 1 - \alpha^2/2 & -\alpha \\ \alpha & 1 - \alpha^2/2 \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ 0 & 1/\sqrt{\beta} \end{pmatrix}. \tag{17}$$

If ϵ is negative, we should define α and β as

$$\alpha = \sqrt{-2\epsilon\sinh\eta\cosh\eta}, \qquad \beta = \sqrt{\frac{-\epsilon\cosh\eta}{2\sinh\eta}}, \tag{18}$$

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and the matrix becomes

$$\begin{pmatrix} 1 + \alpha^2/2 & -\alpha\beta \\ -\alpha/\beta & 1 + \alpha^2/2 \end{pmatrix}, \tag{19}$$

which can be decomposed into

$$\begin{pmatrix} 1/\sqrt{\beta} & 0 \\ 0 & \sqrt{\beta} \end{pmatrix} \begin{pmatrix} 1 + \alpha^2/2 & -\alpha \\ -\alpha & 1 + \alpha^2/2 \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ 0 & 1/\sqrt{\beta} \end{pmatrix}. \tag{20}$$

It is clear now that the transition from the rotation-like Wigner (with $\cos\phi$ as the diagonal element) to the squeeze-like matrix (with $\cosh\chi$ as the diagonal element) is like the transitions from $\sin\alpha$ to $\sinh\alpha$, and from $\cos\alpha$ to $\cosh\alpha$, as shown in Fig. 1. This is a continuous transition, but not analytic. The second derivative is not continuous.

3   Periodic System in Optics

Let us consider an optical beam going through multiple layers consisting of two different refractive indexes. This problem has been extensively discussed in the literature. The two-by-two matrix formulation of this problem is given in the Appendix.

When the beam goes through the first medium, we can use the matrix

$$R(2\phi_1) = \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}. \tag{21}$$

For the second medium, we use ϕ_2 instead of ϕ_1.

If the beam in the first medium hits the second medium, it is partially transmitted and partially reflected. According to the Appendix, the boundary matrix takes the form

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{22}$$

This form is also given in Eq.(6) in connection with the Wigner decomposition. When the beam hits the first medium from the second medium, the boundary matrix is $B(-\eta)$.

Let us consider the cycle which starts from the half-way in the second medium and ends at the second medium as illustrated in Fig 2. Then the beam transfer matrix becomes

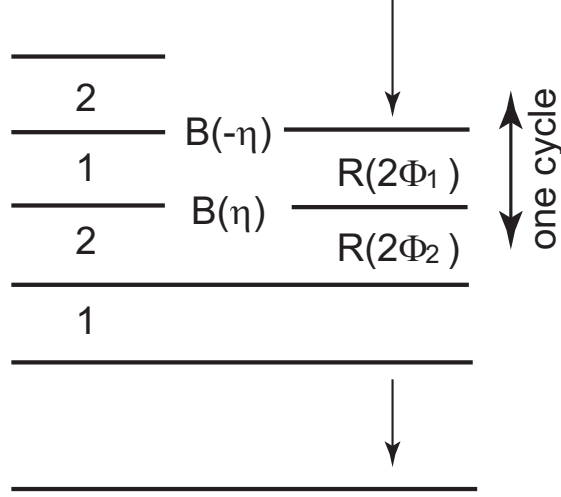$$[ABCA] = R(\phi_2)\,[B(\eta)R(2\phi_1)\,B(-\eta)]\,R(\phi_2). \tag{23}$$

Figure 2: Optical layers. There are phase-shift matrices for their respective layers. There is a boundary matrix for the transition from the first to second medium, and its inverse applies to the transition from the second to first medium. The cycle starts from the middle of the second layer.

The quantity inside the square bracket is takes a form the Wigner decomposition, but it is sandwiched between two rotation matrix. Our problem is to write the entire matrix chain as a Wigner decomposition, convenient for the periodic system. For this purpose, we write the Wigner decomposition in the middle as a Bargmann decomposition

$$R(\theta)S(-2\lambda)R(\theta), \tag{24}$$

where θ and λ are determined by

$$\cosh\lambda = (\cosh\eta)\sqrt{1 - \cos^2\phi_1\tanh^2\eta},$$

$$\cos\theta = \frac{\cos\phi_1}{(\cosh\eta)\sqrt{1 - \cos^2\phi_1\tanh^2\eta}}. \tag{25}$$

We can now write $[ABCA]$ as

$$R(\phi_2)\,R(\theta)S(-2\lambda)R(\theta)R(\phi_2). \tag{26}$$

Since $R(\theta)R(\phi_2) = R(\theta + \phi_2/2)$, the $[ABCA]$ matrix becomes a Bargmann decomposition of the form

$$[ABCA] = R(\theta^*)S(-2\lambda)R(\theta^*), \tag{27}$$

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with

$$\theta^* = \theta + \frac{1}{2}\phi_2. \tag{28}$$

After matrix multiplications, the $[ABCA]$ matrix can be written as

$$\begin{pmatrix} \cosh\lambda\cos\theta^* & -\sinh\lambda - \cosh\lambda\sin\theta^* \\ -\sinh\lambda + \cosh\lambda\sin\theta^* & \cosh\lambda\cos\theta^* \end{pmatrix} \tag{29}$$

For the multilayer system, we need a repeated application of this from. For this purpose, we have to convert this into the Wigner decomposition:

$$B(\eta^*)W(\tau^*)B(-\eta^*). \tag{30}$$

When $\cosh\lambda\cos\theta^*$ is smaller than one,

$$W(\tau^*) = \begin{pmatrix} \cos\phi^* & -\sin\phi^* \\ \sin\phi^* & \cos\phi^* \end{pmatrix}, \tag{31}$$

with

$$\cos\phi^* = \cosh\lambda\cos\theta^*, \qquad \exp(\eta^*) = \sqrt{\frac{\sin\theta^* - \tanh\lambda}{\sin\theta^* + \tanh\lambda}} \tag{32}$$

If the diagonal element is greater than one, the $W(\tau^*)$ matrix is of the form

$$W(\tau*) = \begin{pmatrix} \cosh\chi^* & -\sinh\chi^* \\ -\sinh\chi^* & \cosh\chi^* \end{pmatrix}, \tag{33}$$

with

$$\cosh\chi^* = \cosh\lambda\cos\theta^*, \qquad \exp(\eta) = \sqrt{\frac{\tanh\lambda - \sin\theta^*}{\tanh\lambda + \sinh\theta^*}} \tag{34}$$

The situation is similar if the diagonal element is one, and the matrix is triangular. We can use this Wigner decomposition to calculate

$$[B(\eta^*)W(\tau*)B(-\eta^*)]^N = B(\eta^*)\left[W(\tau^*)\right]^N B(-\eta) \tag{35}$$

It is seen that this periodic system contains the Wigner, Bargmann, and Iwasawa decompositions in its natural language, and it forces us to bring the $[ABCA]$ matrix into its Wigner decomposition for its repeated applications in the periodic system. In Sec 4, we shall study what this Wigner decomposition means in symmetries in particle physics.

4 Space-time Symmetries

In 1939 [2], Eugene Wigner considered subgroups of the Lorentz group whose transformations leave the given momentum of a particle invariant. These subgroups are called Wigner's little groups. While leaving the momentum invariant, the little group transforms internal space-time variables. For instance, if the particle is at rest, rotations do not change its momentum, but they can change the orientations of the particle spin. In this section, we formulate Wigner's little group using the technique of decompositions discussed in Sec. 2 and Sec. 3.

In Sec. 2, we studied the decompositions of the $[ABCD]$ into single-parameter matrices, using the two-by-two matrices $B(\eta), R(\phi)$, and $S(\chi)$. It is known that these two-by-two matrices correspond to the four-by-four Lorentz-transformation matrices applicable to the Minkowskian space-time coordinate variables (z, y, z, t) [3]. The two-by-two matrix $B(\eta)$ can also be written as the the four-by-four matrix for the Lorents boost along the z direction:

$$
B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh \eta & \sinh \eta \\ 0 & 0 & \sinh \eta & \cosh \eta \end{pmatrix}, \tag{36}
$$

which performs a Lorentz boost along the z direction. The two-by-two matrix $R(\phi)$ an be translated into

$$
R(\phi) = \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \rightarrow \begin{pmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{37}
$$

which performs a rotation around the y axis. The $S(\chi)$ corresponds to

$$
S(\chi) = \begin{pmatrix} \cosh(\chi/2) & \sinh(\chi/2) \\ \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix} \rightarrow \begin{pmatrix} \cosh \chi & 0 & 0 & \sinh \chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \chi & 0 & 0 & \cosh \chi \end{pmatrix}, \tag{38}
$$

which performs a Lorentz boost along the x direction.

It is possible now to translate the contents of Secs. 2 and 3 into the language of four-by-four Lorentz transformation matrices applicable to the

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Minkowski space of (z, y, z, t). In this convention, the momentum-energy four-vector is (p_x, p_y, p_z, E). If the particle moves along the z direction, this four-vector becomes

$$\left(0, 0, p, \sqrt{p^2 + m^2}\right), \tag{39}$$

in the unit system where $c = 1$, where m is the particle mass. We can obtain this four-vector by boosting a particle at rest with the four-momentum

$$(0, 0, 0, m), \tag{40}$$

using the four-by-four boost matrix given in Eq.(36), with

$$\tanh(\eta) = \frac{p}{\sqrt{p^2 + m^2}}. \tag{41}$$

Now the four-momentum of Eq.(40) is invariant under the rotation matrix

$$R(2\phi) = \begin{pmatrix} \cos(2\phi z) & 0 & \sin(2\phi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(2\phi) & 0 & \cos(2\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{42}$$

Thus, the matrix

$$B(\eta)R(2\phi)B(-\eta) \tag{43}$$

leaves the four-momentum of Eq.(39) invariant. After making this rotation, we can bring the momentum to its initial state by boosting it by $B(\eta)$. The net effect is the momentum-preserving transformation. This set of transformations is illustrated in Fig. 3, and corresponds to the Wigner decomposition.

If the particle has a space-like momentum, we can start with the four-momentum

$$(0, 0, p, E), \tag{44}$$

where E is smaller than p, which it can be brought to the Lorentz frame where the four-vector becomes

$$(0, 0, p, 0). \tag{45}$$

The boost matrix takes form of Eq.(36), with

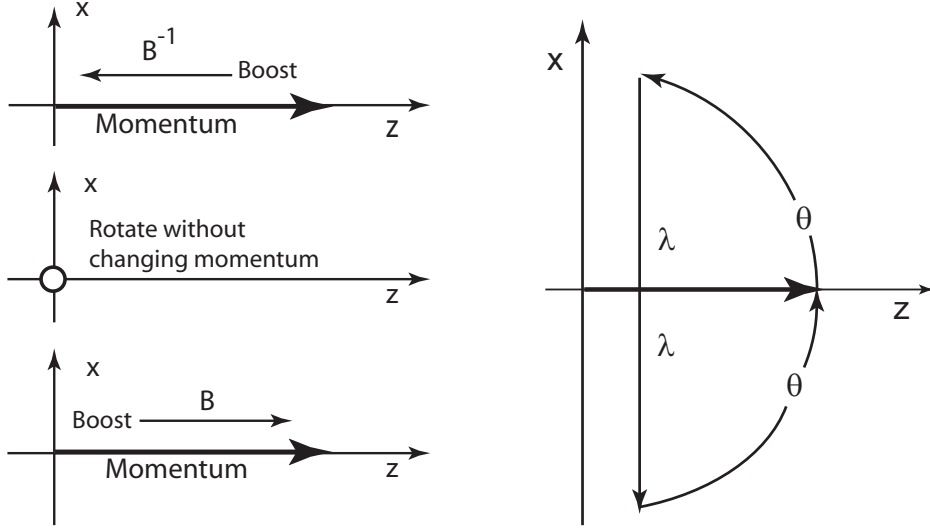$$\tanh(\eta) = \frac{E}{p}. \tag{46}$$

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Figure 3: Illustrations of the Wigner decomposition (left) and the Bargmann decomposition. In both cases, the net transformation leaves the four-momentum of the particle unchanged.

The four-momentum of Eq.(45) is invariant under the boost

$$
S(-2\chi) = \begin{pmatrix} \cosh(2\chi) & 0 & 0 & -\sinh(2\chi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(2\chi) & 0 & 0 & \cosh(2\chi) \end{pmatrix} \tag{47}
$$

along the x direction.

Let us finally consider a massless particle with it four-momentum

$$
(0, 0, p, p). \tag{48}
$$

It is invariant under the rotation around the z axis. In addition, it is invariant under the transformation

$$
\begin{pmatrix} 1 & 0 & -2\gamma & 2\gamma \\ 0 & 1 & 0 & 0 \\ 2\gamma & 0 & 1-2\gamma^2 & 2\gamma^2 \\ 2\gamma & 0 & -2\gamma^2 & 1+2\gamma^2 \end{pmatrix}. \tag{49}
$$

This four-by-four matrix has a stormy history [3, 4], but the bottom line is that it corresponds to the triangular matrix of Eq.(5), and the variable γ performs gauge transformations.

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We can obtain this massless case from the massive or imaginary case using the limiting procedure spelled out in Sec. 2. This procedure is widely known as the group contraction in the literature.

As for the Bargmann decomposition, let us go to Fig. 3. The particle moving along the z direction can be rotated first by $R(\theta)$. It can then be boosted along the negative x axis, and then rotated again by $R(\theta)$ to the original position. Indeed, this is a momentum-preserving transformation. The net transformation can be written as

$$R(\theta)S(-2\lambda)R(\theta). \qquad (50)$$

This Bargmann decomposition is applicable to all three cases of the momentum [14].

In addition, Wigner's little group allows rotations around the momentum which does not change it. This extra degree of freedom does not affect the description of the symmetries given in this section [16]

Concluding Remarks

In this report, we started with a two-by-two matrix with real elements, but we were led to consider three classes of equi-diagonal matrices, with their traces less than two, equal to two, and greater than two. From the mathematical point of view, this process is the construction of group representations according to conjugate classes.

These conjugate classes correspond to Wigner's little groups for the internal space-time symmetries for massive, massless, and imaginary-mass particles in the Lorentz-covariant world. Indeed, this aspect is a very "unreasonable agreement" between mathematics and physics.

It was noted that these equi-diagonal matrices as products of three one-parameter matrices, resulting in the Wigner, Bargmann, and Iwasawa decompositions. It is noted further that the optical periodic systems, such as multilayer optics, can perform these decompositions. Thus, the optical periodic system speaks the language of the fundamental symmetries for elementary particles in Einstein's Lorentz-covariant world.

Appendix

The N-layer optics starts with the boundary matrix of the form [17, 10]

$$B(\eta) = \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}, \tag{51}$$

which, as illustrated in Fig. 2, describes the transition from *medium* 2 to *medium* 1, taking into account both the transmission and reflection of the beam. As the beam goes through the *medium* 1, the beam undergoes the phase shift represented by the matrix

$$P(\phi_1) = \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_1} \end{pmatrix}. \tag{52}$$

When the wave hits the surface of the second medium, the corresponding matrix is

$$B(-\eta) = \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix}, \tag{53}$$

which is the inverse of the matrix given in Eq.(51). Within the second medium, we write the phase-shift matrix as

$$P(\phi_2) = \begin{pmatrix} e^{-i\phi_2} & 0 \\ 0 & e^{i\phi_2} \end{pmatrix}. \tag{54}$$

Then, when the wave goes through one cycle starting from the midpoint in the second medium, the beam transfer matrix becomes

$$M_1 = \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix} \begin{pmatrix} \cosh(\eta/2) & \sinh(\eta/2) \\ \sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} e^{-i\phi_1} & 0 \\ 0 & e^{i\phi_1} \end{pmatrix}$$

$$\times \begin{pmatrix} \cosh(\eta/2) & -\sinh(\eta/2) \\ -\sinh(\eta/2) & \cosh(\eta/2) \end{pmatrix} \begin{pmatrix} e^{-i\phi_2/2} & 0 \\ 0 & e^{i\phi_2/2} \end{pmatrix}. \tag{55}$$

This arrangement of the matrices is illustrated in Fig. 2

The M_1 matrix Eq.(55) contains complex numbers, but we are interested in carrying out calculations with real matrices. This can be done by means of a conjugate or similarity transformation [10]. Let us next consider the matrix

$$C = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{i\pi/4} \\ -e^{-i\pi/4} & e^{-i\pi/4} \end{pmatrix}. \tag{56}$$

This matrix and its inverse can be written as

$$C = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \qquad C = \frac{e^{-i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{57}$$

We can then consider the conjugate transform of the M_1 matrix

$$M_2 = C \, M_1 \, C^{-1}, \tag{58}$$

with

$$M_2 = \begin{pmatrix} \cos(\phi_2/2) & -\sin(\phi_2/2) \\ \sin(\phi_2/2) & \cos(\phi_2/2) \end{pmatrix} \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{pmatrix}$$

$$\times \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \begin{pmatrix} \cos(\phi_2/2) & -\sin(\phi_2/2) \\ \sin(\phi_2/2) & \cos(\phi_2/2) \end{pmatrix}. \tag{59}$$

The conjugate transformation of Eq.(58) changes the boundary matrix $B(\eta)$ of Eq.(51) to a squeeze matrix

$$B(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}, \tag{60}$$

and the phase-shift matrices $P(\phi_1)$ of Eq.(52) and Eq.(54) to rotation matrices

$$R(2\phi_i) = \begin{pmatrix} \cos(\phi_i) & -\sin(\phi_i) \\ \sin(\phi_i) & \cos(\phi_i) \end{pmatrix}, \tag{61}$$

with $i = 1, \ 2$.

References

[1] P. S. Theocaris and E. E. Gdoutos, *Matrix Theory of Photoelasticity*(Springer-Verlag, Berlin, 1979).

[2] E. Wigner, Ann. Math. **40**, 149 (1939).

[3] Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincaré Group* (Reidel, Dordrecht, 1986).

[4] Y. S. Kim and E. P. Wigner, J. Math. Phys. **31**, 55 (1990).

[5] D. Han, Y. S. Kim, and M. E. Noz, Am. J. Phys. **67** 61 (1999).

[6] H. P. Yuen, Phys. Rev. A **13**, 2226 (1976).

[7] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[8] P. A. M. Dirac, J. Math. Phys. **4**, 901 (1963).

[9] V. Bargmann, Ann. Math. **48**, 568–640 (1947).

[10] E. Georgieva and Y. S. Kim Phys. Rev. E **64** 026602 (2001).

[11] M. Hamermesh, *Group Theory and Its Application to Physical Problems* (Addison-Wesley, Reading Massachusetts, U.S.A.).

[12] Y. S. Kim and M. E. Noz, *Phase Space Picture of Quantum Mechanics* (World Scientific, Singapore, 1991).

[13] S. Baskal and Y. S. Kim, J. Mod. Opt. **57**, 1251 (2010).

[14] D. Han and Y. S. Kim, Phys. Rev. A **37**, 4494 (1988).

[15] S. Baskal and Y. S. Kim, J. Opt. Soc. Am. A **26**, 3049 (2009)

[16] D. Han, Y. S. Kim, and D. Son, J. Math. Phys. **27**, 2228 (1986).

[17] J. J. Monzón and L. L. Sánchez-Soto, J. Opt. Soc. Am. A, **17**, 1475 (2000).