

# Direct spreading measures of Laguerre polynomials

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## Abstract

The direct spreading measures of the Laguerre polynomials  $L_n^{(\alpha)}(x)$ , which quantify the distribution of its Rakhmanov probability density  $\rho_{n,\alpha}(x) = \frac{1}{d_n^2} x^\alpha e^{-x} [L_n^{(\alpha)}(x)]^2$  along the positive real line in various complementary and qualitatively different ways, are investigated. These measures include the familiar root-mean-square or standard deviation and the information-theoretic lengths of Fisher, Renyi and Shannon types. The Fisher length is explicitly given. The Renyi length of order  $q$  (such that  $2q \in \mathbb{N}$ ) is also found in terms of  $(n, \alpha)$  by means of two error-free computing approaches; one makes use of the Lauricella function  $F_A^{(2q+1)}\left(\frac{1}{q}, \dots, \frac{1}{q}; 1\right)$ , which is based on the Srivastava-Niukkanen linearization relation of Laguerre polynomials, and another one which utilizes the multivariate Bell polynomials of Combinatorics. The Shannon length cannot be exactly calculated because of its logarithmic-functional form, but its asymptotics is provided and sharp bounds are obtained by use of an information-theoretic optimization procedure. Finally, all these spreading measures are mutually compared and computationally analyzed; in particular, it is found that the apparent quasi-linear relation between the Shannon length and the standard deviation becomes rigorously linear only asymptotically (i.e. for  $n \gg 1$ ).

*Keywords:* Orthogonal polynomials, Laguerre polynomials, spreading lengths, computation of information measures, Shannon entropy, Renyi entropy, Fisher information, Bell polynomials, Lauricella functions.

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## 1. Introduction

The Laguerre polynomials  $\{L_n^{(\alpha)}(x)\}$  are real hypergeometric polynomials orthonormal with respect to the weight function  $\omega_\alpha(x) = x^\alpha e^{-x}$  on the interval  $[-1, 1]$ . Their algebraic properties (orthogonality relation, three-term recurrence relation, second-order differential equation, ladder relation,...) are simple and widely known [1–3], what has enabled to describe a great deal of scientific and technological phenomena. The Laguerre polynomials are known to play a crucial role not only in applied and computational mathematics [1–9], mathematical physics [2, 10], combinatorics [11–14], information theory [15–21], quantum algebras [7, 22], asymptotics [18, 23–26] and theory of special functions [2, 5, 15–20, 27–29], but also in non-relativistic, relativistic and supersymmetric quantum mechanics [28, 30–33], atomic and molecular physics [21, 34–36] and  $D$ -dimensional physics [37–39]. Let us just mention that the Laguerre polynomials control to a great extent the wavefunctions which describe the quantum states of one and many-body systems with a great diversity of quantum-mechanical potentials [17, 21, 26, 30, 33, 34, 36, 38, 40–42]. The Coulomb and Morse potentials are only two particularly relevant cases in atomic and molecular physics (see e.g. [10, 31, 33, 35, 41]) as well as in  $D$ -dimensional physics [37, 43], where the radial wavefunctions are controlled by Laguerre polynomials.

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In this work we study the spreading measures of Laguerre polynomials  $L_n^{(\alpha)}$ , which quantify the distribution of its associated probability density

$$\rho_{n,\alpha}(x) = \frac{1}{d_n^2} \left[ L_n^{(\alpha)}(x) \right]^2 \omega_\alpha(x), \quad (1)$$

where  $d_n^2$  is a constant such that  $\rho_{n,\alpha}(x)$  is normalized to the unity. Heretoforth this distribution is called the Rakhmanov density of the polynomials, in honour to the pioneering work [44] of this mathematician who has shown that this density governs the asymptotic ( $n \rightarrow \infty$ ) behaviour of the ratio of two orthogonal polynomials with contiguous orders  $n$  and  $n + 1$ . Physically, this probability density characterizes the stationary states of a large class of quantum-mechanical potentials [2, 15, 17, 31, 32, 40]. The most familiar spreading measure is the simple root-mean-square or standard deviation

$$(\Delta x)_{n,\alpha} = \left( \langle x^2 \rangle_{n,\alpha} - \langle x \rangle_{n,\alpha}^2 \right)^{\frac{1}{2}}, \quad (2)$$

where the expectation value of a function  $f(x)$  is defined by

$$\langle f(x) \rangle_{n,\alpha} := \int_0^\infty f(x) \rho_{n,\alpha}(x) dx. \quad (3)$$

The information-theoretic-based spreading measures of the Laguerre polynomials are not so well known. We refer to the Fisher information

$$F[\rho_{n,\alpha}] := \left\langle \left[ \frac{d}{dx} \log \rho_{n,\alpha}(x) \right]^2 \right\rangle = \int \frac{[\rho'_{n,\alpha}(x)]^2}{\rho_{n,\alpha}(x)} dx, \quad (4)$$

to the Renyi entropy of order  $q$

$$R_q[\rho_{n,\alpha}] := \frac{1}{1-q} \log \langle [\rho_{n,\alpha}(x)]^{q-1} \rangle, \quad (5)$$

and its ( $q \rightarrow 1$ ) limit, the Shannon entropy

$$S[\rho_{n,\alpha}] := - \int_0^\infty \rho_{n,\alpha}(x) \log \rho_{n,\alpha}(x) dx,$$

which measure the distribution of the Laguerre polynomial  $L_n^{(\alpha)}$  all over the orthogonality interval without reference to any specific point of the interval, so providing alternative and complementary measures for the spreading of the Laguerre polynomials. The knowledge of these measures and some quantum-mechanical applications is reviewed in Ref. [16] up to 2001. Their behaviour for large  $n$  and fixed  $\alpha$  has been recently surveyed [18].

Since the Fisher, Renyi and Shannon measures of a given density  $\rho(x)$  have particular units, which are different from that of the variable  $x$ , it is much more useful to use the related information-theoretic lengths [45–49]; namely, the Fisher length given by

$$(\delta x)_{n,\alpha} = \frac{1}{\sqrt{F[\rho_{n,\alpha}]}}, \quad (6)$$

and the  $q$ th-order Renyi and Shannon lengths defined by

$$\mathcal{L}_q^R[\rho_{n,\alpha}] = \exp(R_q[\rho_{n,\alpha}]); \quad q > 0, q \neq 1, \quad (7)$$

and

$$N[\rho_{n,\alpha}] = \lim_{q \rightarrow 1} \mathcal{L}_q^R[\rho_{n,\alpha}] = \exp(S[\rho_{n,\alpha}]), \quad (8)$$

respectively. Following Hall [45–47], these three quantities together with the standard deviation will be referred as the direct spreading measures of the density  $\rho_{n,\alpha}$  because they share the following properties: linear scaling under adequate boundary conditions, same units as the variable, and vanishing when the density tends to a delta density. Moreover, they have an associated uncertainty property [36, 45–47] and fulfil the inequalities

$$(\delta x)_{n,\alpha} \leq (\Delta x)_{n,\alpha} \quad \text{and} \quad N[\rho_{n,\alpha}] \leq (2\pi e)^{\frac{1}{2}} (\Delta x)_{n,\alpha}, \quad \text{when } \rho_{n,\alpha}(0) = 0. \quad (9)$$

Here we will investigate the direct spreading measures of the Laguerre polynomials mentioned above. First, in Section II, we give the known values of the ordinary moments, the standard deviation and the Fisher length of these polynomials. Second, in Section III, the entropic moments  $\langle [\rho_{n,\alpha}(x)]^k \rangle$  and the Renyi lengths are computed by use of two different approaches; one makes use of the Srivastava-Niukkanen linearization relation [27] of Laguerre polynomials, and another one based on the combinatorial multivariate Bell polynomials [48, 50]. Third, in Section IV, the asymptotics of the Shannon length is given and some sharp bounds to this measure are found. Then, in Section IV, all the four spreading measures are computationally discussed. Finally, some open problems and conclusions are given.

## 2. Ordinary moments, standard deviation and Fisher length

In this section the known values for the moments-around-the-origin  $\langle x^k \rangle_{n,\alpha}$  ( $k \in \mathbb{Z}$ ), the standard deviation  $(\Delta x)_{n,\alpha}$  and the Fisher length  $(\delta x)_{n,\alpha}$  of the Laguerre polynomials are given. Let us start writing the orthonormality relation

$$\int_0^\infty \tilde{L}_n^{(\alpha)}(x) \tilde{L}_m^{(\alpha)}(x) \omega_\alpha(x) dx = \delta_{nm},$$

for the orthonormal Laguerre polynomials

$$\tilde{L}_n^{(\alpha)}(x) = \left[ \frac{n!}{\Gamma(\alpha + n + 1)} \right]^{\frac{1}{2}} L_n^{(\alpha)}(x). \quad (10)$$

Then, the moment-around-the-origin of order  $k \in \mathbb{Z}$  is defined, according to Eq. (3), by

$$\langle x^k \rangle_{n,\alpha} = \int_0^\infty x^k \rho_{n,\alpha}(x) dx = \int_0^\infty x^{k+\alpha} e^{-x} \left[ \tilde{L}_n^{(\alpha)}(x) \right]^2 dx.$$

This integral can be calculated by different means; in particular, by use of the expression [38, 51–54]

$$\int_0^\infty x^s e^{-x} L_n^\alpha(x) L_m^\beta(x) dx = \Gamma(s+1) \sum_{r=0}^{\min(n,m)} (-1)^{n+m} \binom{s-\alpha}{n-r} \binom{s-\beta}{m-r} \binom{s+r}{r},$$

one finds that

$$\langle x^k \rangle_{n,\alpha} = \frac{n! \Gamma(k + \alpha + 1)}{\Gamma(n + \alpha + 1)} \sum_{r=0}^n \binom{k}{n-r}^2 \binom{k + \alpha + r}{r},$$

where the binomial number is  $\binom{a}{b} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$ . Then, taking into account Eq. (2) and the values  $\langle x^k \rangle_{n,\alpha}$  for  $k = 1, 2$ , one has the following expression for the standard deviation of the Laguerre polynomials [19]

$$(\Delta x)_{n,\alpha} = \sqrt{2n^2 + 2(\alpha + 1)n + \alpha + 1}. \quad (11)$$

The Fisher information of the Laguerre polynomials defined by Eq. (4) has been recently shown [15, 20] to have the value

$$F[\rho_{n,\alpha}] = \begin{cases} 4n + 1; & \alpha = 0, \\ \frac{(2n+1)\alpha+1}{\alpha^2-1}; & \alpha > 1, \\ \infty; & \alpha \in (-1, +1], \alpha \neq 0, \end{cases}$$

so that the Fisher length of these polynomials has, according to (6), the value

$$(\delta x)_{n,\alpha} = \begin{cases} \frac{1}{\sqrt{4n+1}}; & \alpha = 0, \\ \sqrt{\frac{\alpha^2-1}{(2n+1)\alpha+1}}; & \alpha > 1, \\ 0; & \alpha \in (-1, +1], \alpha \neq 0. \end{cases}$$

It is worth remarking that the inequality  $(\delta x)_{n,\alpha} \leq (\Delta x)_{n,\alpha}$  is clearly satisfied.

### 3. Renyi lengths

In this section the Renyi lengths of the Laguerre polynomials  $\mathcal{L}_q^R[\rho_{n,\alpha}]$  defined by Eq. (7) will be computed by two different approaches: an algebraic approach which is based on the Srivastava-Niukkanen linearization relation [27], and a combinatorial method which utilizes the multivariate Bell polynomials [50]. Let us advance that the Renyi integrals are computed only for the half-integer values of  $q$ . It is worth noting that the final formulas cannot clearly be extended to all real values of  $q$ , which explains why we cannot take limits to get the Shannon entropy in a straightforward manner.

According to Eqs. (5) and (7), the Renyi length of order  $q$  is given by

$$\mathcal{L}_q^R[\rho_{n,\alpha}] = \{W_q[\rho_{n,\alpha}]\}^{-\frac{1}{q-1}}, \quad q > 0, q \neq 1, \quad (12)$$

where

$$W_q[\rho_{n,\alpha}] := \langle [\rho_{n,\alpha}(x)]^{q-1} \rangle = \int_0^\infty [\rho_{n,\alpha}(x)]^q dx = \int_0^\infty [\tilde{L}_n^\alpha(x)]^{2q} x^{q\alpha} e^{-qx} dx, \quad (13)$$

are the frequency or entropic moments of the Rakhmanov density (1) of the Laguerre polynomials. In spite of the efforts of numerous researchers [3, 6–8, 11–13, 28], these quantities have not yet been calculated. Here we will compute them for  $2q \in \mathbb{N}$  by use of two different approaches.

#### 3.1. Algebraic approach

To calculate the entropic moment  $W_q[\rho_{n,\alpha}]$ , we first use (10) and (13) to write

$$W_q[\rho_{n,\alpha}] := \left[ \frac{n!}{\Gamma(\alpha+n+1)} \right]^q I_q[L_n^{(\alpha)}], \quad (14)$$

where

$$I_q[L_n^{(\alpha)}] := \int_0^\infty x^{\alpha q} e^{-qx} [L_n^{(\alpha)}(x)]^{2q} dx. \quad (15)$$

This functional of the orthogonal Laguerre polynomial can be calculated by use of the linearization formula of Srivastava-Niukkanen [27] for the products of various Laguerre polynomials given by

$$x^\mu L_{m_1}^{(\alpha_1)}(t_1 x) \cdots L_{m_r}^{(\alpha_r)}(t_r x) = \sum_{k=0}^{\infty} \Theta_k(\mu, \beta, r, \{m_i\}, \{\alpha_i\}, \{t_i\}) L_k^{(\beta)}(x),$$

where the coefficients  $\Theta_k(\mu, \beta, r, \{m_i\}, \{\alpha_i\}, \{t_i\})$  can be expressed as

$$\Theta_k(\mu, \beta, r, \{m_i\}, \{\alpha_i\}, \{t_i\}) = (\beta+1)_\mu \binom{m_1+\alpha_1}{m_1} \cdots \binom{m_r+\alpha_r}{m_r} \\ \times F_A^{(r+1)}[\beta+\mu+1, -m_1, \dots, -m_r, -k; \alpha_1+1, \dots, \alpha_r+1, \beta+1; t_1, \dots, t_r, 1],$$

in terms of the Lauricella's hypergeometric functions of  $(r+1)$  variables [9]. The Pochhammer symbol is  $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ . This general relation with the values  $(\beta=0, \alpha_1 = \dots = \alpha_r = \alpha, m_1 = \dots = m_r = n,$

$x = qt$ ,  $t_1 = \dots = t_r = \frac{1}{q}$ ,  $\mu = \alpha q$ ,  $r = 2q$ ) readily yields the following linearization result for the powers of Laguerre polynomials:

$$(qt)^{2q} \left[ L_n^{(\alpha)}(t) \right]^{2q} = \sum_{k=0}^{\infty} \Theta_k \left( \alpha q, 0, 2q, \{n\}, \{\alpha\}, \left\{ \frac{1}{q} \right\} \right) L_k^{(0)}(qt), \quad (16)$$

where

$$\begin{aligned} \Theta_k \left( \alpha q, 0, 2q, \{n\}, \{\alpha\}, \left\{ \frac{1}{q} \right\} \right) &= \Gamma(\alpha q + 1) \binom{n + \alpha}{n}^{2q} \\ &\times F_A^{(2q+1)} \left( \alpha q + 1; -n, \dots, -n; -k; \alpha + 1, \dots, \alpha + 1, 1; \frac{1}{q}, \dots, \frac{1}{q}, 1 \right). \end{aligned}$$

Taking into account (15), (16) and the orthogonality relation of the polynomials  $L_n^{(\alpha)}(x)$ , one finally has that the term with  $k = 0$  is the only non-vanishing contribution to  $I_q \left[ L_n^{(\alpha)} \right]$ , so that

$$I_q \left[ L_n^{(\alpha)} \right] = \frac{1}{q^{\alpha q + 1}} \Theta_0 \left( \alpha q, 0, 2q, \{n\}, \{\alpha\}, \left\{ \frac{1}{q} \right\} \right), \quad (17)$$

with

$$\begin{aligned} \Theta_0 \left( \alpha q, 0, 2q, \{n\}, \{\alpha\}, \left\{ \frac{1}{q} \right\} \right) &= \Gamma(\alpha q + 1) \binom{n + \alpha}{n}^{2q} \\ &\times F_A^{(2q+1)} \left( \alpha q + 1; -n, \dots, -n; 0; \alpha + 1, \dots, \alpha + 1, 1; \frac{1}{q}, \dots, \frac{1}{q}, 1 \right). \end{aligned} \quad (18)$$

Then, the entropic moments of the Laguerre polynomials have, according to Eqs. (14) and (17), the following expression

$$W_q [\rho_{n,\alpha}] = \left[ \frac{n!}{\Gamma(\alpha + n + 1)} \right]^q \frac{1}{q^{\alpha q + 1}} \Theta_0 \left( \alpha q, 0, 2q, \{n\}, \{\alpha\}, \left\{ \frac{1}{q} \right\} \right). \quad (19)$$

Finally, from Eqs. (12), (18) and (19) one has that the Renyi entropy of order  $q$  of the Laguerre polynomials is given by

$$\begin{aligned} \mathcal{L}_q^R [\rho_{n,\alpha}] &= \left[ \left( \frac{n!}{\Gamma(\alpha + n + 1)} \right)^q \frac{1}{q^{\alpha q + 1}} \Theta_0 \left( \alpha q; \frac{1}{q}, \dots, \frac{1}{q} \right) \right]^{-\frac{1}{q-1}} \\ &= \left[ \left( \frac{n!}{\Gamma(\alpha + n + 1)} \right)^q \frac{1}{q^{\alpha q + 1}} \Gamma(\alpha q + 1) \binom{n + \alpha}{n}^{2q} \right. \\ &\quad \left. \times F_A^{(2q+1)} \left( \alpha q + 1, -n, \dots, -n, 0; \alpha + 1, \dots, \alpha + 1, 1; \frac{1}{q}, \dots, \frac{1}{q}, 1 \right) \right]^{-\frac{1}{q-1}}, \end{aligned} \quad (20)$$

for every  $q > 0$ ,  $q \neq 1$ . Some examples follow:

- For  $n = 0$  we have that

$$\mathcal{L}_q^R [\rho_{0,\alpha}] = \left[ \frac{1}{(\Gamma(\alpha + 1))^q} \frac{\Gamma(\alpha q + 1)}{q^{\alpha q + 1}} F_A^{(2q+1)} \left( \alpha q + 1, 0, \dots, 0, 0; \alpha + 1, \dots, \alpha + 1, 1; \frac{1}{q}, \dots, \frac{1}{q}, 1 \right) \right]^{-\frac{1}{q-1}}.$$

By considering the definition of the Lauricella function in [9], this expression can be written down as

$$\begin{aligned}\mathcal{L}_q^R[\rho_{0,\alpha}] &= \left[ \frac{1}{(\Gamma(\alpha+1))^q} \frac{\Gamma(\alpha q+1)}{q^{\alpha q+1}} \right. \\ &\quad \times \left. \sum_{j_1, \dots, j_{2q+1}=0}^{+\infty} \frac{(\alpha q+1)_{j_1+\dots+j_{2q+1}}(0)_{j_1} \cdots (0)_{j_{2q+1}} \left(\frac{1}{q}\right)^{j_1+\dots+j_{2q}}}{(\alpha+1)_{j_1} \cdots (\alpha+1)_{j_{2q}} (1)_{2q+1}} \frac{\left(\frac{1}{q}\right)^{j_1+\dots+j_{2q}}}{j_1! \cdots j_{2q+1}!} \right]^{-\frac{1}{q-1}} \\ &= \left[ \frac{1}{\Gamma(\alpha+1)^q} \frac{\Gamma(\alpha q+1)}{q^{\alpha q+1}} \right]^{-\frac{1}{q-1}},\end{aligned}\tag{21}$$

where we have taken into account that only the term with  $j_1 = \dots = j_{2q+1} = 0$  is different from zero in the previous sum.

- For  $n = 1$  we have

$$\mathcal{L}_q^R[\rho_{1,\alpha}] = \left[ \frac{\Gamma(\alpha q+1)}{(\Gamma(\alpha+2))^q} \frac{(1+\alpha)^{2q}}{q^{\alpha q+1}} F_A^{(2q+1)} \left( \alpha q+1, -1, \dots, -1, 0; \alpha+1, \dots, \alpha+1, 1; \frac{1}{q}, \dots, \frac{1}{q}, 1 \right) \right]^{-\frac{1}{q-1}},$$

that yields

$$\begin{aligned}\mathcal{L}_q^R[\rho_{1,\alpha}] &= \left[ \frac{\Gamma(\alpha q+1)}{(\Gamma(\alpha+2))^q} \frac{(1+\alpha)^{2q}}{q^{\alpha q+1}} \right. \\ &\quad \times \left. \sum_{j_1, \dots, j_{2q+1}=0}^{+\infty} \frac{(\alpha q+1)_{j_1+\dots+j_{2q+1}} (-1)_{j_1} \cdots (-1)_{j_{2q}} (0)_{j_{2q+1}} \left(\frac{1}{q}\right)^{j_1+\dots+j_{2q}}}{(\alpha+1)_{j_1} \cdots (\alpha+1)_{j_{2q}} (1)_{2q+1}} \frac{\left(\frac{1}{q}\right)^{j_1+\dots+j_{2q}}}{j_1! \cdots j_{2q+1}!} \right]^{-\frac{1}{q-1}},\end{aligned}$$

as  $(-1)_0 = 1$ ,  $(-1)_1 = -1$  and  $(-1)_i = 0, i \geq 2$ , only the terms with  $j_1, \dots, j_{2q}$  equal to 0 and 1 are different from zero. Then, by considering the number of terms with a given number  $k$  of indices equal to 1, this expression can be straightforwardly reduced to

$$\mathcal{L}_q^R[\rho_{1,\alpha}] = \left[ \frac{\Gamma(\alpha q+1)}{(\Gamma(\alpha+2))^q} \frac{(1+\alpha)^{2q}}{q^{\alpha q+1}} \sum_{k=0}^{2q} \binom{2q}{k} (\alpha q+1)_k \left( \frac{-1}{q(\alpha+1)} \right)^k \right]^{-\frac{1}{q-1}},\tag{22}$$

which can be further simplified as

$$\mathcal{L}_q^R[\rho_{1,\alpha}] = \left[ \frac{\Gamma(\alpha q+1)(1+\alpha)^{2q}}{(\Gamma(\alpha+2))^q q^{\alpha q+1}} {}_2F_0 \left( \begin{matrix} -2q, \alpha q+1 \\ - \end{matrix}; \frac{-1}{q(\alpha+1)} \right) \right]^{-\frac{1}{q-1}}.$$

### 3.2. Combinatorial approach

In this approach we begin with the explicit expression of the Laguerre polynomials given by

$$\tilde{L}_n^{(\alpha)}(x) = \sum_{k=0}^n c_k x^k,$$

with

$$c_k^{(n,\alpha)} = \sqrt{\frac{\Gamma(n+\alpha+1)}{n!}} \frac{(-1)^k}{\Gamma(\alpha+k+1)} \binom{n}{k},\tag{23}$$

Recently ([48]; see appendix) it has been found that an integer power of a polynomial can be expressed by use of the multivariate Bell polynomials of Combinatorics [50]. This result applied to the Laguerre polynomials gives

$$\left[ \tilde{L}_n^{(\alpha)}(x) \right]^p = \sum_{k=0}^{np} \frac{p!}{(k+p)!} B_{k+p,p}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}) x^k,\tag{24}$$

with  $c_i^{(n,\alpha)} = 0$  for  $i > n$ , and the remaining coefficients are given by Eq. (23). Moreover, the Bell polynomials are given by

$$B_{m,l}(a_1, a_2, \dots, a_{m-l+1}) = \sum_{\hat{\pi}(m,l)} \frac{m!}{j_1! j_2! \cdots j_{m-l+1}!} \left(\frac{a_1}{1!}\right)^{j_1} \left(\frac{a_2}{2!}\right)^{j_2} \cdots \left(\frac{a_{m-l+1}}{(m-l+1)!}\right)^{j_{m-l+1}}, \quad (25)$$

where the sum runs over all partitions  $\hat{\pi}(m, l)$  such that

$$j_1 + j_2 + \cdots + j_{m-l+1} = l, \quad \text{and} \quad j_1 + 2j_2 + \cdots + (m-l+1)j_{m-l+1} = m.$$

The replacement of expression (24) with  $p = 2q$  into Eq. (13) yields the value

$$\begin{aligned} W_q[\rho_{n,\alpha}] &= \sum_{k=0}^{2nq} \frac{(2q)!}{(k+2q)!} B_{k+2q,2q}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}) \int_0^\infty x^{q\alpha} e^{-qx} x^k dx = \\ &= \sum_{k=0}^{2nq} \frac{\Gamma(\alpha q + k + 1)}{q^{\alpha q + k + 1}} \frac{(2q)!}{(k+2q)!} B_{k+2q,2q}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}), \end{aligned} \quad (26)$$

where the parameters  $c_i^{(n,\alpha)}$  are given by Eq. (23), keeping in mind that  $c_i^{(n,\alpha)} = 0$  for every  $i > n$ , so that the only non-vanishing terms correspond to those with  $j_{i+1} = 0$  so that  $(c_i^{(n,\alpha)})^{j_{i+1}} = 1$  for every  $i > n$ .

It is worthwhile to check that for  $q = 1$  one has that

$$W_1[\rho_{n,\alpha}] = \int_0^\infty \rho_{n,\alpha}(x) dx = 1,$$

and that for  $q = 2$  we have

$$W_2[\rho_{n,\alpha}] = \sum_{k=0}^{4n} \frac{\Gamma(2\alpha + k + 1)}{2^{2\alpha + k + 1}} \frac{24}{(k+4)!} B_{k+4,4}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}),$$

for the Onicescu information [55] of the Laguerre polynomials. Finally, from Eqs. (12) and (26) one has the following alternative expression for the  $q$ th-order Renyi length of the Laguerre polynomial for  $q = 2, 3, \dots$ ,

$$\mathcal{L}_q^R[\rho_{n,\alpha}] = \left[ \sum_{k=0}^{2nq} \frac{\Gamma(\alpha q + k + 1)}{q^{\alpha q + k + 1}} \frac{(2q)!}{(k+2q)!} B_{k+2q,2q}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}) \right]^{-\frac{1}{q-1}}, \quad (27)$$

which for  $q = 2$  yields the value

$$\mathcal{L}_2^R[\rho_{n,\alpha}] = \left[ \sum_{k=0}^{4n} \frac{\Gamma(2\alpha + k + 1)}{2^{2\alpha + k + 1}} \frac{24}{(k+4)!} B_{k+4,4}(c_0^{(n,\alpha)}, 2!c_1^{(n,\alpha)}, \dots, (k+1)!c_k^{(n,\alpha)}) \right]^{-1},$$

for the Onicescu or second-order Renyi length [55] of the Laguerre polynomials.

With these expressions we obtain the same values of  $\mathcal{L}_q^R[\rho_{0,\alpha}]$  and  $\mathcal{L}_q^R[\rho_{1,\alpha}]$  as in the previous subsection:

- For  $n = 0$  we have

$$\mathcal{L}_q^R[\rho_{0,\alpha}] = \left[ \frac{\Gamma(\alpha q + 1)}{q^{\alpha q + 1}} B_{2q,2q}(c_0^{(0,\alpha)}, 0, \dots, 0) \right]^{-\frac{1}{q-1}}.$$

Since only the coefficient  $c_0^{(0,\alpha)}$  is different from zero, the previous Bell polynomial contains only one term with index  $j_1 = 2q$  (see Eq. (25)). This yields

$$\mathcal{L}_q^R[\rho_{0,\alpha}] = \left[ \frac{\Gamma(\alpha q + 1)}{q^{\alpha q + 1}} (c_0^{(1,\alpha)})^{2q} \right]^{-\frac{1}{q-1}} = \left[ \frac{1}{(\Gamma(\alpha q + 1))^q} \frac{\Gamma(\alpha q + 1)}{q^{\alpha q + 1}} \right]^{-\frac{1}{q-1}},$$

in accordance with the result (21) obtained with the previous method.

- For  $n = 1$  we have

$$\mathcal{L}_q^R[\rho_{1,\alpha}] = \left[ \sum_{k=0}^{2q} \frac{\Gamma(\alpha q + k + 1)}{q^{\alpha q + k + 1}} \frac{(2q)!}{(k + 2q)!} B_{k+2q,2q}(c_0^{(1,\alpha)}, 2c_1^{(1,\alpha)}0, \dots, 0) \right]^{-\frac{1}{q-1}}.$$

Since the only coefficients different from zero are  $c_0^{(1,\alpha)}$  and  $c_1^{(1,\alpha)}$ , the previous Bell polynomials contain only one term with indices  $j_1 = 2q - k$  and  $j_2 = k$  (see Eq. (25)). This yields

$$\begin{aligned} \mathcal{L}_q^R[\rho_{1,\alpha}] &= \left[ \sum_{k=0}^{2q} \frac{\Gamma(\alpha q + k + 1)}{q^{\alpha q + k + 1}} \frac{(2q)!}{(2q - k)!k!} \left(c_0^{(1,\alpha)}\right)^{2q-k} \left(c_1^{(1,\alpha)}\right)^k \right]^{-\frac{1}{q-1}} \\ &= \left[ \frac{\Gamma(\alpha q + 1)(1 + \alpha)^{2q}}{(\Gamma(\alpha + 2))^q q^{\alpha q + 1}} \sum_{k=0}^{2q} \binom{2q}{k} (\alpha q + 1)_k \left(\frac{-1}{\alpha(q + 1)}\right)^k \right]^{-\frac{1}{q-1}}, \end{aligned}$$

in accordance with the corresponding result (22) obtained with the first method.

The equivalence of Eqs. (20) and (27) for a generic  $n$  can not be easily proved because of the non-trivial special functions involved.

#### 4. Shannon length: Asymptotics and sharp bounds

The goal of this Section is twofold. First, to study the asymptotics of the Shannon spreading length  $N[\rho_{n,\alpha}]$  of the orthonormal Laguerre polynomials  $\tilde{L}_n^{(\alpha)}$  and its relation to the standard deviation  $(\Delta x)_{n,\alpha}$ . Second, to find sharp upper bounds to  $N[\rho_{n,\alpha}]$  by use of an information-theoretic optimization procedure.

Although many results have been recently published in the literature (see e.g. [23, 56]) about the asymptotics of the Laguerre polynomials themselves, they have not yet been successfully used to obtain the asymptotics of functionals of these mathematical functions beyond the  $L_p$ -norm method of Aptekarev et al [24, 25]. Here we use the results provided by this method to fix the asymptotics of the Shannon length of these polynomials and its relation to the standard deviation. From Eq. (8) we have that

$$N[\rho_{n,\alpha}] = \exp(S[\rho_{n,\alpha}]), \quad (28)$$

where

$$S[\rho_{n,\alpha}] := - \int_0^\infty \omega_\alpha(x) \left[\tilde{L}_n^{(\alpha)}(x)\right]^2 \log \left\{ \omega_\alpha(x) \left[\tilde{L}_n^{(\alpha)}(x)\right]^2 \right\} dx = E_n \left[\tilde{L}_n^{(\alpha)}\right] + J_n \left[\tilde{L}_n^{(\alpha)}\right],$$

with the following entropic functionals [16, 25, 26]

$$E_n \left[\tilde{L}_n^{(\alpha)}\right] = - \int_0^\infty \omega_\alpha(x) \left[\tilde{L}_n^{(\alpha)}(x)\right]^2 \log \left[\tilde{L}_n^{(\alpha)}(x)\right]^2 dx, \quad (29)$$

and

$$J_n \left[\tilde{L}_n^{(\alpha)}\right] = - \int_0^\infty \omega_\alpha(x) \left[\tilde{L}_n^{(\alpha)}(x)\right]^2 \log \omega_\alpha(x) dx = 2n + \alpha + 1 - \alpha \psi(\alpha + n + 1). \quad (30)$$

Moreover, the use to the  $L_p$ -norm method of Aptekarev et al [24] has permitted to find [16] the following values for the asymptotics of  $E_n \left[\tilde{L}_n^{(\alpha)}\right]$

$$E_n \left[\tilde{L}_n^{(\alpha)}\right] = -2n + (\alpha + 1) \log(n) - \alpha - 2 + \log(2\pi) + o(1). \quad (31)$$

Then, according to Eqs. (29), (30) and (31), one has that the asymptotical behaviour

$$S[\rho_{n,\alpha}] = (\alpha + 1) \log(n) - \alpha \log(n + \alpha + 1) - 1 + \log(2\pi) + o(1),$$



for the Shannon entropy, and

$$N[\rho_{n,\alpha}] \simeq \frac{2\pi}{e} \frac{n^{\alpha+1}}{(n+\alpha+1)^\alpha}, \quad (32)$$

for the Shannon length of the orthonormal Laguerre polynomials. Where we have used the logarithmic asymptotic behaviour of the digamma function  $\psi$  (see Eq. (6.3.18) from [57]). Moreover, from Eqs. (11) and (32) one finds that

$$N[\rho_{n,\alpha}] = \frac{\pi\sqrt{2}}{e} (\Delta x)_{n,\alpha} + O(1), \quad (33)$$

between the asymptotical values of the Shannon length and the standard deviation of the polynomial  $\tilde{L}_n^{(\alpha)}$ . It is worth noting that this relation fulfils the general inequality (9) which mutually relates the Shannon length and the standard deviation for general densities. Moreover, the relation (33) for the Rakhmanov densities of Laguerre polynomials is also satisfied by the Rakhmanov densities of Hermite [29, 48] and Jacobi [29, 49] polynomials.

Let us now find sharp upper bounds to the Shannon length  $N[\rho_{n,\alpha}]$  by taking into account the non-negativity of the relative Shannon entropy (also called Kullback-Leibler entropy) of two arbitrary probability densities  $\rho(x)$  and  $f(x)$ :

$$KL[\rho, f] = \int \rho(x) \log \frac{\rho(x)}{f(x)} dx.$$

The Jensen inequality implies that  $KL[\rho, f] \geq 0$ . Then, as

$$KL[\rho, f] = \int \rho(x) \log \rho(x) dx - \int \rho(x) \log f(x) dx = -S[\rho] - \int \rho(x) \log f(x) dx,$$

we have that the Shannon entropy of  $\rho_{n,\alpha}(x)$  is bounded from above by means of

$$S[\rho_{n,\alpha}] \leq - \int_0^\infty \rho_{n,\alpha}(x) \log f(x) dx. \quad (34)$$

This expression produces an infinite set of upper bounds to the Shannon entropy of Laguerre polynomials. Furthermore, to obtain an expression in terms of useful expectation values like  $\langle x^b \rangle$  and  $\langle \log x \rangle$ , the choice of  $f(x)$  is

$$f(x) = \frac{ba^{\frac{1+m}{b}}}{\Gamma(\frac{1+m}{b})} x^m e^{-ax^b}; \quad m > -1, a > 0, b \in \mathbb{N}^+, 0 \leq x < \infty, \quad (35)$$

which is normalized to unity. Evaluating now the bound in (34) we obtain

$$- \int_0^{+\infty} \rho(x) \log \rho(x) dx = \log \frac{\Gamma(\frac{1+m}{b})}{ba^{\frac{1+m}{b}}} - m \langle \log x \rangle + a \langle x^b \rangle.$$

Differentiating this expression with respect to the parameter  $a$ , and equating it to zero we obtain the value  $a = \frac{1+m}{b \langle x^b \rangle}$ , that is a minimum given the convexity of the previous expression. This yields the bound

$$S[\rho_{n,\alpha}] \leq \log \left[ \frac{\Gamma(\beta) e^\beta}{b\beta^\beta} \langle x^b \rangle^\beta \right] - m \langle \log x \rangle; \quad b > 0, m > -1, \beta = \frac{1+m}{b}, \quad (36)$$

following the lines of Refs [58, 59]. Then, according to Eqs. (28) and (36), we have the following set of infinite sharp bounds

$$N[\rho_{n,\alpha}] \leq \frac{\Gamma(\beta) e^\beta}{b\beta^\beta} \langle x^b \rangle^\beta e^{-m \langle \log x \rangle}; \quad m > -1, b > 0, \quad (37)$$

for the Shannon length of the Laguerre polynomial  $L_n^{(\alpha)}(x)$ . For  $m = 0$  we have the upper bound

$$N[\rho_{n,\alpha}] \leq \frac{\Gamma(\frac{1}{b}) (be)^{\frac{1}{b}}}{b} \langle x^b \rangle^{\frac{1}{b}}, \quad b > 0. \quad (38)$$

This bound is particularly interesting because it only depends on the expectation value  $\langle x^b \rangle$ ; the expectation value  $\langle \log x \rangle$  is, at times, unavailable or difficult to evaluate.

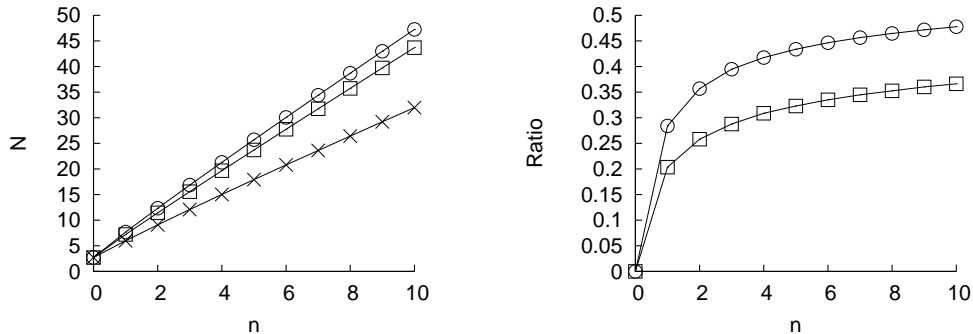


Figure 1: Left: Shannon length ( $\times$ ), upper bound with  $m = 0$  and  $b = b_{\text{opt}}$  ( $\odot$ ), and upper bound with  $m = m_{\text{opt}}$  and  $b = b_{\text{opt}}$  ( $\square$ ) of the Laguerre polynomials  $L_n^{(0)}(x)$ , as a function of the degree  $n$ . Right: Relative ratios of the bounds with  $m = 0$  and  $b = b_{\text{opt}}$  ( $\odot$ ), and with  $m = m_{\text{opt}}$  and  $b = b_{\text{opt}}$  ( $\square$ ), as a function of  $n$ .

$n$	0	1	2	3	4	5	6	7	8	9	10
$b_{\text{opt}}$	1	3	4	6	7	8	9	10	11	12	13

Table 1: Values  $b_{\text{opt}}$  of the parameter  $b$  which yield the best (i.e. lowest) upper bounds (38) to the Laguerre polynomial  $L_n^{(0)}(x)$  for various degrees  $n$ .

## 5. Some computational issues

In this section we study various computational issues of the direct spreading measures of Laguerre polynomials. It is worth pointing out that there is no stable numerical algorithm for the computation of the Renyi and Shannon lengths of these polynomials in contrast to the case of orthogonal polynomials on a finite interval for which an efficient algorithm based on the three-term recurrence relation has been recently found by Buyarov et al [60]. Moreover, a naive numerical evaluation of these Laguerre functionals by means of quadratures is not often convenient except for the lowest-order polynomials since the increasing number of integrable singularities spoils any attempt to achieve a reasonable accuracy for arbitrary  $n$ . Here we carry out the following numerical study. First, we examine the numerical accuracy of the bounds (37) and (38) to the Shannon length of the Laguerre polynomials  $L_n^{(\alpha)}(x)$ ,  $\alpha$  fixed, for various degrees  $n$  by taking into account the optimal values of the parameter  $b$  in (38) and the optimal values of  $(b, m)$  in (37). These optimal values have been obtained by minimizing the corresponding inequalities numerically. Second, we study the mutual comparison of Fisher, Shannon and Onicescu lengths and the standard deviation of  $L_n^{(\alpha)}(x)$  for fixed  $\alpha$  and various degrees  $n$ . Finally, we discuss the correlation of the Shannon length  $N(L_n^{(\alpha)})$  and the standard deviation  $(\Delta x)_n$  for various pairs  $(n, \alpha)$ , which allow us to find, at times, linear relations between their components.

In Figure 1 it is numerically studied the accuracy of the bounds (37) and (38) to the Shannon length of the Laguerre polynomial  $L_n^{(\alpha=0)}(x)$  given by the optimal values  $(b_{\text{opt}}, 0)$  in (38). This is done by comparing the corresponding optimal bounds with the “numerically exact” value of the lengths  $N[\rho_{n,0}]$  for the polynomials with degree  $n$  from 0 to 10. The graph on the right of the figure gives the relative ratio of the bound given by Eq. (38) with  $b = b_{\text{opt}}$ , and the ratio of the bound given by Eq. (37) with  $b = b_{\text{opt}}$  and  $m = m_{\text{opt}}$ . Notice that the latter bound is always better than the former, as we have the parameter  $m$  to adjust. The values of optimal pairs  $(b_{\text{opt}}, 0)$  and  $(b_{\text{opt}}, m_{\text{opt}})$  are shown in Tables 1 and 2, respectively. Note that the optimum value  $b_{\text{opt}}$  is different when considering the bound (37) or (38). Also notice that the optimum values for  $n = 0$  are  $(b_{\text{opt}}, m_{\text{opt}}) = (1, 0)$ , where the density  $f(x)$  defined in (35) equals the Rakhmanov density for  $n = 0$ . Remark that the best bounds are obtained for expectation values  $\langle x^b \rangle$  where  $b = b_{\text{opt}}$  is an increasing function of the degree  $n$  of the polynomial in both cases; this is directly connected with the larger spreading of the polynomial when its degree has higher values.

To study the behaviour of the accuracy of the two previous bounds with respect to  $\alpha$ , we have done

$n$	0	1	2	3	4	5	6	7	8	9	10
$b_{\text{opt}}$	1	4	6	7	9	10	11	12	14	15	16
$m_{\text{opt}}$	0	-0.332	-0.338	-0.322	-0.332	-0.327	-0.324	-0.321	-0.322	-0.320	-0.319

Table 2: Values  $(b_{\text{opt}}, m_{\text{opt}})$  of the parameters  $(b, m)$  which yield the best (i.e. lowest) upper bound (37) to the Laguerre polynomial  $L_n^{(0)}(x)$  for various degrees  $n$ .

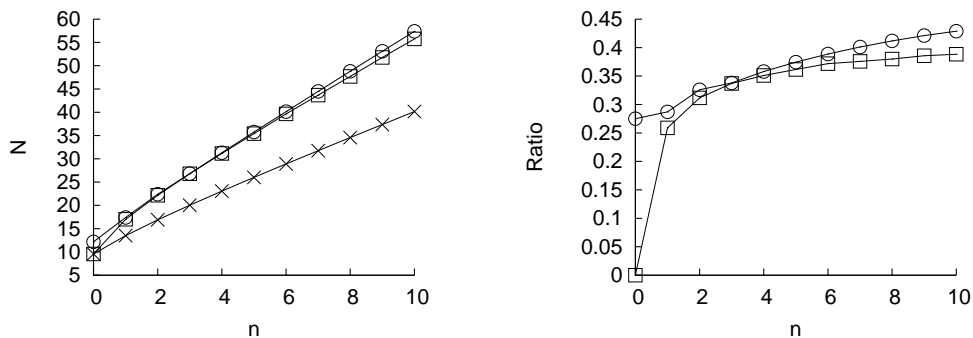


Figure 2: Left: Shannon length ( $\times$ ), upper bound with  $m = 0$  and  $b = b_{\text{opt}}$  ( $\odot$ ), and upper bound with  $m = m_{\text{opt}}$  and  $b = b_{\text{opt}}$  ( $\square$ ) of the Laguerre polynomials  $L_n^{(5)}(x)$ , as a function of the degree  $n$ . Right: Relative ratios of the bounds with  $m = 0$  and  $b = b_{\text{opt}}$  ( $\odot$ ), and with  $m = m_{\text{opt}}$  and  $b = b_{\text{opt}}$  ( $\square$ ), as a function of  $n$ .

in Figure 2 a study of the Shannon lengths  $N[\rho_{n,5}]$  similar to that done in Figure 1 for  $L_n^{(0)}(x)$ . The corresponding values  $(b_{\text{opt}}, 0)$  and  $(b_{\text{opt}}, m_{\text{opt}})$  are given in Tables 3 and 4 respectively. The two graphs of the figure show qualitatively similar and quantitatively better results than those found in Figure 1.

In Figures 3 and 4 we study the mutual comparison of various direct spreading measures (namely, the standard deviation  $\Delta x$  and the Fisher, Shannon and the Onicescu or second-order Renyi lengths) of the Laguerre polynomials  $L_n^{(0)}(x)$  and  $L_n^{(5)}(x)$ , respectively, when the degree  $n$  varies from 0 to 10. Several observations are in order. First, all the measures with global character (standard deviation, Shannon and Renyi lengths) grow linearly or quasilinearly when the degree of the polynomial is increasing; essentially because the polynomial spreads more and more. Moreover, they behave so that  $\Delta x < \mathcal{L}_2 < N$ . Second, the (local) Fisher length decreases when the degree  $n$  is increasing; essentially, because the polynomial becomes more and more oscillatory, so growing its gradient content. Third, the Fisher length has always a value smaller than all the global spreading measures.

Finally, in Figure 5 we have numerically studied the connection of the Shannon length  $N[\rho_{n,\alpha}]$  and the standard deviation  $(\Delta x)_{n,\alpha}$  of the Laguerre polynomials  $L_n^{(\alpha)}(x)$ , with  $\alpha = 0$  and 5, when the degree  $n$  varies from 0 to 20. This apparent quasilinear behaviour of the Shannon length with respect to the standard deviation is in accordance to the rigorous expression (33), i.e.  $N[\rho_{n,\alpha}] \simeq \frac{\pi\sqrt{2}}{e} (\Delta x)_{n,\alpha}$  for  $n \gg 1$ .

## 6. Conclusions and open problems

The global (standard deviation and the Renyi and Shannon lengths) and local (Fisher length) direct spreading measures of the Laguerre polynomials  $L_n^{(\alpha)}(x)$  are analytically and numerically studied. Beyond

$n$	0	1	2	3	4	5	6	7	8	9	10
$b_{\text{opt}}$	5	6	7	8	10	11	12	13	14	15	16

Table 3: Values  $b_{\text{opt}}$  of the parameter  $b$  which yield the best (i.e. lowest) upper bounds (38) to the Laguerre polynomial  $L_n^{(5)}(x)$  for various degrees  $n$ .

$n$	0	1	2	3	4	5	6	7	8	9	10
$b_{\text{opt}}$	1	5	7	9	10	11	13	14	15	16	17
$m_{\text{opt}}$	5	0.288	0.053	-0.049	-0.098	-0.131	-0.160	-0.177	-0.190	-0.201	-0.210

Table 4: Values  $(b_{\text{opt}}, m_{\text{opt}})$  of the parameters  $(b, m)$  which yield the best (i.e. lowest) upper bound (37) to the Laguerre polynomial  $L_n^{(5)}(x)$  for various degrees  $n$ .

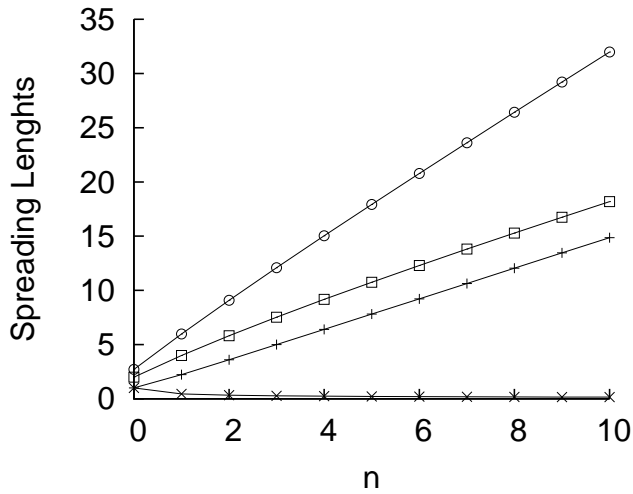


Figure 3: Standard deviation  $\Delta x$  (+), Fisher length  $\delta x$  (×), Onicescu length  $\mathcal{L}_2$  (□), and Shannon length  $N$  (○) of the Laguerre polynomial  $L_n^{(5)}(x)$  as a function of  $n$ .

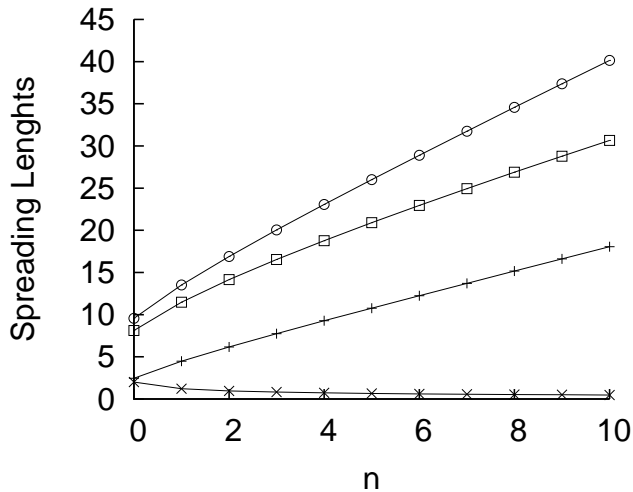


Figure 4: Standard deviation  $\Delta x$  (+), Fisher length  $\delta x$  (×), Onicescu length  $\mathcal{L}_2$  (□), and Shannon length  $N$  (○) of the Laguerre polynomial  $L_n^{(5)}(x)$  as a function of  $n$ .

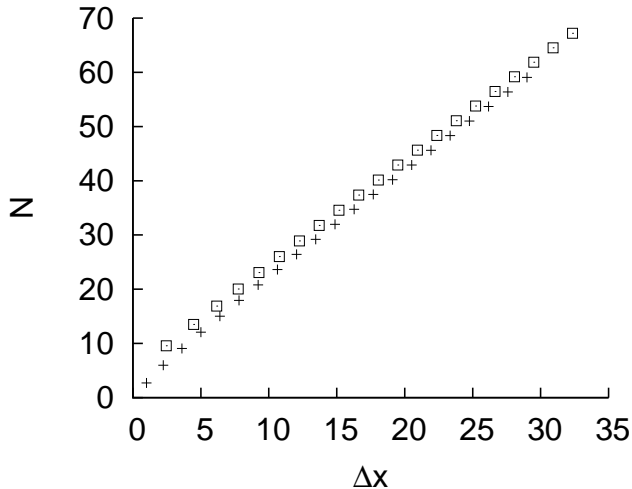


Figure 5: Shannon length  $N$  as a function of the standard deviation  $\Delta x$  for the Laguerre polynomials  $L_n^{(0)}(x)$  (+) and  $L_n^{(5)}(x)$  (□), when the degree  $n$  varies from 0 to 20.

the ordinary moments  $\langle x^k \rangle$ ,  $k \in \mathbb{Z}$ , and the standard deviation  $(\Delta x)_n$ , which have been explicitly given in terms of  $(n, \alpha)$ , we have developed two theoretical approaches of algebraic and combinatorial types to obtain two equivalent analytical expressions for the Renyi lengths of half-integer order. For the Shannon length, whose explicit value is not yet known (in fact, its calculation is a formidable task!), we have found sharp bounds in terms of the expectation value  $\langle x^k \rangle$  and/or the logarithmic expectation value  $\langle \log x \rangle$  by means of an information-theoretic-based optimization procedure.

Moreover, the linear correlation of the Shannon length and the standard deviation for the Laguerre polynomials  $L_n^{(\alpha)}(x)$  with large degree  $n$  is underlined. In fact, the correlation factor is not only independent on the parameter  $\alpha$  but, most importantly, it is the same as for the remaining hypergeometric families on a finite interval (Jacobi polynomials) [29, 49] or on the whole real line (Hermite polynomials) [29, 48].

Then we carried out a numerical study of the four direct spreading measures of Laguerre polynomials. Let us remark, among other results, that the Fisher length has the smallest value, and the Shannon length depends quasilinearly on the standard deviation.

Finally, let us highlight a number of open information-theoretic problems related to Laguerre polynomials: (i) to find the asymptotics of the entropic moments and, subsequently, the Renyi lengths in the spirit of [24, 25], (ii) to identify the most general class of polynomials for which the asymptotical relation (33) of the Shannon length and the standard deviation is fulfilled, and (iii) to characterize the most general class of polynomials for which the ratio between these two direct spreading measures is a constant (i.e., it does not depend on the degree nor the parameters of the polynomials) as already pointed out in [29].

## 7. Acknowledgments

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## References

- [1] T. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach, New York, 1978.
- [2] A. F. Nikiforov, V. B. Uvarov, *Special Functions in Mathematical Physics*, Birkäuser-Verlag, Basel, 1988.
- [3] G. E. Andrews, R. Askey, R. Roy, *Special Functions*. In *Encyclopedia for Mathematics and its Applications*, Cambridge University Press, 1999.

- [4] M. Ismail, Classical and quantum orthogonal polynomials in one variable, Cambridge, Univ. Press., 2005.
- [5] N. M. Temme, Special Functions: An Introduction to the Classical Functions of Mathematical Physics, Wiley-Interscience, 1996.
- [6] K. Chen, C. Chyan, H. Srivastava, Certain classes of polynomials expansions and multiplication formulas, *Math. Comput. Modelling* 37 (2003) 135.
- [7] H. M. Srivastava, A unified theory of polynomial expansions and their applications involving Clebsch-Gordan type linearization relations and Neumann series, *Astrophys. Space Science* 150 (1988) 251.
- [8] M. Hounkonnou, S. Belmehdi, A. Ronveaux, Linearization of arbitrary products of classical orthogonal polynomials, *Applicationes Mathematicae* 27 (2000) 187–196.
- [9] H. M. Srivastava, P. W. Karlsson, Multiple Gaussian Hypergeometric Series, John Wiley and Sons, New York, 1985.
- [10] R. Crandall, R. Bettega, R. Whitnell, A class of exactly soluble three-body problems, *J. Chem. Phys.* 83.
- [11] D. Foata, V. Strehl, Combinatorics of Laguerre polynomials, enumeration and design, Waterloo, Ontario, 1982.
- [12] M. de Sainte-Catherine, G. Viennot, Combinatorial interpretation of integrals of products of Hermite, Laguerre and Tchebycheff polynomials, *Lect. Notes in Math.* 1171 (1985) 120–128.
- [13] D. Foata, D. Zeilberger, Laguerre polynomials, weighted derangements, and positivity, *SIAM J. Discrete Math* 1 (1988) 425–433.
- [14] R. Askey, M. E. H. Ismail, T. Koornwinder, Weighted permutation problems and Laguerre polynomials, *J. Combin. Th.* A25 (1978) 277–287.
- [15] R. J. Yáñez, P. Sánchez-Moreno, A. Zarzo, J. S. Dehesa, Fisher information of special functions and second-order differential equations, *J. Math. Phys.* 49 (2008) 082104.
- [16] J. S. Dehesa, A. Martínez-Finkelshtein, J. Sánchez-Ruiz, Quantum information entropies and orthogonal polynomials, *J. Comput. Appl. Math.* 133 (2001) 23–46.
- [17] P. Sánchez-Moreno, J. Omiste, J. Dehesa, Entropic moments of Laguerre polynomials and complexity properties of the half-line Coulomb potential, *Int. J. Quantum Chemistry* Accepted.
- [18] A. I. Aptekarev, A. Martínez-Finkelshtein, J. S. Dehesa, Asymptotics of orthogonal polynomials entropy, *J. Comput. Appl. Math.* 233 (2009) 2136.
- [19] J. S. Dehesa, P. Sánchez-Moreno, R. J. Yáñez, Cramer-Rao information plane of orthogonal hypergeometric polynomials, *J. Comput. Appl. Math.* 186 (2006) 523–541.
- [20] J. Sánchez-Ruiz, J. S. Dehesa, Fisher information of orthogonal hypergeometric polynomials, *J. Comput. Appl. Math.* 182 (2005) 150–164.
- [21] E. Aydmer, C. Orta, R. Server, Quantum information entropies of the Morse potential ArXiv: quant-ph/0602203v1.
- [22] G. Dattoli, D. Levi, P. Winternitz, Heisenberg, umbral calculus and orthogonal polynomials ArXiv: math-ph/0712.2957v1.
- [23] D. Borwein, J. Borwein, R. Crandall, Effective Laguerre asymptotics, *SIAM J. Numer. Anal.* 46 (2008) 3285.
- [24] A. I. Aptekarev, V. S. Buyarov, J. S. Dehesa, Asymptotic behavior of the  $L^p$ -norms and the entropy for general orthogonal polynomials, *Russian Acad. Sci. Sb. Math.* 82 (1995) 373–395.
- [25] J. S. Dehesa, R. J. Yáñez, A. I. Aptekarev, V. S. Buyarov, Strong asymptotics of Laguerre polynomials and information entropies of two-dimensional harmonic oscillator and one-dimensional Coulomb potentials, *J. Math. Phys.* 39 (1998) 3050–3060.
- [26] J. Sánchez-Ruiz, J. Dehesa, Entropic integrals of orthogonal hypergeometric polynomials with general supports, *J. Comput. Appl. Math.* 118 (2000) 311–322.
- [27] H. M. Srivastava, A. W. Niukkanen, Some Clebsch-Gordan type linearization relations and associated families of Dirichlet integrals, *Math. Comput. Modelling* 37 (2003) 245–250.
- [28] A. Niukkanen, Clebsch-Gordan linearization relations for the products of Laguerre polynomials and hydrogen-like functions, *J. Phys. A: Math. Gen.* 18 (1985) 1399.
- [29] J. I. de Vicente, J. Sánchez-Ruiz, J. S. Dehesa, Standard deviation of probability distributions involving orthogonal polynomials, 2004, iWOP Proceedings, Madrid.
- [30] A. Galindo, P. Pascual, Quantum Mechanics, Springer, Berlin, 1990.
- [31] V. G. Bagrov, D. M. Gitman, Exact Solutions of Relativistic Wavefunctions, Kluwer Acad. Publ., Dordrecht, 1990.
- [32] A. K. F. Cooper, V. Sukkatma, Supersymmetry in Quantum Mechanics, World Sci. Publi., 2001.
- [33] S. Flügge, Practical Quantum Mechanics, Springer, Berlin, 1971.
- [34] J. Sánchez-Ruiz, P. López-Artés, J. Dehesa, Expansions in series of varying Laguerre polynomials and some applications to molecular potentials, *J. Comput. Appl. Math.* 153 (2003) 411–421.
- [35] B. Bransden, C. Joachain, The Physics of Atoms and Molecules, Prentice Hall, 2003.
- [36] J. S. Dehesa, A. Martínez-Finkelshtein, V. N. Sorokin, Information-theoretic measures for Morse and Pöschl-Teller potentials, *Mol. Phys.* 104 (2006) 613–622.
- [37] L. Wang, X. Gu, Z. Ma, S. Dong, Exact solutions to the D-dimensional Schrödinger equation with a pseudoharmonic oscillator, *Found. Phys. Lett.* 15 (2002) 569.
- [38] M. M. Nieto, Hydrogen atom and relativistic pi-mesic atom in N-space dimensions, *Am. J. Phys.* 47 (1979) 1067–1072.
- [39] S. Dong, M. Lozada-Cassou, Generalized hypervirial and recurrence relations for radial matrix elements in arbitrary dimensions, *Modern Phys. Lett.* 20 (2005) 1533–1540.
- [40] J. J. Omiste, J. S. Dehesa, R. J. Yáñez, Information-theoretic properties of the half-line Coulomb potential, *J. Math. Chem.* Available online.
- [41] C. Dunkl, A Laguerre polynomial orthogonality and the hydrogen atom, *Anal. Appl.* 1(2) (2003) 177.
- [42] G. Gangopadhyay, A generating function for the product of the laguerre polynomials: Franck-Condon factor for the multiphoton processes, *J. Phys. A: Math. Gen.* 31 (1998) L771–775.

- [43] N. Alves, E. Drigo, The factorisation method and supersymmetry, *J. Phys. A: Math. Gen.* 21 (1988) 3215–3225.
- [44] E. A. Rakhmanov, On the asymptotics of the ratio of orthogonal polynomials, *Math. USSR Sb.* 32 (1977) 199–213.
- [45] W. Hall, Universal geometric approach to uncertainty, entropy, and information, *Phys. Rev. A* 59 (1999) 2602.
- [46] M. J. W. Hall, Quantum properties of classical Fisher information, *Phys. Rev. A* 62 (2000) 012107.
- [47] M. J. W. Hall, Exact uncertainty relations, *Phys. Rev. A* 64 (2001) 052103.
- [48] P. Sánchez-Moreno, J. Dehesa, D. Manzano, R. J. Yáñez, Spreading lengths of Hermite polynomials, *J. Comput. Appl. Math.* 233 (2010) 2136.
- [49] A. Guerrero, P. Sánchez-Moreno, J. Dehesa, Direct spreading measures of Jacobi polynomials, *J. Phys. A: Math. Theor.* Accepted for publication.
- [50] J. Riordan, *An Introduction to Combinatorial Analysis*, Princeton Univ. Press, New Jersey, 1980.
- [51] E. Schrödinger, Quantisierung als Eigenwertproblem I, *Ann. Physik (Leipzig)* 79 (1926) 361–376.
- [52] E. Schrödinger, Quantisierung als Eigenwertproblem II, *Ann. Physik (Leipzig)* 79 (1926) 489–527.
- [53] E. Schrödinger, Quantisierung als Eigenwertproblem III, *Ann. Physik (Leipzig)* 80 (1926) 437–490.
- [54] E. Schrödinger, Quantisierung als Eigenwertproblem IV, *Ann. Physik (Leipzig)* 81 (1926) 109–139.
- [55] O. Onicescu, Theorie de l’information. Energie informationelle, *C.R. Acad. Sci. Paris A* 263 (1966) 25.
- [56] S. Y. Igashov, Asymptotic approximation and weight estimate for the Laguerre polynomials, *Int. Trans. and Special Funct.* 8(3–4) (1999) 209.
- [57] M. Abramowitz, I. A. S. (Eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, U.S. Government Printing Office, Washington D.C., 1964.
- [58] J. C. Angulo, Information entropy and uncertainty in D-dimensional many-body systems, *Phys. Rev. A* 50 (1994) 311–313.
- [59] J. C. Angulo, J. S. Dehesa, Tight rigorous bounds to atomic information entropies, *J. Chem. Phys.* 97 (1992) 6485, erratum: *J. Chem. Phys.* **98** (1993), 9223.
- [60] V. S. Buyarov, J. S. Dehesa, A. Martínez-Finkelshtein, J. Sánchez-Lara, Computation of the entropy of polynomials orthogonal on an interval, *SIAM J. Sci. Comput.* 26 (2004) 488–509.