ASKEY-WILSON POLYNOMIALS: AN AFFINE HECKE ALGEBRAIC APPROACH

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Abstract. We study Askey-Wilson type polynomials using representation theory of the double affine Hecke algebra. In particular, we prove bi-orthogonality relations for non-symmetric and anti-symmetric Askey-Wilson polynomials with respect to a complex measure. We give duality properties of the non-symmetric Askey-Wilson polynomials, and we show how the non-symmetric Askey-Wilson polynomials can be created from Sahi's intertwiners. The diagonal terms associated to the bi-orthogonality relations (which replace the notion of quadratic norm evaluations for orthogonal polynomials) are expressed in terms of residues of the complex weight function using intertwining properties of the non-symmetric Askey-Wilson transform under the action of the double affine Hecke algebra. We evaluate the constant term, which is essentially the well-known Askey-Wilson integral, using shift operators. We furthermore show how these results reduce to well-known properties of the symmetric Askey-Wilson polynomials, as were originally derived by Askey and Wilson using basic hypergeometric series theory.

1. Introduction

1.1. Due to work of Cherednik [2]–[6], Macdonald [15], Noumi [16] and Sahi [19], one can associate to every irreducible affine root system certain families of orthogonal polynomials (all closely related to the Macdonald polynomials), and prove their basic properties using a fundamental representation of the affine Hecke algebra in terms of difference-reflection operators. In this paper, we consider a rank one example of this theory in detail. The example is connected with a rank one non-reduced irreducible affine root system which has four orbits under the action of the associated affine Weyl group. The family of symmetric orthogonal polynomials associated to this particular affine root system is the celebrated four parameter family of Askey-Wilson polynomials, see [1].

1.2. The four parameter family of Askey-Wilson polynomials has played an important and central role in the theory of basic hypergeometric orthogonal polynomials. In fact, up to date they seem to be the most general family of basic hypergeometric orthogonal polynomials which satisfy the additional requirement that they are joint eigenfunctions of a second-order q-difference operator. We use the link between the Askey-Wilson polynomials and the most general non-reduced affine root system of rank one (see 1.1) to derive in this paper the basic properties of the Askey-Wilson polynomials (and more!) from the algebraic structure of the associated (double) affine Hecke algebra.

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1.3. We introduce a Cherednik-Dunkl type difference-reflection operator Y using the fundamental representation of the affine Hecke algebra of type $\widetilde{A}_1$. The fundamental representation was defined by Noumi [16] in the higher rank case, see also Sahi [19]. Sahi's [19] non-symmetric Askey-Wilson polynomials are defined as the eigenfunctions of the Cherednik-Dunkl operator Y. They form a linear basis of the Laurent polynomials in one variable. We explicitly indicate their connection with the symmetric Askey-Wilson polynomials as originally defined by Askey and Wilson in [1]. In particular, we give explicit expressions for the non-symmetric Askey-Wilson polynomials as a sum of two terminating balanced ${}_4\phi_3$'s (here ${}_r\phi_s$ is the basic hypergeometric series, see Gasper and Rahman [8] for the definition). All the other results in this paper are derived without using the explicit series expansions of the (non-)symmetric Askey-Wilson polynomials in terms of basic hypergeometric series.

1.4. We derive bi-orthogonality relations for the non-symmetric Askey-Wilson polynomials by explicitly computing the adjoint of the Cherednik-Dunkl operator Y with respect to an explicit, complex measure. By a kind of symmetrization procedure, the bi-orthogonality relations imply Askey and Wilson's [1] orthogonality relations for the symmetric Askey-Wilson polynomials.

1.5. We shortly describe Sahi's [19] results how an anti-isomorphism of the double affine Hecke algebra gives rise to a duality between the spectral and the geometric parameter of the (non-)symmetric Askey-Wilson polynomial. We use the duality to determine explicit intertwining properties of the action of the double affine Hecke algebra under the non-symmetric Askey-Wilson transform. Here the non-symmetric Askey-Wilson transform is a generalized Fourier transform which is defined with respect to the bi-orthogonality measure of the non-symmetric Askey-Wilson polynomials. This leads to the evaluation of the diagonal terms of the (bi-)orthogonality relations in terms of certain residues of the complex weight function. This technique is motivated by Cherednik's [4] approach to the study of the diagonal terms for non-symmetric Macdonald polynomials, in which he rewrites part of the action of the double affine Hecke algebra on non-symmetric Macdonald polynomials in terms of explicit operators acting on the spectral parameter.

1.6. To complete the explicit computation of the diagonal terms, we need the evaluation of the constant term and the evaluation of the non-symmetric Askey-Wilson polynomial in a specific point (the latter playing a fundamental role in the duality arguments). We evaluate the constant term, which is essentially the well-known Askey-Wilson integral (see [1]), using shift operators. For the evaluation of the non-symmetric Askey-Wilson polynomial in a specific point we use a Rodrigues type formula for the non-symmetric Askey-Wilson polynomial in terms of Sahi's [19] intertwiners.

1.7. The purpose of this paper is two-fold. First of all, we would like to show the power of the Cherednik-Macdonald theory in the study of basic hypergeometric orthogonal polynomials. It not only shows that all the basic properties of the Askey-Wilson polynomials can be obtained by natural algebraic manipulations, but it also reveals new and important insights in the structure of the Askey-Wilson polynomials.

Secondly, the affine Hecke algebraic approach also works for multivariable Askey-Wilson polynomials, the so called Koornwinder polynomials [11] (which are associated with a higher rank non-reduced affine root system). The structure of the proofs in the higher rank setting are essentially the same, although the technicalities are more involved. Only part of the Cherednik-Macdonald theory associated with non-reduced affine root systems has been written down explicitly at this moment, see Noumi [16] and Sahi [19]. In our opinion, this paper can serve as one of the building blocks for obtaining a full understanding of the Cherednik-Macdonald theory in the case of non-reduced affine root systems. The higher rank case will be treated in an upcoming paper of the second author.

1.8. In view of our aims described in 1.7, we have chosen to make this paper fairly self-contained. In particular, we have included some of the proofs of Noumi [16] and Sahi [19], restricted to our present rank one setting. Furthermore, we have included several proofs which are fairly straightforward modifications from Cherednik's [2]–[6] and Macdonald's [15] work in case of Macdonald polynomials, see also for instance Opdam's [17] lecture notes for the classical $q = 1$ setting.

1.9. Finally we would like to point out the close connection with the paper [9] of Kalnins and Miller. In [9] the Askey-Wilson second order q-difference operator is written as the composition of a first order q-difference operator with its adjoint. This decomposition leads naturally to proofs of the orthogonality relations and of the quadratic norm evaluations for the symmetric Askey-Wilson polynomials (using shift principles). In our paper we use similar techniques, but we decompose the Askey-Wilson second order q-difference operator now as a sum of a difference-reflection operator (the Cherednik-Dunkl operator Y) and its inverse. This decomposition has the advantage that the Cherednik-Dunkl operator Y itself satisfies a self-adjointness property with respect to a suitable extension of the Askey-Wilson orthogonality measure to a complex measure space for *non-symmetric* functions. This extra property of Y naturally leads to the introduction of non-symmetric and anti-symmetric Askey-Wilson polynomials and to their bi-orthogonality relations.

1.10. *Notations:* We use Gasper and Rahman's [8] notations for basic hypergeometric series and q-shifted factorials. We write $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ for the positive integers and $\mathbb{N} = \{1, 2, \dots\}$ for the strictly positive integers.

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2. The Dunkl-Cherednik difference-reflection operators

2.1. Let $\widehat{\mathbb{R}}$ be the vector space consisting of affine, linear transformation from $\mathbb{R}$ to $\mathbb{R}$. We identify $\widehat{\mathbb{R}}$ with $\mathbb{R} \oplus \mathbb{R}\delta$, where δ is the function identically one, and where $\mathbb{R}$ acts by multiplication on itself. We introduce and study in this section a particular example of a rank one affine root system $S \subset \widehat{\mathbb{R}}$. See Macdonald [14] for the general discussion of affine root systems.

2.2. Let $\langle .,. \rangle$ be the positive semi-definite form on $\widehat{\mathbb{R}}$ defined by

$$\langle \lambda+\mu\delta, \lambda'+\mu'\delta \rangle = \lambda\lambda', \qquad \lambda, \lambda', \mu, \mu' \in \mathbb{R}.$$

Then we can associate to every $0 \neq f = \lambda + \mu\delta \in \widehat{\mathbb{R}} \setminus \mathbb{R}\delta$ the involution $s_f : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$ defined by

$$s_f(g) = g - \langle g, f^\vee \rangle f, \qquad g \in \widehat{\mathbb{R}},$$

where $f^\vee := 2f/\langle f, f \rangle$. Observe that s_f is an isometry with respect to $\langle .,. \rangle$. In fact, we have $s_f(g) = g \circ \widetilde{s}_f$, where $\widetilde{s}_f \in \widehat{\mathbb{R}}$ is the reflection in $f^{-1}(0)$.

2.3. We define now a subset $S \subset \widehat{\mathbb{R}}$ by

$$S = \{\pm 1 + \frac{m}{2}\delta, \ \pm 2 + m\delta \mid m \in \mathbb{Z}\},$$

and we write $\mathcal{W} = \mathcal{W}(S)$ for the subgroup of invertible linear transformations of $\widehat{\mathbb{R}}$ generated by s_f, $f \in S$. Then $S \subset \widehat{\mathbb{R}}$ is an affine root system. In particular, $\langle f, g^\vee \rangle \in \mathbb{Z}$ for all $f, g \in S$, and S is stable under the action of the affine Weyl group $\mathcal{W}(S)$.

2.4. The *gradient root system* Σ of S is the projection of S on $\mathbb{R}$ along the direct sum decomposition $\widehat{\mathbb{R}} = \mathbb{R} \oplus \mathbb{R}\delta$. Here $\Sigma = \{\pm 1, \pm 2\}$, which is a non-reduced root system of type BC_1, with associated Weyl group $W = \{1, s_1\} = \{\pm 1\}$. Observe that $W \subset \mathcal{W}$ acts on $\widehat{\mathbb{R}}$ by $(\pm 1)(\lambda + \mu\delta) = \pm\lambda + \mu\delta$.

2.5. Let $a_0 = \delta - 2 \in S$ and $a_1 = 2 \in S$. Observe that $a_0^\vee = a_0/2 = \frac{1}{2}\delta - 1 \in S$, and $a_1^\vee = a_1/2 = 1 \in S$. Then $\{a_0^\vee, a_1^\vee\}$ forms a basis of the affine root system S. We write S^+ for the positive roots in S with respect to $\{a_0^\vee, a_1^\vee\}$, so that $S = S^+ \cup (-S^+)$ disjoint union. We furthermore set $\Sigma^+ = \{a_1^\vee, a_1\}$, which are the positive roots of Σ with respect to the basis $\{a_1^\vee\}$. Observe that $S^+ = \Sigma^+ \cup \{f \in S \mid f(0) > 0\}$.

2.6. The affine Weyl group $\mathcal{W}$ is generated by the simple reflections $s_0 := s_{a_0} = s_{a_0^\vee}$ and $s_1 = s_{a_1} = s_{a_1^\vee}$, while W is generated by s_1. Observe that $s_1 s_0 = \tau(1)$, where $\tau(\mu)$ is the translation operator $\tau(\mu)f = f + \langle \mu, f \rangle \delta$ for $f \in \widehat{\mathbb{R}}$ and $\mu \in \mathbb{R}$. In particular, $s_1 s_0$ has infinite order in $\mathcal{W}$ and

$$\mathcal{W} = W \ltimes \tau(\mathbb{Z}).$$

Furthermore, $\mathcal{W}$ is isomorphic to the Coxeter group with two generators s_0, s_1 and relations $s_0^2 = 1$, $s_1^2 = 1$.

2.7. The affine root system S has four $\mathcal{W}$-orbits, namely

$$S_s^1 = \mathcal{W}a_0^\vee = \big(\frac{1}{2} + \mathbb{Z}\big)\delta \pm 1, \qquad S_s^2 = \mathcal{W}a_1^\vee = \mathbb{Z}\delta \pm 1,$$
$$S_l^1 = \mathcal{W}a_0 = \big(1 + 2\mathbb{Z}\big)\delta \pm 2, \qquad S_l^2 = \mathcal{W}a_1 = 2\mathbb{Z}\delta \pm 2.$$

2.8. We have the disjoint union $S = R^\vee \cup R$, with $R = S_l^1 \cup S_l^2$ a reduced, irreducible affine root system with basis $\{a_0, a_1\}$, affine Weyl group $\mathcal{W}$, and gradient root system $\{\pm a_1\}$, and with $R^\vee = S_s^1 \cup S_s^2$ the corresponding affine co-root system. The co-root system $R^\vee$ is a reduced, affine root system with basis $\{a_0^\vee, a_1^\vee\}$, affine Weyl group $\mathcal{W}$, and gradient root system $\{\pm a_1^\vee\}$. Similarly as for S, see 2.5, the fixed choice of basis give rise to a decomposition of R and $R^\vee$ in positive and negative roots (the positive roots are denoted by R^+ and $R^{\vee,+}$, respectively). For example, we have $R^+ = \{a_1\} \cup \{\pm a_1 + \mathbb{N}\delta\}$.

2.9. Let $\omega \in \widehat{\mathbb{R}}$ be the involution $\omega(x) := \frac{1}{2} - x$ ($x \in \mathbb{R}$), and consider ω as an involution of $\widehat{\mathbb{R}}$ by $\omega : f \mapsto f \circ \omega$ for $f \in \widehat{\mathbb{R}}$. Then ω preserves S. Furthermore, $\omega = s_1\tau(\frac{1}{2})$, $\omega(a_i) = a_{1-i}$ and $\omega s_i \omega = s_{1-i}$ for $i = 0, 1$. The subgroup $\mathcal{W}^e$ of the invertible linear transformations of $\widehat{\mathbb{R}}$ generated by $\mathcal{W}$ and ω is called the *extended affine Weyl group*. It is isomorphic to $\mathcal{W} \rtimes \Omega$, where Ω is the subgroup of order two generated by ω.

2.10. Set $\mathcal{A} := \mathbb{C}[x^{\pm 1}]$ for the algebra of Laurent polynomials in one indeterminate x. We set $x^f := q^\lambda x^\mu \in \mathcal{A}$ for $f = \mu + \lambda\delta \in \mathbb{Z} + \mathbb{R}\delta$, where q is a fixed non-zero complex number. Observe that $x^a \in \mathcal{A}$ is well defined for $a \in S$ and that $\mathcal{W}^e$ preserves $\mathbb{Z} + \mathbb{R}\delta$. Furthermore, $w(x^\mu) := x^{w(\mu)}$ for $\mu \in \mathbb{Z}$ and $w \in \mathcal{W}^e$ extends to an action of $\mathcal{W}^e$ on $\mathcal{A}$ by linearity. In particular,

$$s_0(x^m) = q^m x^{-m}, \qquad s_1(x^m) = x^{-m}, \qquad m \in \mathbb{Z}.$$

Observe that $\tau(\mu)$ ($\mu \in \mathbb{Z}$) acts as a q-difference operator: $\tau(\mu)(x^m) = q^{\mu m} x^m$ for all $m \in \mathbb{Z}$.

2.11. A *multiplicity function* $\underline{t} = \{t_a\}_{a \in S}$ of S is a choice of non-zero complex numbers t_a ($a \in S$) such that $t_{w(a)} = t_a$ for all $a \in S$ and all $w \in W$. We use the convention that $t_f = 1$ for all $f \in \widehat{\mathbb{R}} \setminus S$. A multiplicity function $\underline{t}$ of S is determined by the values $k_0 := t_{a_0}$, $u_0 := t_{a_0^\vee}$, $k_1 := t_{a_1}$ and $u_1 := t_{a_1^\vee}$, see 2.7. Later on it will be necessary to impose (generic) conditions on the parameters t_f ($f \in S$) and on the deformation parameter q. Until section 6 it suffices to assume that $|q| \neq 1$ and that $k_0^2, k_1^2, u_1^2 \notin \pm q^{\mathbb{Z}}$. These conditions are assumed to hold throughout the remainder of the paper, unless specified explicitly otherwise.

2.12. The *Hecke algebra* $H_0 = H_0(k_1)$ of type A_1 is the unital, associative $\mathbb{C}$-algebra with generator T_1 and relation $(T_1 - k_1)(T_1 + k_1^{-1}) = 0$. Observe that $\{1, T_1\}$ is a linear basis of H_0 and that T_1 is invertible in H_0 with inverse $T_1^{-1} = T_1 + k_1^{-1} - k_1$.

2.13. The *affine Hecke algebra* $H = H(R; k_0, k_1)$ of type $\widetilde{A}_1$ is the unital $\mathbb{C}$-algebra with generators T_0 and T_1 and relations

$$(T_i - k_i)(T_i + k_i^{-1}) = 0, \qquad i = 0, 1.$$

Similarly as for H_0 we have that T_i is invertible in H with inverse $T_i^{-1} = T_i + k_i^{-1} - k_i$.

2.14. For $w \in \mathcal{W}$, let $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ be a reduced expression, i.e. a minimal expression of w as product of the simple reflections s_0 and s_1. Then $T_w := T_{i_1} T_{i_2} \cdots T_{i_r}$ is well-defined and $\{T_w\}_{w \in \mathcal{W}}$ is a linear basis of $H(R; k_0, k_1)$, see [13]. In particular, we may regard H_0 as a subalgebra of H.

2.15. We set $Y := T_{\tau(1)} = T_1 T_0 \in H$. By [13], we known that Y is algebraically independent in H. Let $\mathbb{C}[Y^{\pm 1}] \subset H$ be the commutative subalgebra generated by $Y^{\pm 1}$. Then

$$H_0(k_1) \otimes \mathbb{C}[Y^{\pm 1}] \simeq H(R; k_0, k_1) \simeq \mathbb{C}[Y^{\pm 1}] \otimes H_0(k_1)$$

as linear spaces, where the isomorphisms are given by multiplication. In particular, $\{Y^m, Y^n T_1\}_{m,n \in \mathbb{Z}}$ and $\{Y^m, T_1 Y^n\}_{m,n \in \mathbb{Z}}$ are linear bases of $H(R; k_0, k_1)$, see [13, proposition 3.7].

In the remainder of the paper we identify $\mathcal{A}$ with $\mathbb{C}[Y^{\pm 1}]$ as algebra by identifying the indeterminate x of $\mathcal{A}$ with Y. In particular, we write $f(Y)=\sum_k c_k Y^k \in \mathbb{C}[Y^{\pm 1}]$ for $f(x)=\sum_k c_k x^k \in \mathcal{A}$.

2.16. By Lusztig [13, proposition 3.6], we have the fundamental commutation relations

$$T_1 f(Y) - f(Y^{-1})T_1 = \big((k_1-k_1^{-1})Y^2+(k_0-k_0^{-1})Y\big)\left(\frac{f(Y^{-1})-f(Y)}{1-Y^2}\right)$$

in $H(R;k_0,k_1)$ for all $f(Y)\in\mathbb{C}[Y^{\pm 1}]$. Indeed, observe that if the formula holds for $f(Y)$ and $g(Y)$, then it also holds for $f(Y)g(Y)$. It thus suffices to prove it for $f(Y)=Y^{\pm 1}$, in which case it follows from an elementary computation using the definition of Y and the quadratic relations for the T_i, see 2.15 and 2.13, respectively.

2.17. The following result was observed by Sahi [19, theorem 5.1] in the higher rank setting.

Corollary (The non-affine intertwiner)**.** *Set $S_1=[T_1,Y]=T_1Y-YT_1\in H$. Then $f(Y)S_1=S_1f(Y^{-1})$ in H for all $f(Y)\in\mathbb{C}[Y^{\pm 1}]$.*

Proof. This follows immediately from the definition of S_1 and from Lusztig's commutation relation 2.16. □

2.18. Another important consequence of Lusztig's commutation relation 2.16 is the following result.

Corollary. *The affine Hecke algebra $H=H(R;k_0,k_1)$ acts on $\mathbb{C}[Y^{\pm 1}]$ by*

$$\begin{aligned}T_1.g(Y) &= k_1 g(Y^{-1}) + \big((k_1-k_1^{-1})Y^2+(k_0-k_0^{-1})Y\big)\left(\frac{g(Y^{-1})-g(Y)}{1-Y^2}\right)\\ &= k_1 g(Y) + k_1^{-1}\frac{(1-k_0k_1Y^{-1})(1+k_0^{-1}k_1Y^{-1})}{(1-Y^{-2})}(g(Y^{-1})-g(Y)),\\ f(Y).g(Y) &= f(Y)g(Y)\end{aligned}$$

for all $f(Y),g(Y)\in\mathbb{C}[Y^{\pm 1}]$.

Proof. Let χ be the character of $H_0(k_1)$ which maps T_1 to k_1. By 2.15 we may identify the representation space of the induced representation $\mathrm{Ind}_{H_0}^{H}(\chi)=H\otimes_\chi\mathbb{C}$ with $\mathbb{C}[Y^{\pm 1}]$. By 2.16 the corresponding induced action of H on $\mathbb{C}[Y^{\pm 1}]$ is as indicated in the statement of the corollary. □

2.19. We define linear operators $\widehat{T}_i\in\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ by

$$\begin{aligned}\widehat{T}_i :=& k_i + k_i^{-1}\frac{\big(1-k_iu_ix^{a_i^\vee}\big)\big(1+k_iu_i^{-1}x^{a_i^\vee}\big)}{1-x^{a_i}}(s_i-\mathrm{id})\\ =& k_is_i + \frac{(k_i-k_i^{-1})+(u_i-u_i^{-1})x^{a_i^\vee}}{(1-x^{a_i})}(\mathrm{id}-s_i), \qquad i=0,1.\end{aligned}$$

The following theorem was proved by Noumi [16, section 3] in the higher rank setting, see also [19, section 2.3].

Theorem. *The application $T_i\mapsto\widehat{T}_i$ $(i=0,1)$ extends uniquely to an algebra homomorphism $\pi_{\underline{t},q}:H(R;k_0,k_1)\rightarrow End_{\mathbb{C}}(\mathcal{A})$.*

Proof. We identify $\mathbb{C}[Y^{\pm 1}] \subset H(R; u_1, k_1)$ with $\mathcal{A}$ as algebra by identifying Y with x^{-1}. Then it follows from corollary 2.18 (applied to $H(R; u_1, k_1)$) that $(\widehat{T}_1 - k_1)(\widehat{T}_1 + k_1^{-1}) = 0$ in $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$. Conjugating $\widehat{T}_1$ with the involution ω, see 2.9, and replacing k_1 and u_1 by k_0 and u_0 respectively, we see that $(\widehat{T}_0 - k_0)(\widehat{T}_0 + k_0^{-1}) = 0$ in $\mathrm{End}(\mathcal{A})$. The theorem follows, since we have shown that all the defining relations 2.13 of $H(R; k_0, k_1)$ are satisfied by the linear operators $\widehat{T}_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ $(i = 0, 1)$. □

2.20. Observe that the linear operator $\widehat{T}_0$ on $\mathcal{A}$ has a reflection and a q-difference part, while $\widehat{T}_1$ has only a reflection part, see 2.10. The operators $\widehat{T}_i \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ $(i = 0, 1)$ are called the *difference-reflection operators associated with* S. In the remainder of the paper, we simply write T_i for the difference-reflection operators $\widehat{T}_i$ $(i = 0, 1)$ if no confusion is possible. The operator $Y = T_1 T_0 \in \mathrm{End}_{\mathbb{C}}(\mathcal{A})$ is called the *Cherednik-Dunkl operator* associated with S.

2.21. The representation $\pi_{\underline{t},q}$ has two extra degrees of freedom u_0 and u_1 besides the deformation parameter q (which already lives on the affine Weyl group level, see 2.10). The motivation to label these two degrees of freedom in this particular way comes from the theory of double affine Hecke algebras. The *double affine Hecke algebra* $\mathcal{H}(S; \underline{t}; q)$ associated with the affine root system S (see [19]) is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ generated by $\pi_{\underline{t},q}(H(R; k_0, k_1))$ and $\mathcal{A}$, where we consider $\mathcal{A}$ as a subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ via its regular representation. We write $f(z) \in \mathrm{End}\big(\mathcal{A}\big)$ for the Laurent polynomial $f(x) \in \mathcal{A}$ regarded as a linear endomorphism of $\mathcal{A}$. By the second formula for the difference-reflection operators T_i in 2.19 we have that

$$f(z)T_i - T_i(s_i f)(z) = \frac{(k_i - k_i^{-1}) + (u_i - u_i^{-1}) z^{a_i^\vee}}{(1 - z^{a_i})}\big(f(z) - (s_i f)(z)\big), \qquad f \in \mathcal{A}$$

in $\mathcal{H}(S; \underline{t}, q)$ for $i = 0, 1$.

2.22. The labeling of the extra degrees of freedom in the representation $\pi_{\underline{t},q}$ is now justified by the following theorem, together with 2.11.

Theorem. *$\mathcal{H}(S; \underline{t}; q)$ is isomorphic as algebra to the unital, associative $\mathbb{C}$-algebra $\mathcal{F}(\underline{t}; q)$ with generators $V_0^\vee, V_0, V_1, V_1^\vee$ and relations:*

1. *The application $T_i \mapsto V_i$ for $i = 0, 1$ extends to an algebra homomorphism $H(R; k_0, k_1) \to \mathcal{F}(\underline{t}; q)$.*
2. *The application $T_i \mapsto V_i^\vee$ for $i = 0, 1$ extends to an algebra homomorphism $H(R; u_0, u_1) \to \mathcal{F}(\underline{t}; q)$.*
3. *(Compatibility). $V_1^\vee V_1 V_0 V_0^\vee = q^{-1/2}$.*

The isomorphism $\phi : \mathcal{F}(\underline{t}; q) \to \mathcal{H}(S; \underline{t}; q)$ is explicitly given by $\phi(V_i) = T_i$ $(i = 0, 1)$, $\phi(V_0^\vee) = T_0^{-1} z^{-a_0^\vee} = q^{-1/2} T_0^{-1} z$ and $\phi(V_1^\vee) = z^{-a_1^\vee} T_1^{-1} = z^{-1} T_1^{-1}$.

The existence of the algebra homomorphism ϕ follows by direct computations using 2.21. It is immediate that ϕ is surjective. The injectivity of ϕ requires a detailed study of the difference-reflection operators T_i associated with S. We give the proof in 8.3.

3. Non-symmetric Askey-Wilson polynomials

3.1. For $a \in R$, let $\mathcal{R}(a) \in \mathrm{End}(\mathcal{A})$ be the difference-reflection operator defined by

$$\mathcal{R}(a) := t_a s_a + t_a^{-1}\frac{(1-t_at_{a/2}x^{a/2})(1+t_at_{a/2}^{-1}x^{a/2})}{(1-x^a)}\big(1-s_a\big).$$

Then it is immediate that $\mathcal{R}(a_i) = T_i s_i$ for $i = 0, 1$, where T_i is the difference-reflection operator associated with S (see 2.19). Furthermore, we have $w\mathcal{R}(a)w^{-1} = \mathcal{R}(w(a))$ for all $w \in \mathcal{W}$ and all $a \in R$. Since any $a \in R$ is conjugate to a_0 or a_1 under the action of $\mathcal{W}$, we obtain from the quadratic relations for the difference-reflection operators T_i (see 2.13 and 2.19) that

$$\mathcal{R}(a)^{-1} = \mathcal{R}(-a) + (t_a - t_a^{-1})s_a, \qquad a \in R.$$

3.2. We define a total order $\preceq$ on the basis of monomials $\{x^m\}_{m\in\mathbb{Z}}$ of $\mathcal{A}$ by

$$1 \prec x^{-1} \prec x \prec x^{-2} \prec x^2 \prec \cdots .$$

Observe that this order is not well behaved under multiplication of the monomials: if $x^{m_i} \preceq x^{n_i}$ $(i = 1, 2)$, then not necessarily $x^{m_1+m_2} \preceq x^{n_1+n_2}$.

3.3. Let $\epsilon : \mathbb{Z} \to \{\pm 1\}$ be the function which sends a positive integer to 1 and a strictly negative integer to -1.

Lemma. *Let $a \in R$ be of the form $a = 2 + k\delta$ with $k \in \mathbb{Z}$ (see 2.8) and let $m \in \mathbb{Z}$. Then*

$$\mathcal{R}(a)\big(x^m\big) = t_a^{\epsilon(m)}x^m + \text{lower order terms w.r.t. } \preceq .$$

Proof. For $a \in R$, let $D_a \in \mathrm{End}(\mathcal{A})$ be the divided difference operator defined by

$$D_a f := \frac{f - s_a f}{1 - x^a}, \qquad f \in \mathcal{A}.$$

Then $D_a(1) = 0$ and

$$D_a\big(x^m\big) = \begin{cases} -x^{m-a} - x^{m-2a} - \cdots - x^{m-\langle m,a^\vee\rangle a} & \text{if } \langle m, a^\vee\rangle \in \mathbb{N},\\ x^m + x^{m+a} + \cdots + x^{m-(1+\langle m,a^\vee\rangle)a} & \text{if } \langle m, a^\vee\rangle \in -\mathbb{N}.\end{cases}$$

Observe that $\langle m, a^\vee\rangle = m$ when $a = 2 + k\delta$ for some $k \in \mathbb{Z}$. The lemma is now immediate when $m \in \mathbb{Z}_+$. For $m \in -\mathbb{N}$, we first observe that the coefficient of x^{-m} in the expansion of $\mathcal{R}(a)(x^m)$ in terms of monomials is zero. Indeed, the coefficient of x^{-m} in

$$t_a^{-1}(1 - t_at_{a/2}x^{a/2})(1 + t_at_{a/2}^{-1}x^{a/2})D_a(x^m)$$

is $-t_a q^{-mk}$, which cancels with the coefficient of $t_a s_a(x^m) = t_a q^{-mk}x^{-m}$. Hence the highest order term of $\mathcal{R}(a)(x^m)$ is $t_a^{-1}x^m$ when $m \in -\mathbb{N}$. This completes the proof of the lemma. □

3.4. Lemma 3.3 implies the following triangularity property of the Cherednik-Dunkl operator Y. Set $\gamma_m := k_0^{\epsilon(m)}k_1^{\epsilon(m)}q^m$ for $m \in \mathbb{Z}$.

Proposition. *For all $m \in \mathbb{Z}$, we have*

$$Y(x^m) = \gamma_m x^m + \text{lower order terms w.r.t. } \preceq .$$

Proof. Observe that $Y = T_1T_0 = \mathcal{R}(a_1)s_1\mathcal{R}(a_0)s_0 = \mathcal{R}(a_1)\mathcal{R}(s_1(a_0))\tau(1)$. Now $s_1(a_0) = 2 + \delta$ and $\tau(1)(x^m) = q^m x^m$ (see 2.10), so the proposition follows from lemma 3.3. □

3.5. The diagonal terms γ_m ($m \in \mathbb{Z}$) of the triangular operator Y are pair-wise different by the generic conditions 2.11 on q and on the multiplicity function $\underline{t}$. Hence proposition 3.4 leads immediately to the following proposition (compare with Sahi [19, section 6] for the higher rank setting).

Proposition. *There exists a unique basis* $\{P_m(\cdot) = P_m(\cdot; \underline{t}; q) \mid m \in \mathbb{Z}\}$ *of* $\mathcal{A}$ *such that*

1. $P_m(x) = x^m +$ *lower order terms with respect to* $\preceq$,
2. $Y(P_m) = \gamma_m P_m$

for all $m \in \mathbb{Z}$.

Definition. *The Laurent polynomial* $P_m = P_m(\cdot; \underline{t}; q)$ ($m \in \mathbb{Z}$) *is called the monic, non-symmetric Askey-Wilson polynomial of degree* m.

We will justify this terminology in section 5, where we relate the non-symmetric Askey-Wilson polynomials with the well-known symmetric Askey-Wilson polynomials by a kind of symmetrization procedure.

4. The fundamental representation

4.1. In the previous section we have diagonalized the action of the "translation part" $\mathbb{C}[Y^{\pm 1}]$ of the affine Hecke algebra $H = H(R; k_0, k_1)$ under the fundamental representation $\pi_{\underline{t},q}$ (see 2.19). The corresponding eigenfunctions are exactly the non-symmetric Askey-Wilson polynomials. Since H is generated as algebra by Y and the difference-reflection operator T_1, see 2.15, it suffices to understand the action of T_1 on the non-symmetric Askey-Wilson polynomials in order to completely decompose $\mathcal{A}$ as an H-module. Recall the notation $\gamma_m = k_0^{\epsilon(m)} k_1^{\epsilon(m)} q^m$ ($m \in \mathbb{Z}$) for the eigenvalues of Y.

Proposition. *For* $m \in \mathbb{Z}$ *we have*

$$T_1 P_m = \alpha_m P_m + \beta_m P_{-m},$$

with

$$\alpha_m = \frac{(k_1^{-1} - k_1)\gamma_m^2 + (k_0^{-1} - k_0)\gamma_m}{1 - \gamma_m^2}.$$

If $m \in -\mathbb{N}$ *then* $\beta_m = k_1$, *and*

$$\beta_m = k_1 \prod_{\xi = \pm 1} \frac{(1 + k_0 k_1^{-1} \gamma_m^{\xi})(1 - k_0^{-1} k_1^{-1} \gamma_m^{\xi})}{(1 - \gamma_m^{2\xi})}$$

if $m \in \mathbb{Z}_+$.

Proof. The formula for $m = 0$ reduces to $T_1(P_0) = k_1 P_0$, which is clear. For $0 \neq m \in \mathbb{Z}$, we derive from Lusztig's formula 2.16 and from the definition 3.5 of the non-symmetric Askey-Wilson polynomials that for all $f(Y) \in \mathbb{C}[Y^{\pm 1}]$,

$$\big(f(Y) - f(\gamma_{-m})\big) T_1 P_m = \alpha_m \big(f(\gamma_m) - f(\gamma_{-m})\big) P_m$$

with α_m as given in the statement of the proposition. Since γ_m ($m \in \mathbb{Z}$) are mutually different by the conditions 2.11 on the parameters, we derive from proposition 3.5 that $T_1 P_m = \alpha_m P_m + \beta_m P_{-m}$ for some β_m.

If $m \in -\mathbb{N}$, then we have $x^m \prec s_1(x^m) = x^{-m}$. Combined with the formula $T_1 = s_1 \mathcal{R}(a_1)^{-1} + k_1 - k_1^{-1}$ (see 3.1) and with the triangularity of $\mathcal{R}(a_1)$ (see lemma 3.3), we obtain that the coefficient of x^{-m} in the expansion of $T_1(x^m)$ with respect to the basis of monomials is equal to k_1. By the definition 3.5 of the non-symmetric Askey-Wilson polynomials, we conclude that $\beta_m = k_1$ for $m \in -\mathbb{N}$.

Now act by T_1 on both sides of the formula $T_1 P_m = \alpha_m P_m + \beta_m P_{-m}$ and use the quadratic relation for T_1, see 2.12. It follows that the α_m and the β_m satisfy the relation

$$\beta_m \beta_{-m} = (k_1 - \alpha_m)(k_1^{-1} + \alpha_m), \qquad 0 \neq m \in \mathbb{Z}.$$

This allows us to compute β_m with $m \in \mathbb{N}$ from the known expressions for α_m and β_{-m}, which yields the desired result. □

4.2. A uniform formula for the action of T_1 on the non-symmetric Askey-Wilson polynomials can be obtained by renormalizing the non-symmetric Askey-Wilson polynomials in a suitable way. A natural renormalization, together with a new proof of proposition 4.1, is given in section 10.

4.3. As a consequence of proposition 4.1, we can compute the action of the non-affine intertwiner S_1 (see 2.17) on the non-symmetric Askey-Wilson polynomials explicitly.

Corollary. *We have $S_1(P_m) = \big(\gamma_m - \gamma_{-m}\big)\beta_m P_{-m}$ for $m \in \mathbb{Z}$, where β_m is as in proposition 4.1.*

Proof. By proposition 4.1 and by the definition 3.5 of the non-symmetric Askey-Wilson polynomial, we have

$$\begin{aligned} S_1 P_m =& (T_1 Y - Y T_1) P_m = \big(\gamma_m - Y\big) T_1 P_m \\ =& \big(\gamma_m - Y\big)\big(\alpha_m P_m + \beta_m P_{-m}\big) = \big(\gamma_m - \gamma_{-m}\big)\beta_m P_{-m}. \end{aligned}$$

□

4.4. We set $\mathcal{A}(0) = \mathrm{span}\{P_0\}$ and $\mathcal{A}(m) = \mathrm{span}\{P_m, P_{-m}\}$ for $m \in \mathbb{N}$.

Theorem. (i) *The representation $(\pi_{\underline{t},q}, H(R; k_0, k_1))$ is faithful.*

(ii) *The center $\mathcal{Z}(H)$ of $H = H(R; k_0, k_1)$ is equal to $\mathbb{C}[Y^{\pm 1}]^W = \mathbb{C}[Y + Y^{-1}]$.*

(iii) *The decomposition $\mathcal{A} = \oplus_{m \in \mathbb{Z}_+} \mathcal{A}(m)$ is the multiplicity-free, irreducible decomposition of $\mathcal{A}$ as $(\pi_{\underline{t},q}, H)$-module. It is also the decomposition of $\mathcal{A}$ in isotypical components for the action of the center, where the central character of $\mathcal{A}(m)$ is given by $\chi_m(f(Y)) = f\big(\gamma_m\big)$ for $f(Y) \in \mathcal{Z}(H) = \mathbb{C}[Y^{\pm 1}]^W$.*

Proof. **(i)** Suppose that $h = f(Y) + T_1 g(Y)$ acts as zero on $\mathcal{A}$ under the representation $\pi_{\underline{t},q}$, where $f, g \in \mathcal{A}$. Let h act on the non-symmetric Askey-Wilson polynomials P_m $(m < 0)$ and use proposition 4.1 together with the fact that the coefficients β_m $(m < 0)$ in 4.1 are non-zero. Then we conclude that $g(\gamma_m) = 0$ for all $m \in -\mathbb{N}$. By the conditions 2.11 on the parameters, this implies $g = 0$ in $\mathcal{A}$. But $h = f(Y)$ acting on P_m shows that $f\big(\gamma_m\big) = 0$ for all m, hence $f = 0$ in $\mathcal{A}$. Combined with 2.15, this shows that $\pi_{\underline{t},q}$ is faithful.

(ii) Clearly any element from $\mathbb{C}[Y + Y^{-1}]$ commutes with $\mathbb{C}[Y^{\pm 1}]$, but also with T_1 by Lusztig's formula 2.16. Hence 2.15 gives $\mathbb{C}[Y + Y^{-1}] \subset \mathcal{Z}(H)$. Suppose $0 \neq h = f(Y) + T_1 g(Y) \in \mathcal{Z}(H)$, where $f, g \in \mathcal{A}$. Then h acts as a constant on each of the P_m $(m \in \mathbb{Z})$. In view of proposition 4.1, this implies that $g(\gamma_m) = 0$ for all $m \neq 0$, hence $g = 0$ in $\mathcal{A}$. By corollary 4.3 we then have for $m \neq 0$,

$$f\big(\gamma_m\big) S_1 P_m = S_1(h P_m) = h(S_1 P_m) = f\big(\gamma_m^{-1}\big) S_1 P_m.$$

Furthermore, $S_1P_m \neq 0$ by the conditions 2.11 on the parameters. Hence $f(\gamma_m) = f(\gamma_m^{-1})$ for $0 \neq m \in \mathbb{Z}$, i.e. $h = f(Y) \in \mathbb{C}[Y+Y^{-1}]$.

(iii) The second statement follows directly from proposition 3.5 and from the fact that the central character values $\chi_m(Y+Y^{-1}) = \gamma_m + \gamma_m^{-1}$ $(m \in \mathbb{Z}_+)$ are pairwise different by the conditions 2.11 on the parameters. For the first statement, it then suffices to show that $\mathcal{A}(m)$ $(m \in \mathbb{Z}_+)$ are irreducible H-modules. This follows without difficulty from 2.15, proposition 4.1, corollary 4.3 and the fact that $\beta_m \neq 0$ for all $0 \neq m \in \mathbb{Z}$ by the conditions 2.11 on the parameters. □

5. The (anti-)symmetric Askey-Wilson polynomials

5.1. In the present rank one setting, the representation theory of the underlying two-dimensional Hecke algebra $H_0 = H_0(k_1)$ is extremely simple: the trivial representation χ_+ and the alternating representation χ_- exhaust its irreducible representations, where $\chi_\pm$ are uniquely determined by $\chi_\pm(T_1) = \pm k_1^{\pm 1}$. The corresponding mutually orthogonal, primitive idempotents are given by

$$C_+ = \frac{1}{1+k_1^2}\big(1 + k_1T_1\big), \qquad C_- = \frac{1}{1+k_1^{-2}}\big(1 - k_1^{-1}T_1\big).$$

So $\{C_-, C_+\}$ is a partition of the unity for H_0. In particular, we have $C_- + C_+ = 1$.

5.2. The partition of the unity of H_0 introduced in 5.1 gives the decomposition $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ of $\mathcal{A}$ in isotypical components for the action of $(\pi_{\underline{t},q}|_{H_0}, H_0)$, where $\mathcal{A}_\pm = C_\pm\mathcal{A}$. Observe that $\mathcal{A}_\pm$ consists of the Laurent polynomials $f \in \mathcal{A}$ which satisfy $(T_1 \mp k_1^{\pm 1})f = 0$. By the explicit expression 2.19 for the difference-reflection operator T_1 we have $T_1 - k_1 = \phi_1(x)(s_1 - \mathrm{id})$ for some non-zero rational function $\phi_1(x)$, so that $\mathcal{A}_+$ coincides with the algebra $\mathcal{A}^W = \mathbb{C}[x+x^{-1}]$ consisting of W-invariant Laurent polynomials. We call $\mathcal{A}_-$ the subspace of *anti-symmetric* Laurent polynomials. The decomposition $f = C_-f + C_+f$ for $f \in \mathcal{A}$ is its unique decomposition as a sum of an anti-symmetric and a symmetric Laurent polynomial.

5.3. The irreducible H-module $\mathcal{A}(m) \subset \mathcal{A}$ decomposes under the action of H_0 by $\mathcal{A}(m) = \mathcal{A}_-(m) \oplus \mathcal{A}_+(m)$, where $\mathcal{A}_\pm(m) = C_\pm\mathcal{A}(m)$.

Proposition. **(i)** *Let $m \in \mathbb{Z}_+$, then dim$(\mathcal{A}_+(m)) = 1$. More precisely, there exists a unique $P_m^+ \in \mathcal{A}_+(m)$ of the form $P_m^+(x) = x^m +$ lower order terms with respect to $\preceq$. In terms of non-symmetric Askey-Wilson polynomials, we have*

$$P_m^+ = P_m + \frac{(1+k_0k_1^{-1}\gamma_m)(1-k_0^{-1}k_1^{-1}\gamma_m)}{(1-\gamma_m^2)}P_{-m}, \qquad m \in \mathbb{Z}_+.$$

(ii) *We have $\mathcal{A}_-(0) = \{0\}$ and dim$(\mathcal{A}_-(m)) = 1$ for $m \in \mathbb{N}$. More precisely, there exists for all $m \in \mathbb{N}$ a unique $P_m^- \in \mathcal{A}_-(m)$ of the form $P_m^-(x) = x^m +$ lower order terms with respect to $\preceq$. In terms of non-symmetric Askey-Wilson polynomials, we have*

$$P_m^- = P_m - \frac{(1+k_0k_1^{-1}\gamma_m^{-1})(1-k_0^{-1}k_1^{-1}\gamma_m^{-1})}{(1-\gamma_m^{-2})}P_{-m}, \qquad m \in \mathbb{N}.$$

Proof. The statements for $m = 0$ are immediate since $T_1(1) = k_11$, where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. For $m \in \mathbb{N}$, we can write

$C_\pm P_m \in \mathcal{A}_\pm(m)$ explicitly as

$$\begin{aligned} C_\pm P_m &= \frac{1 \pm k_1^{\pm 1}\alpha_m}{1+k_1^{\pm 2}} P_m \pm \frac{k_1^{\pm 1}\beta_m}{1+k_1^{\pm 2}} P_{-m} \\ &= k_1^{\pm 1} \frac{(k_1^{\mp 1} \pm \alpha_m)}{(1+k_1^{\pm 2})} \big(P_m \pm k_1^{-1}(k_1^{\pm 1} \mp \alpha_m) P_{-m}\big) \end{aligned}$$

in view of (the proof of) proposition 4.1. Observe that the coefficient of P_m is non-zero by the conditions 2.11 on the parameters. In particular, $\mathcal{A}_\pm(m) = \mathrm{span}\{C_\pm P_m\}$ are one-dimensional subspaces for all $m \in \mathbb{N}$. Dividing out the non-zero coefficient of P_m in the expansion of $C_\pm P_m$ and using proposition 3.5, we conclude that there exist unique elements $P_m^\pm \in \mathcal{A}_\pm(m)$ ($m \in \mathbb{N}$) satisfying $P_m^\pm(x) = x^m +$ lower order terms w.r.t. $\preceq$. The explicit formulas for $P_m^\pm$ in terms of non-symmetric Askey-Wilson polynomials follow now by substituting the explicit expression for α_m in the above expansion of $C_\pm P_m$, see proposition 4.1. □

Definition. **(i)** *The polynomial* $P_m^+ = P_m^+(\cdot;\underline{t};q) \in \mathcal{A}_+$ ($m \in \mathbb{Z}_+$) *is called the monic, symmetric Askey-Wilson polynomial of degree* m.

(ii) *The polynomial* $P_m^- = P_m^-(\cdot;\underline{t};q) \in \mathcal{A}_-$ ($m \in \mathbb{N}$) *is called the monic, anti-symmetric Askey-Wilson polynomial of degree* m.

Askey and Wilson [1] defined a very general family of basic hypergeometric orthogonal polynomials which are nowadays known as the Askey-Wilson polynomials. In 5.9 we justify our terminology for the Laurent polynomials $P_m^\pm$ by showing that the P_m^+ ($m \in \mathbb{Z}$) coincide with the Askey-Wilson polynomials as defined in [1].

5.4. We can also express the non-symmetric Askey-Wilson polynomial in terms of the symmetric Askey-Wilson polynomial in the following way.

Lemma. *We have* $P_0 = P_0^+$, *and for* $m \in \mathbb{N}$,

$$\begin{aligned} P_m &= \frac{1}{\gamma_m - \gamma_{-m}} \big(Y - \gamma_{-m}\big) P_m^+, \\ P_{-m} &= \frac{\gamma_m}{(1+k_0k_1^{-1}\gamma_m)(1-k_0^{-1}k_1^{-1}\gamma_m)} \big(Y - \gamma_m\big) P_m^+. \end{aligned}$$

Proof. The statement for $m = 0$ is trivial. For $m \in \mathbb{N}$ the formulas follow directly from proposition 3.5 and from the expansion of P_m^+ as linear combination of non-symmetric Askey-Wilson polynomials, see proposition 5.3. □

5.5. Observe that the affine Weyl group $\mathcal{W}$ acts on $\mathcal{A}$ by algebra automorphisms (see 2.10). This action can be uniquely extended to an action (by automorphisms) of $\mathcal{W}$ on the rational functions $\mathbb{C}(x)$ in the indeterminate x. Since $|q| \neq 1$ (see 2.11), we have

$$\bigoplus_{w \in \mathcal{W}} \mathbb{C}(x) w = \bigoplus_{m \in \mathbb{Z}, \sigma \in W} \mathbb{C}(x)\tau(m)\sigma$$

as a subalgebra of $\mathrm{End}_{\mathbb{C}}\big(\mathbb{C}(x)\big)$.

5.6. Any $X \in \mathrm{Im}(\pi_{\underline{t},q}) \subset \mathrm{End}_{\mathbb{C}}\big(\mathcal{A}\big)$ can be uniquely written as a finite $\mathbb{C}(x)$-linear combination of the automorphisms $w \in \mathcal{W}$ of $\mathcal{A}$. By 5.5, we may regard X as a linear endomorphism of $\mathbb{C}(x)$. We thus have a unique decomposition $X = X_- s_1 + X_+$ where $X_\pm \in \bigoplus_{m \in \mathbb{Z}} \mathbb{C}(x)\tau(m)$ are q-difference operators with rational coefficients. We write $X_{sym} := X_- + X_+$, so that $Xf = X_{sym}f$ for all $f \in \mathcal{A}_+ = \mathcal{A}^W$.

5.7. In order to make the connection between the symmetric Askey-Wilson polynomials P_m^+ ($m \in \mathbb{Z}$) and the four parameter family of Askey-Wilson polynomials as originally defined in [1], it is convenient to reparametrize the multiplicity function $\underline{t} \simeq (u_0, u_1, k_0, k_1)$ (see 2.11) by

$$a = k_1 u_1, \quad b = -k_1 u_1^{-1}, \quad c = q^{\frac{1}{2}} k_0 u_0, \quad d = -q^{\frac{1}{2}} k_0 u_0^{-1}.$$

5.8. Using the parameters a, b, c, d (see 5.7), we can give the following explicit expression for the q-difference operator $(Y + Y^{-1})_{sym}$, see [16] for the higher rank result.

Proposition. *We have*

$$\big(Y + Y^{-1}\big)_{sym} = A(x)\big(\tau(1) - 1\big) + A(x^{-1})\big(\tau(-1) - 1\big) + k_0 k_1 + k_0^{-1} k_1^{-1},$$

with

$$A(x) = k_0^{-1} k_1^{-1} \frac{(1 - ax)(1 - bx)(1 - cx)(1 - dx)}{(1 - x^2)(1 - qx^2)}.$$

Proof. We write $T_i = k_i + \phi_i(x)(s_i - 1)$ with

$$\phi_0(x) = k_0^{-1} \frac{(1 - cx^{-1})(1 - dx^{-1})}{(1 - qx^{-2})}, \quad \phi_1(x) = k_1^{-1} \frac{(1 - ax)(1 - bx)}{(1 - x^2)}$$

for the difference-reflection operators associated with S, see 2.19. Since $Y = T_1 T_0$, $s_0 = t(-1)s_1$ and $T_i^{-1} = T_i + k_i^{-1} - k_i$ for $i = 0, 1$, we have

$$(Y + Y^{-1})_{sym} = B(x)(\tau(1) - 1) + C(x)(\tau(-1) - 1) + D(x)$$

for unique coefficients $B, C, D \in \mathbb{C}(x)$. Observe that $D(x) = (Y + Y^{-1})(1) = k_0 k_1 + k_0^{-1} k_1^{-1}$ where $P_0 = 1 \in A_+$ is the Laurent polynomial identically equal to one, since $T_i(1) = k_i 1$ for $i = 0, 1$. To compute the coefficient $B(x)$ (respectively $C(x)$), we need to compute the coefficient of $\tau(1)$ (respectively $\tau(-1)$) in $(Y + Y^{-1})_{sym}$. Recall that $s_0 = s_1 \tau(1) = \tau(-1)s_1$, so that the $\tau(1)$-term (respectively $\tau(-1)$-term) of Y_{sym} has coefficient $\phi_1(x)\phi_0(x^{-1}) = A(x)$ (respectively has coefficient $(k_1 - \phi_1(x))\phi_0(x)$). The $\tau(1)$-term (respectively $\tau(-1)$-term) of $(Y^{-1})_{sym}$ is zero (respectively has coefficient $\phi_0(x)\phi_1(qx^{-1}) + \phi_0(x)(k_1^{-1} - \phi_1(qx^{-1})) = k_1^{-1}\phi_0(x)$). Adding the contributions, we see that $B(x) = A(x)$ and that $C(x) = \phi_0(x)\big(k_1 + k_1^{-1} - \phi_1(x)\big) = A(x^{-1})$, which completes the proof of the proposition. □

The second order q-difference operator $L = (Y + Y^{-1})_{sym}$ is called the Askey-Wilson second-order q-difference operator, cf. [1, (5.7)].

5.9. The symmetric Askey-Wilson polynomial P_m^+ ($m \in \mathbb{Z}_+$) lies in the irreducible $H(R; k_0, k_1)$-module $\mathcal{A}(m)$, hence the central element $Y + Y^{-1} \in \mathcal{Z}(H)$ acts on P_m^+ as the scalar $\gamma_m + \gamma_m^{-1}$, see theorem 4.4. Combined with proposition 5.8, we conclude that P_m^+ is an eigenfunction of the Askey-Wilson second-order q-difference operator L with eigenvalue $\gamma_m + \gamma_m^{-1}$. The eigenvalues $\gamma_m + \gamma_m^{-1}$ ($m \in \mathbb{Z}_+$) are mutually different by the conditions 2.11 on the parameters and $\{P_m^+\}_{m \in \mathbb{Z}_+}$ is a linear basis of $\mathcal{A}_+ = \mathcal{A}^W$, so that P_m^+ is the unique W-invariant Laurent polynomial satisfying $LP_m^+ = (\gamma_m + \gamma_m^{-1})P_m^+$. A comparison with [1, (5.7)] yields the following result.

Theorem. *The W-invariant Laurent polynomial P_m^+ ($m \in \mathbb{Z}_+$) coincides with the monic Askey-Wilson polynomial of degree m as described in* [1]*. In particular, we have in terms of basic hypergeometric series,*

$$P_m^+(x) = \frac{\big(ab, ac, ad; q\big)_m}{a^m\big(abcdq^{m-1}; q\big)_m}\,{}_4\phi_3\left(\begin{matrix} q^{-m}, q^{m-1}abcd, ax, ax^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

5.10. Theorem 5.9 and lemma 5.4 can be used to write the non-symmetric and the anti-symmetric Askey-Wilson polynomials as a sum of two terminating balanced ${}_4\phi_3$'s. It is convenient to write $P_m^+(x) = P_m^+(x; a, b, c, d)$ for the symmetric Askey-Wilson polynomial $P_m^+(x) = P_m^+(x; \underline{t}; q)$ when we want to emphasize the dependence of P_m^+ on the (reparametrized) multiplicity function (a, b, c, d), see 5.7.

Proposition. **(i)** *For $m \in \mathbb{Z}_+$ we have*

$$\begin{aligned} P_m(x) &= q^m\frac{(1-abcdq^{m-1})}{(1-abcdq^{2m-1})}P_m^+(x; a, b, c, d) \\ &+ q^{(m-1)/2}\frac{(1-cx^{-1})(1-dx^{-1})x(1-q^m)}{(1-abcdq^{2m-1})}P_{m-1}^+(q^{-1/2}x; q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d), \end{aligned}$$

where the second term should be read as zero when $m = 0$.

(ii) *For $m \in \mathbb{N}$ we have*

$$\begin{aligned} P_{-m}(x) &= \frac{1}{(1-cdq^{m-1})}P_m^+(x; a, b, c, d) \\ &- q^{(m-1)/2}\frac{(1-cx^{-1})(1-dx^{-1})x}{(1-cdq^{m-1})}P_{m-1}^+(q^{-1/2}x; q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d). \end{aligned}$$

(iii) *For $m \in \mathbb{N}$ we have*

$$\begin{aligned} P_m^-(x) &= \frac{(1-abcdq^{m-1})}{ab(1-cdq^{m-1})}P_m^+(x; a, b, c, d) \\ &+ q^{(m-1)/2}\frac{(1-cx^{-1})(1-dx^{-1})x(ab-1)}{ab(1-cdq^{m-1})}P_{m-1}^+(q^{-1/2}x; q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d). \end{aligned}$$

Proof. Recall the rational function $\phi_0(x)$ defined in the proof of proposition 5.8. The proof of proposition 5.8 shows that $(Y^{-1})_{sym} = k_1^{-1}\phi_0(x)\big(\tau(-1)-1\big)+k_0^{-1}k_1^{-1}$.

(i) The formula for $m = 0$ is trivial. Let $m \in \mathbb{N}$. By lemma 5.4, we have $P_m = (\gamma_m - \gamma_{-m})^{-1}(\gamma_m - (Y^{-1})_{sym})P_m^+$. In view of the explicit formula for the q-difference operator $(Y^{-1})_{sym}$, we need to write $(t(-1)-1)P_m^+$ as a terminating balanced ${}_4\phi_3$. By a direct computation using the explicit expression of P_m^+ in terms of a terminating balanced ${}_4\phi_3$, see theorem 5.9, we have

$$\begin{aligned} \big((\tau(-1)-1)P_m^+(.; a, b, c, d)\big)(x) =& (q^{1/2}x^{-1} - q^{-1/2}x)(q^{m/2} - q^{-m/2}) \\ &.P_{m-1}^+(q^{-1/2}x; q^{1/2}a, q^{1/2}b, q^{1/2}c, q^{1/2}d), \end{aligned}$$

cf. [1, (5.6)] or [8, (7.7.7)]. This leads to the desired result.

(ii) We have $P_{-m}(x) = (1+k_0k_1^{-1}\gamma_m)^{-1}(1-k_0^{-1}k_1^{-1}\gamma_m)^{-1}(1-\gamma_m(Y^{-1})_{sym})P_m^+$ by lemma 5.4. The proof is now similar to the proof of **(i)**.

(iii) This follows from **(i)** and **(ii)**, together with proposition 5.3. □

6. (Bi-)orthogonality relations

6.1. We assume in this section that the multiplicity function $\underline{t}$ and the deformation parameter q satisfy the additional conditions that $0 < |q| < 1$ and that $ef \notin q^{\mathbb{Z}}$ for all $e, f \in \{a, b, c, d\}$.

6.2. Let $C \subset \mathbb{C}$ be a continuous rectifiable Jordan curve such that aq^k, bq^k, cq^k, dq^k ($k \in \mathbb{Z}_+$) are in the interior of C and such that $a^{-1}q^{-k}, b^{-1}q^{-k}, c^{-1}q^{-k}, d^{-1}q^{-k}$ ($k \in \mathbb{Z}_+$) are in the exterior of C. By the conditions 6.1 on the parameters, such a contour exists. We give C the counterclockwise orientation. Let $\langle .,. \rangle = \langle .,. \rangle_{\underline{t},q}$ and $\big(.,.\big) = \big(.,.\big)_{\underline{t},q}$ be the bilinear forms on $\mathcal{A}$ defined by

$$\langle f, g\rangle = \frac{1}{2\pi i}\int_{x\in C} f(x)g(x^{-1})\Delta(x)\frac{dx}{x}, \qquad \big(f,g\big) = \frac{1}{2\pi i}\int_{x\in C} f(x)g(x^{-1})\Delta_+(x)\frac{dx}{x},$$

where the weight functions $\Delta(x) = \Delta(x;\underline{t};q)$ and $\Delta_+(x) = \Delta_+(x;\underline{t};q)$ are given by the infinite products

$$\Delta(x) = \prod_{a\in R^+}\frac{\big(1-x^a\big)}{\big(1-t_at_{a/2}x^{a/2}\big)\big(1+t_at_{a/2}^{-1}x^{a/2}\big)},$$
$$\Delta_+(x) = \prod_{a\in R: a(0)\geq 0}\frac{\big(1-x^a\big)}{\big(1-t_at_{a/2}x^{a/2}\big)\big(1+t_at_{a/2}^{-1}x^{a/2}\big)}.$$

The conditions 6.1 on the parameters ensure that the weight functions are well-defined. In terms of q-shifted factorials, we can rewrite the weight function $\Delta_+(x)$ as

$$\Delta_+(x) = \frac{\big(x^2, x^{-2};q\big)_\infty}{\big(ax, ax^{-1}, bx, bx^{-1}, cx, cx^{-1}, dx, dx^{-1};q\big)_\infty}$$

using 5.7, 2.7 and 2.10. Hence $\Delta_+(\cdot)$ coincides with the weight function of the orthogonality measure of the symmetric Askey-Wilson polynomials as defined in [1] (see also proposition 6.9). Observe that $\Delta(x) = \alpha(x)\Delta_+(x)$ with $\alpha(x) = \alpha(x;k_1,u_1)$ given by

$$\alpha(x) = \frac{(1-k_1u_1x^{-1})(1+k_1u_1^{-1}x^{-1})}{(1-x^{-2})} = \frac{(1-ax^{-1})(1-bx^{-1})}{(1-x^{-2})}.$$

6.3. Using Cauchy's theorem we can rewrite $\langle .,. \rangle$ and $\big(.,.\big)$ as an integral over the unit circle T in the complex plane plus a finite sum of residues of the integrand. The residues of the weight functions $\Delta(\cdot)$ and $\Delta_+(\cdot)$ can be computed explicitly, see [1, section 2] or [8, section 7.5] for more details.

6.4. Observe that the factor $\alpha(x)$ in the weight function $\Delta(x)$ satisfies the identity $\alpha(x) + \alpha(x^{-1}) = 1 - ab$. Since $\Delta_+(x)$ is furthermore invariant under $x \mapsto x^{-1}$, we see that the restrictions of the bilinear forms $\langle .,. \rangle$ and $\big(.,.\big)$ to $\mathcal{A}^W$ coincide up to a constant:

Lemma. *For $f, g \in \mathcal{A}^W$, we have* $\langle f, g\rangle = \frac{1}{2}(1-ab)\big(f,g\big) = \frac{1}{2}(1+k_1^2)\big(f,g\big)$.

6.5. Let T be a linear endomorphism of $\mathcal{A}$. Then there exists at most one linear endomorphism T^* of $\mathcal{A}$ such that $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in \mathcal{A}$, since the bilinear form $\langle .,. \rangle$ is non-degenerate. If T^* exists, then we call T^* the *adjoint* of T with respect to $\langle .,. \rangle$.

6.6. We write T_0', T_1' for the difference-reflection operators associated to S with respect to inverse parameters $(\underline{t}^{-1}, q^{-1})$, where $\underline{t}^{-1}$ is the multiplicity function $(t_a^{-1})_{a\in S}$. More precisely, T_i' is the image of the fundamental generator $T_i \in H(R; k_0^{-1}, k_1^{-1})$ under the (faithful) representation $\pi_{\underline{t}^{-1},q^{-1}}$, see 2.19. Furthermore, we set $Y' = T_1'T_0'$ for the associated Cherednik-Dunkl operator.

Proposition. *The adjoint of the difference-reflection operator T_i $(i = 0, 1)$ and of the Dunkl operator Y exists. More precisely, we have $T_i^* = (T_i')^{-1}$ $(i = 0, 1)$ and $Y^* = (Y')^{-1}$.*

Proof. We use the notation $T_i = k_i + \phi_i(x)(s_i - 1)$ and $T_i' = k_i^{-1} + \phi_i'(x)(s_i' - 1)$ $(i = 0, 1)$ for the difference reflection operator with respect to the parameters $(\underline{t}, q)$ and $(\underline{t}^{-1}, q^{-1})$ respectively. Here $s_1' = s_1$, $(s_0'f)(x) = f(q^{-1}x^{-1})$ and $\phi_i(x)$ is as in the proof of proposition 5.8, while $\phi_i'(x)$ is $\phi_i(x)$ with the parameters $(\underline{t}, q)$ replaced by $(\underline{t}^{-1}, q^{-1})$. In view of the analytic dependence on the parameters $\underline{t}$ and q, we may assume without loss of generality that $0 < q < 1$ and that the Jordan curve C in the definition 6.2 of $\langle ., . \rangle$ satisfies the following additional properties: C has a parametrization of the form $r_C(x)e^{2\pi ix}$ with $r_C : [0, 1] \to (0, \infty)$, and C is W-invariant: $C^{-1} := \{z^{-1} \mid z \in C\} = C$.

For $i = 1$ it follows now from the W-invariance of $\Delta_+(x)$ that $\langle T_1f, g\rangle = \langle f, T_1^*g\rangle$ for all $f, g \in \mathcal{A}$ with

$$T_1^* = k_1 - \phi_1(x^{-1}) + \frac{\alpha(x)}{\alpha(x^{-1})}\phi_1(x)s_1.$$

Now $\alpha(x)\phi_1(x) = \alpha(x^{-1})\phi_1'(x)$ and $\phi_1(x^{-1}) = \phi_1'(x)$, so that

$$T_1^* = k_1 + \phi_1'(x)(s_1' - 1) = (T_1')^{-1}.$$

For $i = 0$, let $f, g \in \mathcal{A}$ and set $h(x) = f(qx^{-1})g(x^{-1})$. Observe that

$$(T_0f)(x)g(x^{-1}) - f(x)\big((T_0')^{-1}g\big)(x^{-1}) = \phi_0(x)\big(h(x) - (s_0h)(x)\big)$$

and that

$$\phi_0(x)\Delta(x) = k_0^{-1}\frac{\big(x^2, q^2x^{-2}; q\big)_\infty}{\big(ax, bx, cx, dx, qax^{-1}, qbx^{-1}, qcx^{-1}, qdx^{-1}; q\big)_\infty}$$

is invariant under $x \mapsto qx^{-1}$. By the specific properties of C, we obtain

$$\begin{aligned}\langle T_0f, g\rangle - \langle f, (T_0')^{-1}g\rangle &= \frac{1}{2\pi i}\int_{x\in C}\big(h(x) - (s_0h)(x)\big)\phi_0(x)\Delta(x)\frac{dx}{x}\\ &= \frac{1}{2\pi i}\int_{x\in C-qC} h(x)\phi_0(x)\Delta(x)\frac{dx}{x} = 0,\end{aligned}$$

where the last equality follows from Cauchy's theorem since $\phi_0(x)\Delta(x)$ is analytic on and within $C - qC$.

The statement for the Dunkl operator Y is immediate since $Y = T_1T_0$. □

6.7. We write P_m' $(m \in \mathbb{Z})$ for the non-symmetric Askey-Wilson polynomials with respect to the inverse parameters $(\underline{t}^{-1}, q^{-1})$.

Proposition. *The two bases $\{P_m\}_{m\in\mathbb{Z}}$ and $\{P_n'\}_{n\in\mathbb{Z}}$ of $\mathcal{A}$ form a bi-orthogonal system with respect to the non-degenerate bilinear form $\langle ., . \rangle$, i.e. $\langle P_m, P_n'\rangle = 0$ for $m, n \in \mathbb{Z}$ if $m \neq n$.*

Proof. By proposition 6.6 and proposition 3.5 we have

$$\gamma_m\langle P_m, P_n'\rangle = \langle Y\,P_m, P_n'\rangle = \langle P_m, (Y')^{-1}P_n'\rangle = \gamma_n\langle P_m, P_n'\rangle.$$

It follows that $\langle P_m, P_n'\rangle = 0$ if $m \neq n$ since the eigenvalues γ_m ($m \in \mathbb{Z}$) of Y are pair-wise different by the conditions 2.11 on the parameters. □

6.8. We write $P_m^{+}{}'$ and $P_m^{-}{}'$ for the symmetric and anti-symmetric Askey-Wilson polynomial with respect to inverse parameters $(\underline{t}^{-1}, q^{-1})$.

Let $\mathcal{B}$ be the basis of $\mathcal{A}$ consisting of P_m^{+} ($m \in \mathbb{Z}_+$) and P_n^{-} ($n \in \mathbb{N}$), and let $\mathcal{B}'$ be the basis of $\mathcal{A}$ consisting of $P_m^{+}{}'$ ($m \in \mathbb{Z}_+$) and $P_m^{-}{}'$ ($m \in \mathbb{N}$).

Proposition. *The pair $(\mathcal{B}, \mathcal{B}')$ forms a bi-orthogonal system of $\mathcal{A}$ with respect to the non-degenerate bilinear form $\langle .,. \rangle$.*

Proof. It follows from proposition 6.7 and from the fact that $P_m^{\pm} \in \mathcal{A}(m) = \mathrm{span}\{P_m, P_{-m}\}$ for $m \in \mathbb{Z}_+$ (with the convention $P_0^{-} \equiv 0$), that $\langle P_m^{\pm}, P_n^{\pm}{}'\rangle = 0$ if $m \neq n$.

By proposition 6.6 we have $\langle C_{\pm}f, g\rangle = \langle f, C_{\pm}'g\rangle$ for all $f, g \in \mathcal{A}$, where $C_{\pm}' = (1 + k_1^{\mp 2})^{-1}(1 \pm k_1^{\mp 1}T_1')$ are the mutually orthogonal, primitive idempotents of $H_0(k_1^{-1})$ (see 5.1) which act on $\mathcal{A}$ via $\pi_{\underline{t}^{-1},q^{-1}}$. Hence $\langle P_m^{\pm}, P_n^{\mp}{}'\rangle = 0$ for all $m, n \in \mathbb{Z}_+$. □

6.9. The bi-orthogonality relations of proposition 6.8 restricted to $\mathcal{A}_+ = \mathcal{A}^W$ reduce to the well-known orthogonality relations [1, theorem 2.3] of the symmetric Askey-Wilson polynomials:

Proposition. *For all $m \in \mathbb{Z}_+$, we have $P_m^{+}{}' = P_m^{+}$. In particular, $\langle P_m^{+}, P_n^{+}\rangle = \big(P_m^{+}, P_n^{+}\big) = 0$ if $m \neq n$.*

Proof. Recall that P_m^{+} is the unique W-invariant Laurent polynomial of the form x^m+ lower order terms with respect to $\preceq$ which is an eigenfunction of the Askey-Wilson q-difference operator $L = (Y + Y^{-1})_{sym}$ with eigenvalue $\gamma_m + \gamma_m^{-1}$. Then $P_m^{+} = P_m^{+}{}'$ follows from the fact that L and the eigenvalue $\gamma_m + \gamma_m^{-1}$ are invariant under $(\underline{t}, q) \mapsto (\underline{t}^{-1}, q^{-1})$. The second statement follows now from proposition 6.8 and lemma 6.4. □

7. The generalized Weyl character formula

7.1. The generalized Weyl character formula relates the anti-symmetric Askey-Wilson polynomial with the symmetric Askey-Wilson polynomial via the generalized Weyl denominator. The generalized Weyl denominator $\delta(\cdot)$, which we define in the following lemma, is an explicit anti-symmetric Laurent polynomial of minimal degree with respect to the total order $\preceq$ on the monomials $\{x^m\}_{m\in\mathbb{Z}}$.

Lemma. *We have $\mathcal{A}_- = \delta(z)\big(\mathcal{A}_+\big)$, where $\delta = \delta(\,\cdot\,; k_0, k_1) \in \mathcal{A}_-$ is given by*

$$\delta(x) = x^{-1}(x - k_0^{-1}k_1^{-1})(x + k_0k_1^{-1}) = x^{-1}(x - a^{-1})(x - b^{-1}).$$

Proof. By 2.21 we have $(T_1 + k_1^{-1})\delta(z) = \delta(z^{-1})(T_1 - k_1)$. Combined with 5.2 this implies $\delta(z)\big(\mathcal{A}_+\big) \subseteq \mathcal{A}_-$. Let now $f \in \mathcal{A}_-$, and set $g = \delta^{-1}f \in \mathbb{C}(x)$. Using the extended action of T_1 on $\mathbb{C}(x)$, see 5.6, we derive that

$$\delta(x^{-1})\big((T_1 - k_1)g\big)(x) = \big((T_1 + k_1^{-1})f\big)(x) = 0,$$

so that $\big((T_1 - k_1)g\big)(x) = 0$. Since $T_1 - k_1 = \phi_1(x)(s_1 - 1)$ with $0 \neq \phi_1(x) \in \mathbb{C}(x)$, we conclude that g is W-invariant in $\mathbb{C}(x)$. In particular, $\delta(x^{-1})f(x) = \delta(x)f(x^{-1})$

in $\mathcal{A}$. Since $\delta(x^{-1})=a^{-1}b^{-1}x^{-1}(x-a)(x-b)$ and $\delta(x)$ are relative coprime in the unique factorisation domain $\mathcal{A}$ by the conditions 2.11 on the parameters, we conclude that f is divisible by δ in $\mathcal{A}$. Hence, $g=\delta^{-1}f\in\mathcal{A}$. Since g is furthermore W-invariant, we conclude that $g\in\mathcal{A}_+$, hence $\mathcal{A}_-\subseteq\delta(z)\big(\mathcal{A}_+\big)$. □

7.2. The bilinear form $\langle .,.\rangle$ restricted to $\mathcal{A}_-$ can now be related to the bilinear form $\big(.,.\big)$ on $\mathcal{A}_+$ using lemma 7.1. We identify $\underline{t}$ with (k_0,k_1,u_0,u_1) in accordance with 2.11.

Lemma. *Assume that the parameters satisfy the additional conditions 6.1. Let* $\delta'(x)=\delta(x;k_0^{-1},k_1^{-1})$*. Then for all* $f,g\in\mathcal{A}^W$*,*

$$\langle\delta(z)f,\delta'(z)g\rangle_{\underline{t},q}=\frac{1}{2}(1+k_1^{-2})\big(f,g\big)_{k_0,qk_1,u_0,u_1,q}.$$

Proof. Set $\alpha'(x)=\alpha(x;k_1^{-1},u_1^{-1})$, see 6.2, then

$$\delta(x)\delta'(x^{-1})\alpha(x)=(1-ax)(1-bx)(1-ax^{-1})(1-bx^{-1})\alpha'(x).$$

By the explicit expression for the W-invariant weight function $\Delta_+(x;\underline{t};q)$, see 6.2, we obtain

$$\langle\delta(z)f,\delta'(z)g\rangle_{\underline{t},q}=\frac{1}{2\pi i}\int_{x\in C}f(x)g(x^{-1})\alpha'(x)\Delta_+(x;k_0,qk_1,u_0,u_1;q)\frac{dx}{x}$$

for $f,g\in\mathcal{A}^W$. The result follows now by symmetrizing the integrand, cf. 6.4. □

7.3. We are now in a position to relate the anti-symmetric Askey-Wilson polynomial P_m^- ($m\in\mathbb{N}$) with the symmetric Askey-Wilson polynomial P_{m-1}^+ via the generalized Weyl denominator δ. The result is as follows.

Proposition (Generalized Weyl character formula)**.** *For* $m\in\mathbb{N}$ *we have*

$$P_m^-(x;\underline{t};q)=\delta(x;k_0,k_1)P_{m-1}^+(x;k_0,qk_1,u_0,u_1;q).$$

Proof. We first prove the proposition when $|q|<1$.

We assume for the moment that the multiplicity function $\underline{t}$ satisfies the additional conditions 6.1 and that $\big(.,.\big)_{k_0,qk_1,u_0,u_1,q}$ restricts to a non-degenerate bilinear form on $\mathcal{A}_m^W:=\mathrm{span}\{m_n\,|\,n=0,\ldots,m-1\}$, where $m_0(x)=1$ and $m_n(x)=x^n+x^{-n}$ for $n\in\mathbb{N}$. These are generic conditions on the parameters, which can be removed by continuity at the end of the proof. Indeed, observe that the restriction of $\big(.,.\big)_{\underline{t},q}$ to $\mathcal{A}_m^W$ is non-degenerate when $0<a,b,c,d,q<1$, since then the bilinear form can be given as integration over the unit circle with respect to the positive weight function $\Delta_+(x)$. By analytic continuation, it follows that the restriction of $\big(.,.\big)_{\underline{t},q}$ to $\mathcal{A}_m^W$ is non-degenerate for generic parameter values satisfying the conditions 6.1.

By proposition 6.9 we conclude that $P_{m-1}^+(x;k_0,qk_1,u_0,u_1;q)$ is the unique W-invariant Laurent polynomial of the form $x^{m-1}+$ lower order terms w.r.t. $\preceq$ which is orthogonal to m_n for $n=0,\ldots,m-2$ with respect to the bilinear form $\big(.,.\big)_{k_0,qk_1,u_0,u_1,q}$. We show that $p(x)=\delta(x;k_0,k_1)^{-1}P_m^-(x;\underline{t};q)$ satisfies the same characterizing conditions.

By lemma 7.1 we have $p\in\mathcal{A}^W$. Since $P_m^-(x)=x^m+$ lower order terms w.r.t. $\preceq$ and $\delta(x)=x+$ lower order terms w.r.t $\preceq$, we have $p(x)=x^{m-1}+$ lower order terms w.r.t. $\preceq$. By the triangularity properties of the anti-symmetric Askey-Wilson polynomials (see proposition 5.3**(ii)**) and by lemma 7.1, we see that $\delta'(z)m_n\in$

$\mathrm{span}\{P_k^{-\prime} \,|\, k=1,\ldots,m-1\}$ for $n=0,\ldots,m-2$. By lemma 7.2 and proposition 6.8 we conclude that

$$\frac{1}{2}(1+k_1^{-2})\big(p,m_n\big)_{k_0,qk_1,u_0,u_1,q} = \langle P_m^-, \delta'(z)m_n\rangle_{\underline{t},q} = 0, \quad (n=0,\ldots,m-2).$$

Hence $p(x) = P_{m-1}^+(x;k_0,qk_1,u_0,u_1;q)$, as desired.

The proof for $|q|>1$ is a slight modification of the arguments for $|q|<1$. One uses now that $P_m^{+\prime} = P_m^+$ for $m\in\mathbb{Z}_+$ (see proposition 6.9) and a characterization of $P_m^+(x;\underline{t};q)$ in terms of the bilinear form $\big(.,.\big)_{\underline{t}^{-1},q^{-1}}$. □

7.4. We have now all the ingredients to express the diagonal terms $\langle P_m, P_m'\rangle$ ($m\in\mathbb{Z}$) and $\langle P_m^-, P_m^{-\prime}\rangle$ ($m\in\mathbb{N}$) of the bi-orthogonality relations in proposition 6.7 and proposition 6.8 in terms of the "quadratic norms" $\langle P_m^+, P_m^+\rangle = \frac{1}{2}(1+k_1^2)\big(P_m^+,P_m^+\big)$ ($m\in\mathbb{Z}_+$).

Indeed, by the generalized Weyl character formula, lemma 7.2 and proposition 6.9 we have for $m\in\mathbb{N}$ that

$$\begin{aligned}&\langle P_m^-, P_m^{-\prime}\rangle_{\underline{t},q} = \\ &\frac{1}{2}(1+k_1^{-2})\big(P_{m-1}^+(\,\cdot\,;k_0,qk_1,u_0,u_1;q), P_{m-1}^+(\,\cdot\,;k_0,qk_1,u_0,u_1;q)\big)_{k_0,qk_1,u_0,u_1,q}.\end{aligned}$$

On the other hand, for $m\in\mathbb{N}$ we have $P_m^- = -(1+k_1^{-2})C_-P_{-m}$ (cf. the proof of proposition 5.3), so that

$$\langle P_{-m}, P_{-m}'\rangle = \frac{(1+k_1^2)(1-\gamma_m^{-2})}{(1+k_0k_1^{-1}\gamma_m^{-1})(1-k_0^{-1}k_1^{-1}\gamma_m^{-1})}\langle P_m^-, P_m^{-\prime}\rangle$$

by proposition 5.3**(ii)**. Here we have used that $\langle C_+f,g\rangle = \langle f, C_-'g\rangle$ for all $f,g\in\mathcal{A}$ (see the proof of proposition 6.8) and that $C_-P_m^- = P_m^-$. Similarly, we can relate $\langle P_m, P_m'\rangle$ for $m\in\mathbb{N}$ to $\langle P_m^-, P_m^{-\prime}\rangle$ using (the proof of) proposition 5.3**(ii)**.

7.5. The generalized Weyl character formula plays a crucial role in the study of shift operators for the symmetric Askey-Wilson polynomials. In turn, shift operators can be used to explicitly evaluate the quadratic norms $\big(P_m^+,P_m^+\big)_{\underline{t},q}$ ($m\in\mathbb{Z}_+$). Combined with 7.4, this leads to explicit evaluations of all the diagonal terms of the bi-orthogonality relations in proposition 6.7 and proposition 6.8.

In section 11 we present another method for deriving explicit expressions of the diagonal terms, which uses the double affine Hecke algebra in an essential way. This method gives more insight in the particular structure of the diagonal terms. Namely, it shows that the diagonal terms can be naturally expressed in terms of the residue of the weight function in a certain specific simple pole, the constant term $\langle 1,1\rangle$, and the value of the Askey-Wilson polynomial at the point $a^{-1}=k_1^{-1}u_1^{-1}$.

We return to shift operators in section 12 in order to evaluate the constant term $\langle 1,1\rangle$ (which is the well-known Askey-Wilson integral, see [1]).

8. The double affine Hecke algebra

8.1. Recall from 2.21 that the double affine Hecke algebra $\mathcal{H}(S;\underline{t};q)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}\big(\mathcal{A}\big)$ generated by $\pi_{\underline{t},q}\big(H(R;k_0,k_1)\big)$ and by $\mathcal{A}$, where $\mathcal{A}$ is regarded as subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathcal{A})$ via its regular representation.

8.2. We have observed in 2.22 that there is a unique surjective algebra homomorphism $\phi : \mathcal{F}(\underline{t}; q) \to \mathcal{H}(S; \underline{t}; q)$ satisfying the conditions as stated in theorem 2.22. In particular, we have

$$z^{a_0^\vee} T_0 = T_0^{-1} z^{-a_0^\vee} + u_0^{-1} - u_0, \qquad z^{-a_1^\vee} T_1^{-1} = T_1 z^{a_1^\vee} + u_1 - u_1^{-1}$$

in $\mathcal{H}(S; \underline{t}; q)$. Combined with 2.15, we conclude that $\mathcal{H}(S; \underline{t}; q)$ is spanned by the elements $z^m Y^n$ and $z^m T_1 Y^n$ where $m, n \in \mathbb{Z}$. In fact, we have the following stronger result, see Sahi [19, theorem 3.2] for the higher rank setting.

Proposition. *The set $\{z^m Y^n, z^m T_1 Y^n \mid m, n \in \mathbb{Z}\}$ is a linear basis of $\mathcal{H}(S; \underline{t}; q)$.*

Proof. Assume that $X = \sum_{m,n \in \mathbb{Z}} \big(c^1_{m,n} z^m Y^n + c^2_{m,n} z^m T_1 Y^n\big) = 0$ in $\mathrm{End}_{\mathbb{C}}\big(\mathcal{A}\big)$ with only finitely many coefficients $c^j_{m,n}$ non-zero. Since z is invertible in $\mathrm{End}_{\mathbb{C}}\big(\mathcal{A}\big)$, we may assume without loss of generality that $c^j_{m,n} = 0$ unless $m \in \mathbb{Z}_+$. Suppose that not all coefficients $c^j_{m,n}$ are zero. Let $m_0 \in \mathbb{Z}_+$ be the largest positive integer such that $c^j_{m_0,n}$ is non-zero for some $n \in \mathbb{Z}$ and some $j \in \{1, 2\}$.

Let X act on the non-symmetric Askey-Wilson polynomial P_{-l} ($l \in \mathbb{N}$), and consider the coefficient of x^{m_0+l} in the resulting expression using proposition 3.5 and proposition 4.1. We obtain $\sum_{n \in \mathbb{Z}_+} k_1 c^2_{m_0,n} \gamma_l^{-n} = 0$ for all $l \in \mathbb{N}$, hence $c^2_{m_0,n} = 0$ for all $n \in \mathbb{Z}$.

Let now X act on P_l with $l \in \mathbb{Z}_+$, and again consider the coefficient of x^{m_0+l} in the resulting expression. Then $\sum_{n \in \mathbb{Z}_+} c^1_{m_0,n} \gamma_l^n = 0$ for all $l \in \mathbb{Z}_+$, hence $c^1_{m_0,n} = 0$ for all $n \in \mathbb{Z}$. This gives the desired contradiction. □

8.3. We can finish now the proof of theorem 2.22 using proposition 8.2. It suffices to show that the surjective algebra homomorphism $\phi : \mathcal{F}(\underline{t}; q) \to \mathcal{H}(S; \underline{t}; q)$ defined in theorem 2.22 is injective. Set $w := V_1^{-1} (V_1^\vee)^{-1} = q^{1/2} V_0 V_0^\vee \in \mathcal{F}(\underline{t}; q)$, then $\phi(w) = z$. Observe that $V_0^\vee = q^{1/2} w^{-1} V_0 + u_0 - u_0^{-1}$ and that $V_1^\vee = w^{-1} V_1^{-1}$ in $\mathcal{F}(\underline{t}; q)$, so that $\mathcal{F}(\underline{t}; q)$ is generated by $w^{\pm 1}, V_0$ and V_1 as an algebra. Furthermore,

$$V_0 w = q w^{-1} V_0 + (k_0 - k_0^{-1}) w + q^{1/2} (u_0 - u_0^{-1}), \qquad V_1 w = w^{-1} V_1^{-1} + u_1^{-1} - u_1$$

in $\mathcal{F}(\underline{t}; q)$, so that any element in $\mathcal{F}(\underline{t}; q)$ can be written as a finite linear combination of elements of the form $f(w) X$, where $f(w)$ is a Laurent polynomial in w and X is an element in the subalgebra of $\mathcal{F}(\underline{t}; q)$ generated by V_0 and V_1. By 2.15 and by the relations (1) in theorem 2.22 for the generators $V_0, V_1 \in \mathcal{F}(\underline{t}; q)$ it follows that $\mathcal{F}(\underline{t}; q)$ is spanned by $\{w^m Z^n, w^m V_1 Z^n \mid m, n \in \mathbb{Z}\}$, where $Z := V_1 V_0$. Since the image of these elements under ϕ are linear independent by proposition 8.2, we conclude that ϕ is injective.

8.4. In the remainder of the paper we use the notations $T_0^\vee := T_0^{-1} z^{-a_0^\vee} \in \mathcal{H}(S; \underline{t}, q)$ and $T_1^\vee := z^{-a_1^\vee} T_1^{-1} \in \mathcal{H}(S; \underline{t}; q)$ for the images of $V_0^\vee$ and $V_1^\vee$ respectively under the algebra isomorphism $\phi : \mathcal{F}(\underline{t}; q) \to \mathcal{H}(S; \underline{t}; q)$ (see theorem 2.22).

8.5. We associate with the multiplicity function $\underline{t} \simeq (k_0, k_1, u_0, u_1)$ a dual multiplicity function $\underline{\tilde{t}}$ by interchanging k_0 and u_1, so $\underline{\tilde{t}} \simeq (u_1, k_1, u_0, k_0)$. We write $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_0^\vee, \widetilde{T}_1^\vee$ for the generators of $\mathcal{H}(S; \underline{\tilde{t}}; q)$ (cf. 2.22 and 8.4), and we write $\widetilde{Y} = \widetilde{T}_1 \widetilde{T}_0$ for the associated Dunkl operator and $\widetilde{z} = \widetilde{T}_1^{-1} (\widetilde{T}_1^\vee)^{-1} = q^{1/2} \widetilde{T}_0 \widetilde{T}_0^\vee$ for the corresponding "multiplication by x" operator. The first part of the following proposition is a special case of Sahi's results in [19, section 7].

Proposition. **(i)** *The application* $T_0 \mapsto \widetilde{T}_1^\vee$, $T_1 \mapsto \widetilde{T}_1$, $T_0^\vee \mapsto \widetilde{T}_0^\vee$ *and* $T_1^\vee \mapsto \widetilde{T}_0$ *uniquely extend to an anti-algebra isomorphism* $\nu = \nu_{\underline{t},q} : \mathcal{H}(S;\underline{t};q) \to \mathcal{H}(S;\underline{\tilde{t}};q)$. *Furthermore,* $\nu_{\underline{t},q}^{-1} = \nu_{\underline{\tilde{t}},q}$ *and* $\nu(z) = \widetilde{Y}^{-1}$, $\nu(Y) = \widetilde{z}^{-1}$.

(ii) *The application* $T_0 \mapsto \widetilde{T}_1^{-1}\widetilde{T}_1^\vee\widetilde{T}_1$, $T_1 \mapsto \widetilde{T}_1$, $T_0^\vee \mapsto \widetilde{T}_0\widetilde{T}_0^\vee\widetilde{T}_0^{-1}$ *and* $T_1^\vee \mapsto \widetilde{T}_0$ *uniquely extend to an algebra isomorphism* $\mu = \mu_{\underline{t},q} : \mathcal{H}(S;\underline{t};q) \to \mathcal{H}(S;\underline{\tilde{t}};q)$. *Furthermore,* $\mu(Y) = \widetilde{z}^{-1}$.

Proof. By theorem 2.22 it suffices to check that μ (respectively ν) is compatible with the defining relations in $\mathcal{F}(\underline{t};q) \simeq \mathcal{H}(S;\underline{t};q)$. This can be done by direct computations. It is immediate that $\nu_{\underline{\tilde{t}},q}$ is the inverse of $\nu_{\underline{t},q}$.

Observe that the application $\widetilde{T}_0 \to T_1^\vee$, $\widetilde{T}_1 \mapsto T_1$, $\widetilde{T}_0^\vee \mapsto (T_1^\vee)^{-1}T_0^\vee T_1^\vee$ and $\widetilde{T}_1^\vee \mapsto T_1T_0T_1^{-1}$ uniquely extend to an algebra homomorphism from $\mathcal{H}(S;\underline{\tilde{t}};q)$ to $\mathcal{H}(S;\underline{t};q)$. It is immediate that this homomorphism is the inverse of μ. □

8.6. Following the terminology of Sahi [19, section 7], we call $\nu = \nu_{\underline{t},q}$ the *duality anti-isomorphism*. Furthermore, we call $\mu = \mu_{\underline{t},q}$ the *duality isomorphism*. These duality isomorphisms play a fundamental role in the theory of non-symmetric Askey-Wilson polynomials. In particular, the duality anti-isomorphism can be used to show that the geometric parameter x and the spectral parameter γ of the non-symmetric Askey-Wilson polynomial are in a sense interchangeable (see Sahi [19, section 7] or section 10). The duality isomorphism describes the intertwining properties of the action of the double affine Hecke algebra under the non-symmetric Askey-Wilson transform, see section 11.

8.7. We write T_i' and $T_i^{\vee\prime}$ $(i = 0, 1)$ for the generators of $\mathcal{H}(S;\underline{t}^{-1};q^{-1})$, cf. 2.22, 6.6 and 8.4.

Proposition. *There exists a unique anti-algebra isomorphism* $^* : \mathcal{H}(S;\underline{t};q) \to \mathcal{H}(S;\underline{t}^{-1};q^{-1})$ *such that* $T_i^* = (T_i')^{-1}$ *and* $(T_i^\vee)^* = (T_i^{\vee\prime})^{-1}$ *for* $i = 0, 1$. *Furthermore,* T^* *coincides with the adjoint of* $T \in \mathcal{H}(S;\underline{t};q)$ *if the parameters satisfy the additional conditions 6.1.*

Proof. The first statement follows easily from theorem 2.22. For the second statement, it suffices to compute the adjoint of $T_i^\vee$ $(i = 0, 1)$ in view of proposition 6.6. Let $z' = q^{-1/2}T_0'T_0^{\vee\prime}$ be the "multiplication by x" operator in $\mathcal{H}(S;\underline{t}^{-1};q^{-1})$. It is immediate that $z^* = (z')^{-1}$. Combined with proposition 6.6 we obtain $(T_0^\vee)^* = q^{-1/2}(z')^{-1}T_0' = (T_0^{\vee\prime})^{-1}$ and $(T_1^\vee)^* = T_1'z' = (T_1^{\vee\prime})^{-1}$. This gives the desired result. □

9. Intertwiners as creation operators

9.1. In corollary 2.17 we have introduced the non-affine intertwiner S_1 and derived its basic property. The results of the previous section allow us to derive the following analogous result for the commutator $[Y, T_1^\vee] \in \mathcal{H}(S;\underline{t};q)$, see Sahi [19, theorem 5.1] for the result in the higher rank setting.

Corollary (The affine intertwiner)**.** *Set* $S_0 := [Y, T_1^\vee] = YT_1^\vee - T_1^\vee Y \in \mathcal{H}(S;\underline{t};q)$. *Then* $g(Y)S_0 = S_0 g(q^{-1}Y^{-1})$ *for all* $g(Y) \in \mathbb{C}[Y^{\pm 1}]$.

Proof. By 2.21 we have $f(z)[T_0, z^{-1}] = [T_0, z^{-1}](s_0f)(z)$ in $\mathcal{H}(S;\underline{t};q)$ for any Laurent polynomial f. Apply now the duality anti-isomorphism $\nu_{\underline{t},q}$ to this equality, and replace the parameters by dual parameters in the resulting identity. This gives

$S_0 f(Y^{-1}) = (s_0 f)(Y^{-1})S_0$ for any Laurent polynomial f. The corollary is now immediate. □

9.2. Recall from corollary 4.3 that the action of S_1 on the non-symmetric Askey-Wilson polynomials is completely explicit. In the following lemma we give the analogous result for the action of the affine intertwiner S_0 on P_m ($m \in \mathbb{Z}_+$).

Lemma. *Let $m \in \mathbb{Z}_+$, then $S_0(P_m) = \big(\gamma_{-m-1} - \gamma_m\big)k_1^{-1}P_{-m-1}$.*

Proof. Let $m \in \mathbb{Z}_+$. By proposition 4.1 we have

$$S_0(P_m) = (Y - \gamma_m)\big((\alpha_m + k_1^{-1} - k_1)z^{-1}P_m + \beta_m z^{-1}P_{-m}\big).$$

It follows then from proposition 3.4 that the leading term of $S_0(P_m)$ with respect to the total order $\preceq$ on the monomials equals

$$(\gamma_{-m-1} - \gamma_m)\big((\alpha_m + k_1^{-1} - k_1)c_m + \beta_m\big)x^{-m-1},$$

where $c_0 := 1$ and where c_m ($m \in \mathbb{N}$) is the unique constant such that $P_m(x) = x^m + c_m x^{-m}+$ lower order terms w.r.t $\preceq$. By proposition 5.3**(i)** we have

$$c_m = 1 - \frac{(1 + k_0k_1^{-1}\gamma_m)(1 - k_0^{-1}k_1^{-1}\gamma_m)}{(1 - \gamma_m^2)} = k_1^{-1}\alpha_m.$$

Furthermore, recall from the proof of proposition 4.1 that $\beta_m = k_1^{-1}(k_1 - \alpha_m)(k_1^{-1} + \alpha_m)$. Hence the leading term of $S_0(P_m)$ reduces to $(\gamma_{m-1} - \gamma_m)k_1^{-1}x^{-m-1}$. On the other hand, corollary 9.1 implies that $S_0(P_m) = d_m P_{-m-1}$ for some constant d_m. By the leading term considerations, we conclude that $d_m = (\gamma_{m-1} - \gamma_m)k_1^{-1}$. □

9.3. The intertwiners S_0 and S_1 can be used to create the non-symmetric Askey-Wilson polynomial P_m ($m \in \mathbb{Z}$) from the unit polynomial $1 \in \mathcal{A}$ in the following way.

Proposition. *We have $(S_1S_0)^m(1) = d_m P_m$ for $m \in \mathbb{Z}_+$ and $\big(S_0(S_1S_0)^{m-1}\big)(1) = d_{-m}P_{-m}$ for $m \in \mathbb{N}$, with the constants d_m ($m \in \mathbb{Z}$) given by*

$$d_m = q^{-(m+1)m}k_0^{-2m}k_1^{-2m}\big(qk_0^2k_1^2;q\big)_{2m}, \qquad m \in \mathbb{Z}_+,$$
$$d_{-m} = q^{-m^2}k_0^{1-2m}k_1^{-2m}\big(qk_0^2k_1^2;q\big)_{2m-1}, \qquad m \in \mathbb{N}.$$

Proof. By corollary 2.17 and corollary 9.1 we have $g(Y)(S_1S_0) = (S_1S_0)g(qY)$ for all $g \in \mathcal{A}$. It follows that $F_m := (S_1S_0)^m(1) \in \mathcal{A}$ for $m \in \mathbb{Z}_+$ satisfies $g(Y)F_m = g(\gamma_m)F_m$ for all $m \in \mathbb{Z}$, so $F_m = d_mP_m$ for some constant d_m. Similarly, we obtain $F_{-m} := S_0(S_1S_0)^{m-1}(1) = d_{-m}P_{-m}$ for some constant d_{-m} when $m \in \mathbb{N}$. By corollary 4.3 and lemma 9.2, we have the recurrence relations

$$d_m = (\gamma_{-m} - \gamma_m)k_1d_{-m}, \qquad m \in \mathbb{N},$$
$$d_{-m-1} = (\gamma_{-m-1} - \gamma_m)k_1^{-1}d_m, \qquad m \in \mathbb{Z}_+.$$

Together with the initial condition $d_0 = 1$, we obtain the explicit expressions for d_m ($m \in \mathbb{Z}$) by complete induction with respect to m. □

10. Evaluation formula and duality

10.1. Let $\mathrm{Ev} = \mathrm{Ev}_{\underline{t},q} : \mathcal{H}(S;\underline{t};q) \to \mathbb{C}$ be the linear map defined by $\mathrm{Ev}(X) := \big(X(1)\big)(k_1^{-1}u_1^{-1})$, where $1 \in \mathcal{A}$ is the Laurent polynomial identically equal to one. Observe that Ev satisfies

$$\mathrm{Ev}\big(T_1^{\pm 1}X\big) = k_1^{\pm 1}\mathrm{Ev}(X), \qquad X \in \mathcal{H}(S;\underline{t};q),$$

since $(T_1 f)(k_1^{-1}u_1^{-1}) = k_1 f(k_1^{-1}u_1^{-1})$ for all $f \in \mathcal{A}$ by the explicit expression 2.19 for the difference-reflection operator T_1.

10.2. Observe that we can evaluate $\mathrm{Ev}(P_m(z)) = P_m(k_1^{-1}u_1^{-1})$ explicitly using proposition 5.10, since the two ${}_4\phi_3$'s in the right hand side of the formula for $P_m(x)$ are equal to one when $x = a^{-1} = k_1^{-1}u_1^{-1}$. We give here an alternative, inductive proof for the evaluation which only uses the Rodrigues type formula for the non-symmetric Askey-Wilson polynomials in terms of the intertwiners S_0 and S_1, see proposition 9.3. We abuse notation by writing $\mathrm{Ev}(f) = \mathrm{Ev}(f(z)) = f(k_1^{-1}u_1^{-1})$ for $f \in \mathcal{A}$.

Proposition. (i) *For $m \in \mathbb{Z}_+$, we have*

$$\begin{aligned} Ev\big(P_m\big) &= k_1^{-m}u_1^{-m}\frac{\big(-qk_1^2, q^{1/2}k_0k_1u_0u_1, -q^{1/2}k_0k_1u_0^{-1}u_1;q\big)_m}{\big(q^{m+1}k_0^2k_1^2;q\big)_m}\\ &= a^{-m}\frac{\big(qab, ac, ad;q\big)_m}{\big(q^m abcd;q\big)_m}. \end{aligned}$$

(ii) *For $m \in \mathbb{N}$, we have*

$$\begin{aligned} Ev\big(P_{-m}\big) &= \frac{k_1^{-m}u_1^{-m}}{1+k_1^{-2}}\frac{\big(-k_1^2, q^{1/2}k_0k_1u_0u_1, -q^{1/2}k_0k_1u_0^{-1}u_1;q\big)_m}{\big(q^m k_0^2k_1^2;q\big)_m}\\ &= \frac{a^{-m}}{(1-a^{-1}b^{-1})}\frac{\big(ab, ac, ad;q\big)_m}{\big(q^{m-1}abcd;q\big)_m}. \end{aligned}$$

Proof. We write $F_m = (S_1S_0)^m(1)$ for $m \in \mathbb{Z}_+$ and $F_{-m} = (S_0(S_1S_0)^{m-1})(1)$ for $m \in \mathbb{N}$, so that $F_m = d_mP_m$ for $m \in \mathbb{Z}$ with the specific constants d_m as given in proposition 9.3. The proposition follows then from the explicit evaluation of the d_m, see proposition 9.3, and from the recurrence relations

$$\mathrm{Ev}(F_m) = k_1^{-1}\gamma_{-m}(1-k_0k_1\gamma_m)(1+k_0^{-1}k_1\gamma_m)\mathrm{Ev}(F_{-m}), \qquad m \in \mathbb{N},$$

respectively

$$\mathrm{Ev}(F_{-m}) = u_1^{-1}\gamma_{-m}(1-u_0u_1q^{-1/2}\gamma_m)(1+u_0^{-1}u_1q^{-1/2}\gamma_m)\mathrm{Ev}(F_{m-1}), \quad m \in \mathbb{N},$$

by complete induction with respect to m. Let $m \in \mathbb{N}$. For the first recurrence relation, observe that by formula 10.1 and by $YF_{-m} = \gamma_{-m}F_{-m}$ we have

$$\mathrm{Ev}(F_m) = \mathrm{Ev}(S_1F_{-m}) = \mathrm{Ev}\big((k_1\gamma_{-m} - k_1T_0T_1)F_{-m}\big).$$

To reduce the T_0T_1-term, we use the relation

$$T_0T_1 = Y^{-1} + (k_0-k_0^{-1})T_1 + (k_1-k_1^{-1})T_1^{-1}Y - (k_0-k_0^{-1})(k_1-k_1^{-1})$$

in $\mathcal{H}$ and formula 10.1, which yields

$$\mathrm{Ev}(F_m) = k_1^{-1}\gamma_{-m}(1-k_0k_1\gamma_m)(1+k_0^{-1}k_1\gamma_m)\mathrm{Ev}(F_{-m})$$

after a direct computation. For the second recurrence relation, observe that

$$\mathrm{Ev}(F_{-m}) = \mathrm{Ev}(S_0F_{m-1}) = \mathrm{Ev}\big((Yz^{-1}T_1^{-1} - u_1\gamma_{m-1})F_{m-1}\big)$$

by formula 10.1, since $YF_{m-1} = \gamma_{m-1}F_{m-1}$. To reduce the $Yz^{-1}T_1^{-1}$-term, we use the relation

$$Yz^{-1}T_1^{-1} = q^{-1}z^{-1}T_1^{-1}Y^{-1} + q^{-1}(u_1^{-1} - u_1)Y^{-1} + q^{-1/2}(u_0^{-1} - u_0)$$

in $\mathcal{H}$ and formula 10.1, which gives the desired recursion

$$\mathrm{Ev}(F_{-m}) = u_1^{-1}\gamma_{-m}(1 - u_0u_1q^{-1/2}\gamma_m)(1 + u_0^{-1}u_1q^{-1/2}\gamma_m)\mathrm{Ev}(F_{m-1})$$

after a direct computation. This completes the proof of the proposition. □

10.3. The explicit expression for the (anti-)symmetric Askey-Wilson polynomial as linear combination of non-symmetric Askey-Wilson polynomials (see proposition 5.3) can be used to express $\mathrm{Ev}(P_m^{\pm})$ as a linear combination of $\mathrm{Ev}(P_m)$ and $\mathrm{Ev}(P_{-m})$. Combined with proposition 10.2, this leads to explicit evaluation formulas for the (anti-)symmetric Askey-Wilson polynomials $P_m^{\pm}$. In particular, it follows that

$$\mathrm{Ev}(P_m^+) = P_m^+(a) = a^{-m}\frac{\big(ab, ac, ad; q\big)_m}{\big(q^{m-1}abcd; q\big)_m}, \qquad m \in \mathbb{Z}_+.$$

This result can also be obtained directly from the explicit expression of P_m^+ in terms of a terminating, balanced ${}_4\phi_3$, see theorem 5.9.

10.4. The evaluation mapping Ev and the duality anti-isomorphism ν are compatible in the following sense.

Lemma. *For all $X \in \mathcal{H}(S;\underline{t};q)$ we have $Ev_{\underline{\tilde{t}},q}\big(\nu_{\underline{t},q}(X)\big) = Ev_{\underline{t},q}\big(X\big)$.*

Proof. For $X = f(z)g(Y)$ with f and g Laurent polynomials, we have

$$\mathrm{Ev}_{\underline{\tilde{t}},q}(\nu_{\underline{t},q}(X)) = \big(g(\tilde{z}^{-1})f(\widetilde{Y}^{-1})(1)\big)(k_1^{-1}k_0^{-1}) = f(k_1^{-1}u_1^{-1})g(k_1k_0) = \mathrm{Ev}_{\underline{t},q}(X),$$

and for $X = f(z)T_1g(Y)$ we have

$$\mathrm{Ev}_{\underline{\tilde{t}},q}(\nu_{\underline{t},q}(X)) = f(k_1^{-1}u_1^{-1})k_1g(k_1k_0) = \mathrm{Ev}_{\underline{t},q}(X).$$

Combined with proposition 8.2 we obtain the desired result. □

10.5. We associate with the evaluation mappings $\mathrm{Ev}_{\underline{t},q}$ and $\mathrm{Ev}_{\underline{\tilde{t}},q}$ two bilinear forms

$$B : \mathcal{H}(S;\underline{t};q) \times \mathcal{H}(S;\underline{\tilde{t}};q) \to \mathbb{C}, \qquad \widetilde{B} : \mathcal{H}(S;\underline{\tilde{t}};q) \times \mathcal{H}(S;\underline{t};q) \to \mathbb{C},$$

which are defined by $B(X,\widetilde{X}) = \mathrm{Ev}_{\underline{t},q}\big(\nu_{\underline{\tilde{t}},q}(\widetilde{X})X\big)$ and $\widetilde{B}\big(\widetilde{X},X\big) = \mathrm{Ev}_{\underline{\tilde{t}},q}\big(\nu_{\underline{t},q}(X)\widetilde{X}\big)$ for $X \in \mathcal{H}(S;\underline{t};q)$ and $\widetilde{X} \in \mathcal{H}(S;\underline{\tilde{t}};q)$.

Lemma. *Let $X, X_1, X_2 \in \mathcal{H}(S;\underline{t};q)$ and $\widetilde{X}, \widetilde{X}_1, \widetilde{X}_2 \in \mathcal{H}(S;\underline{\tilde{t}};q)$. Let $f \in \mathcal{A}$.*

- **(i)** *$B(X,\widetilde{X}) = \widetilde{B}(\widetilde{X},X)$.*
- **(ii)** *$B(X_1X_2,\widetilde{X}) = B\big(X_2,\nu_{\underline{t},q}(X_1)\widetilde{X}\big)$, and $B(X,\widetilde{X}_1\widetilde{X}_2) = B\big(\nu_{\underline{\tilde{t}},q}(\widetilde{X}_1)X,\widetilde{X}_2\big)$.*
- **(iii)** *$B\big((X(f))(z),\widetilde{X}\big) = B\big(X.f(z),\widetilde{X}\big)$ and $B\big(X,(\widetilde{X}(f))(\tilde{z})\big) = B\big(X,\widetilde{X}.f(\tilde{z})\big)$.*
- **(iv)** *$B\big(XT_i,\widetilde{X}\big) = k_iB(X,\widetilde{X})$ for $i = 0, 1$.*

Proof. **(i)** This follows from lemma 10.4 and the fact that $\nu_{\underline{t},q}$ is an anti-algebra homomorphism with inverse $\nu_{\underline{\tilde{t}},q}$, see proposition 8.5.

(ii) This is an immediate consequence of proposition 8.5.

(iii) The first equality is a direct consequence of the identity $(X(f))(z)(1) = X(f) = \big(X.f(z)\big)(1)$ in $\mathcal{A}$. The second identity follows from the first and from **(i)**.

(iv) By the explicit expressions 2.19 for the difference-reflection operators T_i, we have $T_i(1) = k_i 1$ for $i = 0$ and $i = 1$. The identities are now immediate. □

10.6. We write $x_m = k_1^{\epsilon(m)} u_1^{\epsilon(m)} q^m$ ($m \in \mathbb{Z}$) for the eigenvalues of the Cherednik-Dunkl operator $\widetilde{Y} \in \mathcal{H}(S; \underline{\tilde{t}}; q)$, see 3.5. We assume from now on that the parameters $(\underline{t}, q)$ are such that $P_m(x_0^{-1}; \underline{t}; q) = \mathrm{Ev}_{\underline{t},q}(P_m(\cdot; \underline{t}; q)) \neq 0$ and such that $P_m(\gamma_0^{-1}; \underline{\tilde{t}}; q) = \mathrm{Ev}_{\underline{\tilde{t}},q}(P_m(\cdot; \underline{\tilde{t}}; q)) \neq 0$ for all $m \in \mathbb{Z}$, and similarly for P_m^+ ($m \in \mathbb{Z}_+$). By proposition 10.2 and 10.3, the corresponding generic conditions on the parameters can be specified explicitly. We write $s(\gamma) := \gamma + \gamma^{-1}$ for all $\gamma \in \mathbb{C}^*$.

Definition. **(i)** *The renormalized non-symmetric Askey-Wilson polynomials are defined by*

$$E_{\gamma_m}(x; \underline{t}; q) := \frac{P_m(x; \underline{t}; q)}{P_m(x_0^{-1}; \underline{t}; q)}, \qquad m \in \mathbb{Z}.$$

In other words, the non-symmetric Askey-Wilson are normalized such that they take the value one at $x = x_0^{-1}$.

(ii) *The renormalized symmetric Askey-Wilson polynomials are defined by*

$$E^+_{s(\gamma_m)}\big(x; \underline{t}; q\big) := \frac{P_m^+(x; \underline{t}; q)}{P_m^+(x_0; \underline{t}; q)}, \qquad m \in \mathbb{Z}_+.$$

In other words, the symmetric Askey-Wilson polynomials are normalized such that they take the value one at $x = x_0^{\pm 1}$.

Observe that $C_+ E_{\gamma_m} = E^+_{s(\gamma_m)}$ for $m \in \mathbb{Z}$, where $C_+ = (1 + k_1^2)^{-1}(1 + k_1 T_1)$ is the idempotent defined in 5.1, since $(T_1 f)(k_1^{-1} u_1^{-1}) = k_1$ for all $f \in \mathcal{A}$.

10.7. In the following theorem we prove the duality between the geometric parameter $x = x_n$ and the spectral parameter $\gamma = \gamma_m$ for the renormalized (non-)symmetric Askey-Wilson polynomials, see Sahi [19, section 7] for the result in the higher rank setting.

Theorem (Duality)**.** **(i)** *For all $m, n \in \mathbb{Z}$ and $f \in \mathcal{A}$, we have*

$$f(\gamma_m^{-1}) = \widetilde{B}\big(f(\widetilde{z}), E_{\gamma_m}(z; \underline{t}; q)\big), \qquad f(x_n^{-1}) = B\big(f(z), E_{x_n}(\widetilde{z}; \underline{\tilde{t}}; q)\big).$$

In particular, $E_{\gamma_m}(x_n^{-1}; \underline{t}; q) = E_{x_n}(\gamma_m^{-1}; \underline{\tilde{t}}; q)$ for all $m, n \in \mathbb{Z}$.

(ii) *For all $m, n \in \mathbb{Z}$ and $f \in \mathcal{A}^W$, we have*

$$f(\gamma_m) = \widetilde{B}\big(f(\widetilde{z}), E^+_{s(\gamma_m)}\big(z; \underline{t}; q\big)\big), \qquad f(x_n) = B\big(f(z), E^+_{s(x_n)}\big(\widetilde{z}; \underline{\tilde{t}}; q\big)\big).$$

In particular, $E^+_{s(\gamma_m)}\big(x_n; \underline{t}; q\big) = E^+_{s(x_n)}\big(\gamma_m; \underline{\tilde{t}}; q\big)$ for all $m, n \in \mathbb{Z}$.

Proof. **(i)** The second statement follows from the first by taking $f = E_{x_n}(\cdot; \underline{\tilde{t}}; q)$ in the first equality and $f = E_{\gamma_m}(\cdot; \underline{t}; q)$ in the second equality and using lemma 10.5**(i)**.

For the first equality, observe that

$$\widetilde{B}\big(f(\widetilde{z}), E_{\gamma_m}(z)\big) = \widetilde{B}\big(1, f(Y^{-1})E_{\gamma_m}(z)\big) = \widetilde{B}\big(1, (f(Y^{-1})E_{\gamma_m})(z)\big)$$
$$= f(\gamma_m^{-1})\widetilde{B}(1, E_{\gamma_m}(z)\big) = f(\gamma_m^{-1})E_{\gamma_m}(x_0^{-1}) = f(\gamma_m^{-1})$$

by application of lemma 10.5**(i)**, **(ii)** and **(iii)**. The second equality is proved in a similar manner.

(ii) The proof is similar to the proof of **(i)**, taking account of the fact that $f(Y)E^+_{s(\gamma)} = f(\gamma)E^+_{s(\gamma)}$ for all $f(Y) \in \mathbb{C}[Y + Y^{-1}]$ by theorem 4.4 and proposition 5.3. □

10.8. We write $\sigma = \{\gamma_m\}_{m\in\mathbb{Z}}$ for the spectrum of the Dunkl operator $Y \in \mathcal{H}(S;\underline{t};q)$, see 3.5. We define an action of the affine Weyl group $\mathcal{W}$ on σ by $s_0(\gamma_m) = \gamma_{-m-1}$ and $s_1(\gamma_m) = \gamma_{-m}$ for all $m \in \mathbb{Z}$.

The duality between the geometric and spectral parameter of the renormalized non-symmetric Askey-Wilson polynomials can be used to explicitly compute $X(E_\gamma)$ for $X \in \mathcal{H}(S;\underline{t};q)$ as linear combination of non-symmetric Askey-Wilson polynomials, cf. 4.1 and 4.2 for $X = T_1$. Observe that $Y(E_\gamma) = \gamma E_\gamma$ and that $T_0^\vee = q^{-1/2}Y^{-1}(T_1^\vee)^{-1}$ by theorem 2.22 and 8.4. Hence it suffices to expand $T_1(E_\gamma)$ and $T_1^\vee(E_\gamma)$ as linear combination of renormalized non-symmetric Askey-Wilson polynomials. The result is as follows.

Proposition. *Let $\gamma \in \sigma$, then*

$$T_1(E_\gamma) = k_1E_\gamma + k_1^{-1}\frac{(1-k_0k_1\gamma^{-1})(1+k_0^{-1}k_1\gamma^{-1})}{(1-\gamma^{-2})}\big(E_{s_1\gamma} - E_\gamma\big),$$
$$T_1^\vee(E_\gamma) = u_1E_\gamma + u_1^{-1}\frac{(1-u_0u_1q^{1/2}\gamma)(1+u_0^{-1}u_1q^{1/2}\gamma)}{(1-q\gamma^2)}\big(E_{s_0\gamma} - E_\gamma\big).$$

Proof. The first formula is obviously correct for $\gamma = \gamma_0$. Let $\gamma_0 \neq \gamma \in \sigma$. By theorem 10.7 and lemma 10.5**(ii)** and **(iii)** we have

$$(T_1E_\gamma)(x_m^{-1}) = B\big(E_\gamma(z), \widetilde{T}_1.E_{x_m}(\widetilde{z};\underline{\widetilde{t}};q)\big).$$

By the commutation relation 2.21 between T_1 and $f(z)$ ($f \in \mathcal{A}$) and by the identity $B(X, \widetilde{X}\widetilde{T}_1) = k_1B(X, \widetilde{X})$ (see lemma 10.5**(i)** and **(iv)**), we obtain

$$(T_1E_\gamma)(x_m^{-1}) = k_1B\big(E_\gamma(z), E_{x_m}(\widetilde{z}^{-1})\big)$$
$$+ \frac{(k_1-k_1^{-1}) + (k_0-k_0^{-1})\gamma^{-1}}{\big(1-\gamma^{-2}\big)}\Big(B\big(E_\gamma(z), E_{x_m}(\widetilde{z})\big) - B\big(E_\gamma(z), E_{x_m}(\widetilde{z}^{-1})\big)\Big).$$

Here we have used lemma 10.5**(ii)**, as well as the short-hand notation $E_{x_m}(\widetilde{z}) = E_{x_m}(\widetilde{z};\underline{\widetilde{t}};q)$. By lemma 10.5**(ii)**, **(iii)** and theorem 10.7, we have

$$B\big(E_\gamma(z), E_{x_m}(\widetilde{z}^{-1})\big) = E_{x_m}(\gamma;\underline{\widetilde{t}};q) = E_{\gamma^{-1}}(x_m^{-1};\underline{t};q)$$

since $\gamma \neq \gamma_0$. By theorem 10.7 it follows that the formula for $(T_1E_\gamma)(x)$ as stated in the proposition is correct for $x = x_m^{-1}$ ($m \in \mathbb{Z}$), hence it is correct as identity in $\mathcal{A}$.

For the second formula we proceed in a similar manner. First of all, observe that $(T_1^\vee E_\gamma)(x_m^{-1}) = B\big(E_\gamma(z), \widetilde{T}_0.E_{x_m}(\widetilde{z};\underline{\widetilde{t}};q)\big)$. We use now the commutation relation

2.21 between T_0 and $f(z)$ ($f \in \mathcal{A}$) and the identity $B\big(X, \widetilde{X}\widetilde{T}_0\big) = u_1 B\big(X, \widetilde{X}\big)$, which follows from lemma 10.5**(i)** and **(iv)**. Then

$$
\begin{aligned}
(T_1^\vee E_\gamma)(x_m^{-1}) &= u_1 B\big(E_\gamma(z), E_{x_m}(q\widetilde{z}^{-1})\big)\\
&+ \frac{(u_1-u_1^{-1})+(u_0-u_0^{-1})q^{1/2}\gamma}{(1-q\gamma^2)}\Big(B\big(E_\gamma(z), E_{x_m}(\widetilde{z})\big) - B\big(E_\gamma(z), E_{x_m}(q\widetilde{z}^{-1})\big)\Big).
\end{aligned}
$$

By lemma 9.2**(ii)**, **(iii)** and theorem 10.7, we have

$$B\big(E_\gamma(z), E_{x_m}(q\widetilde{z}^{-1})\big) = E_{x_m}(q\gamma; \underline{\tilde{t}}; q) = E_{s_0\gamma}(x_m^{-1}; \underline{t}; q)$$

since $q\gamma_n = \gamma_{-n-1}^{-1}$ for all $n \in \mathbb{Z}$. It follows then by direct computations that the formula for $(T_1^\vee E_\gamma)(x)$ as stated in the proposition is correct for $x = x_m^{-1}$ ($m \in \mathbb{Z}$), hence it is correct as identity in $\mathcal{A}$. □

10.9. The duality for the (non-)symmetric Askey-Wilson polynomials (see theorem 10.7) can also be used to derive recurrence relations for the (non-)symmetric Askey-Wilson polynomials from the difference(-reflection) equations $LE^+_{s(\gamma)} = s(\gamma)E^+_{s(\gamma)}$ (respectively $YE_\gamma = \gamma E_\gamma$). The Askey-Wilson q-difference equation $LE_{s(\gamma)} = s(\gamma)E_\gamma$ then gives the three term recurrence relation [1, (1.24)–(1.27)] for the symmetric Askey-Wilson polynomials (see van Diejen [7, section 4] for the argument in the higher rank setting).

11. The non-symmetric Askey-Wilson transform and its inverse

11.1. We assume in this section that the parameters $(\underline{t}, q)$ and $(\underline{\tilde{t}}, q)$ satisfy the additional conditions 6.1.

Let σ' be the spectrum of Y', so $\sigma' = \{\gamma'_m\}_{m\in\mathbb{Z}}$ with $\gamma'_m = \gamma_m^{-1}$ for all $m \in \mathbb{Z}$, see 3.5. Let $F = F_{k_0,k_1,q}$ be the functions $g : \sigma' \to \mathbb{C}$ with finite support. By the non-degeneracy of the bilinear form $\langle ..,.\rangle_{\underline{t},q}$ on $\mathcal{A}$ and by the bi-orthogonality relations 6.7 for the non-symmetric Askey-Wilson polynomials, we have a bijective linear map $\mathcal{F} = \mathcal{F}_{\underline{t},q} : \mathcal{A} \to F$ defined by

$$\big(\mathcal{F}_{\underline{t},q}(f)\big)(\gamma) := \langle f, E'_\gamma(\,\cdot\,)\rangle_{\underline{t},q}, \qquad f \in \mathcal{A},\ \ \gamma \in \sigma',$$

where $E'_\gamma(\,\cdot\,) = E_\gamma(\,\cdot\,; \underline{t}^{-1}; q^{-1})$ ($\gamma \in \sigma'$) are the renormalized non-symmetric Askey-Wilson polynomials with respect to inverse parameters.

Definition. *The bijective map $\mathcal{F} : \mathcal{A} \to F$ is called the non-symmetric Askey-Wilson transform.*

11.2. Recall the action of $\mathcal{W}$ on σ' defined by $s_0\gamma'_m = \gamma'_{-m-1}$ and $s_1\gamma'_m = \gamma'_{-m}$ for all $m \in \mathbb{Z}$. This induces a left action of $\mathcal{W}$ on F by $(w\,g)(\gamma) = g(w^{-1}\gamma)$ for $w \in \mathcal{W}$, $g \in F$ and $\gamma \in \sigma'$. Let $\widetilde{T}_i$ ($i = 0, 1$) and $\widetilde{z}$ be the linear endomorphisms of F defined by $(\widetilde{z}g)(\gamma) = \gamma g(\gamma)$,

$$
\begin{aligned}
\big(\widetilde{T}_0 g\big)(\gamma) &= u_1 g(\gamma) + u_1^{-1}\frac{(1-u_0u_1q^{1/2}\gamma^{-1})(1+u_0^{-1}u_1q^{1/2}\gamma^{-1})}{(1-q\gamma^{-2})}\big((s_0g)(\gamma) - g(\gamma)\big),\\
\big(\widetilde{T}_1 g\big)(\gamma) &= k_1 g(\gamma) + k_1^{-1}\frac{(1-k_0k_1\gamma)(1+k_0^{-1}k_1\gamma)}{(1-\gamma^2)}\big((s_1g)(\gamma) - g(\gamma)\big)
\end{aligned}
$$

for all $g \in F$ and all $\gamma \in \sigma'$. Observe that these formulas can be obtained from the standard action of the generators $\widetilde{T}_i$ ($i = 0, 1$) and $\widetilde{z}$ of the double affine Hecke algebra $\mathcal{H}(S;\tilde{\underline{t}};q)$ on $\mathcal{A}$ (see 2.19 and 2.22) by formally replacing the $\mathcal{W}$-module $\mathcal{A}$ by the $\mathcal{W}$-module F.

Proposition. *There is a unique action of $\mathcal{H}(S;\tilde{\underline{t}};q)$ on F such that the generators $\widetilde{z}$ and $\widetilde{T}_i$ $(i=0,1)$ of $\mathcal{H}(S;\tilde{\underline{t}};q)$ act as the linear endomorphisms defined above. Furthermore,*

$$\mathcal{F}(Xf)=\mu(X)\mathcal{F}(f), \qquad X\in\mathcal{H}(S;\underline{t};q),\;\; f\in\mathcal{A},$$

where μ is the duality isomorphism defined in proposition 8.5.

Proof. By proposition 8.7 we have

$$\mathcal{F}(Yf)(\gamma)=\langle f,(Y')^{-1}E'_\gamma\rangle=\gamma^{-1}(\mathcal{F}f)(\gamma)=\big(\widetilde{z}^{-1}\mathcal{F}(f)\big)(\gamma)=\big(\mu(Y)\mathcal{F}(f)\big)(\gamma)$$

for all $f\in\mathcal{A}$ and all $\gamma\in\sigma'$.

Again by proposition 8.7, we have $\mathcal{F}(T_1f)(\gamma)=\langle f,(T_1')^{-1}E'_\gamma\rangle$. Combined with proposition 10.8, we derive that $\mathcal{F}(T_1f)=\widetilde{T}_1\mathcal{F}(f)=\mu(T_1)\mathcal{F}(f)$ for all $f\in\mathcal{A}$.

In a similar manner, we derive from proposition 8.7 and proposition 10.8 that $\mathcal{F}(T_1^\vee f)=\widetilde{T}_0\mathcal{F}(f)=\mu(T_1^\vee)\mathcal{F}(f)$ for all $f\in\mathcal{A}$. The proposition now follows since $\mathcal{F}$ is bijective and $\mathcal{H}(S;\underline{t};q)$ is generated as an algebra by Y, T_1 and $T_1^\vee$. □

11.3. Observe that the weight function $\Delta(\gamma;\tilde{\underline{t}};q)$ (see 6.2) has simple poles at $\gamma\in\sigma'$. We define now a linear map $\mathcal{G}=\mathcal{G}_{\underline{t},q}:F\to\mathcal{A}$ by

$$\mathcal{G}_{\underline{t},q}(g)(x)=\sum_{\gamma\in\sigma'}g(\gamma)E_{\gamma^{-1}}(x;\underline{t};q)w(\gamma;\tilde{\underline{t}};q), \qquad g\in F,$$

where $w(\gamma)=w(\gamma;\tilde{\underline{t}};q)$ is defined by

$$w(\gamma;\tilde{\underline{t}};q)=\operatorname*{Res}_{y=\gamma}\left(\frac{\Delta(y;\tilde{\underline{t}};q)}{y}\right)\operatorname{sgn}(\gamma), \qquad \gamma\in\sigma'$$

and $\operatorname{sgn}(\gamma'_m)=\epsilon(m)$ for $m\in\mathbb{Z}$ (see 3.3 for the definition of ϵ). Observe that $w(\gamma;\tilde{\underline{t}};q)=\alpha(\gamma;k_1,k_0)w_+(\gamma;\tilde{\underline{t}};q)$ for all $\gamma\in\sigma'$, where $w_+(\gamma)=w_+(\gamma;\tilde{\underline{t}};q)$ is defined by

$$w_+(\gamma;\tilde{\underline{t}};q)=\operatorname*{Res}_{y=\gamma}\left(\frac{\Delta_+(y;\tilde{\underline{t}};q)}{y}\right)\operatorname{sgn}(\gamma), \qquad \gamma\in\sigma',$$

see 6.2. The weight functions $w(\gamma)$ and $w_+(\gamma)$ can be written out explicitly in terms of q-shifted factorials, see [1, section 2] or [8, section 7.5] for $w_+(\gamma)$.

11.4. In the following proposition we determine the intertwining properties of the $\mathcal{H}(S;\tilde{\underline{t}};q)$-action on F under the linear map $\mathcal{G}:F\to\mathcal{A}$.

Proposition. *We have*

$$\mathcal{G}(Xg)=\mu^{-1}(X)\,\mathcal{G}(g), \qquad X\in\mathcal{H}(S;\tilde{\underline{t}};q),\;\; g\in F,$$

where μ is the duality isomorphism defined in proposition 8.5.

Proof. We write $\widetilde{T}_0=u_1+\widetilde{\phi}_0(\,\cdot\,)(s_0-1)$ and $\widetilde{T}_1=k_1+\widetilde{\phi}_1(\,\cdot\,)(s_1-1)$ with

$$\widetilde{\phi}_0(\gamma)=u_1^{-1}\frac{(1-u_0u_1q^{1/2}\gamma^{-1})(1+u_0^{-1}u_1q^{1/2}\gamma^{-1})}{(1-q\gamma^{-2})},$$

$$\widetilde{\phi}_1(\gamma)=k_1^{-1}\frac{(1-k_0k_1\gamma)(1+k_0^{-1}k_1\gamma)}{(1-\gamma^2)}.$$

The weight function $w(\gamma;\underline{\tilde{t}};q)$ $(\gamma\in\sigma')$ satisfies the fundamental relations

$$\widetilde{\phi}_i(\gamma)w(\gamma;\underline{\tilde{t}};q)=\widetilde{\phi}_i(s_i\gamma)w(s_i\gamma;\underline{\tilde{t}};q),\qquad \gamma\in\sigma',\;\; i=0,1.$$

This follows easily from the explicit expression for the weight function Δ, see 6.2 (compare also with the proof of proposition 6.6). It follows that

$$\begin{aligned}\mathcal{G}(\widetilde{T}_0g)(x)&=\sum_{\gamma\in\sigma'}(\widetilde{T}_0g)(\gamma)E_{\gamma^{-1}}(x)\,w(\gamma)\\&=\sum_{\gamma\in\sigma}g(\gamma^{-1})\big(u_1E_\gamma(x)+\widetilde{\phi}_0(\gamma^{-1})(E_{s_0\gamma}(x)-E_\gamma(x))\big)\,w(\gamma^{-1})\\&=\sum_{\gamma\in\sigma}g(\gamma^{-1})\big(T_1^\vee E_\gamma\big)(x)\,w(\gamma^{-1})=\big(T_1^\vee\mathcal{G}(g)\big)(x)\end{aligned}$$

for all $g\in F$ by proposition 10.8. Similarly, we obtain $\mathcal{G}(\widetilde{T}_1g)(x)=\big(T_1\mathcal{G}(g)\big)(x)$ for all $g\in F$ by proposition 10.8. Furthermore, it is immediate that $\mathcal{G}(\widetilde{z}g)(x)=\big(Y^{-1}(\mathcal{G}(g)\big)(x)$ for all $g\in F$. We conclude that $\mathcal{G}(Xg)=\mu^{-1}(X)\mathcal{G}(g)$ for all $g\in F$ if $X=\widetilde{z}$ or $X=\widetilde{T}_i$ for $i=0,1$. The proposition follows, since these elements generate $\mathcal{H}(S;\underline{\tilde{t}};q)$ as an algebra. □

11.5. Combining proposition 11.2 and proposition 11.4 leads to the following main result of this section.

Theorem. **(i)** *We have* $\mathcal{G}_{\underline{t},q}\circ\mathcal{F}_{\underline{t},q}=c_{\underline{t},q}Id_{\mathcal{A}}$ *and* $\mathcal{F}_{\underline{t},q}\circ\mathcal{G}_{\underline{t},q}=c_{\underline{t},q}Id_F$ *with the constant* $c_{\underline{t},q}$ *given by* $c_{\underline{t},q}=w(\gamma_0^{-1};\underline{\tilde{t}};q)\,\langle 1,1\rangle_{\underline{t},q}$.

(ii) *For all* $\gamma\in\sigma$ *we have*

$$\frac{\langle E_\gamma,E'_{\gamma^{-1}}\rangle_{\underline{t},q}}{\langle 1,1\rangle_{\underline{t},q}}=\frac{w(\gamma_0^{-1};\underline{\tilde{t}};q)}{w(\gamma^{-1};\underline{\tilde{t}};q)}.$$

Proof. **(i)** Let $f\in\mathcal{A}$, then

$$\mathcal{G}\big(\mathcal{F}f\big)=\mathcal{G}\big(\mathcal{F}(f(z)1)\big)=f(z)\big(\mathcal{G}(\mathcal{F}(1))\big)$$

by proposition 11.2 and proposition 11.4, where $1\in\mathcal{A}$ is the function identically equal to one. By the definitions of $\mathcal{F}$ and $\mathcal{G}$ and by the bi-orthogonality relations for the non-symmetric Askey-Wilson polynomials (see proposition 6.7) we have $\mathcal{G}_{\underline{t},q}(\mathcal{F}_{\underline{t},q}(1))=c_{\underline{t},q}1$ with the constant $c_{\underline{t},q}$ as given in the statement of the theorem. Hence $\mathcal{G}\circ\mathcal{F}=c\,\mathrm{Id}_{\mathcal{A}}$. The identity $\mathcal{F}\circ\mathcal{G}=c\,\mathrm{Id}_F$ follows then immediately from the fact that $\mathcal{F}:\mathcal{A}\to F$ is a bijection.

(ii) Let $\gamma\in\sigma$. By **(i)**, we have $\mathcal{G}(\mathcal{F}(E_\gamma))=c\,E_\gamma$. On the other hand, by the explicit definitions of $\mathcal{F}$ and $\mathcal{G}$ and by the bi-orthogonality relations for the non-symmetric Askey-Wilson polynomials (see proposition 6.7), we have

$$\mathcal{G}_{\underline{t},q}(\mathcal{F}_{\underline{t},q}(E_\gamma))=w(\gamma^{-1};\underline{\tilde{t}};q)\langle E_\gamma,E'_{\gamma^{-1}}\rangle_{\underline{t},q}\,E_\gamma.$$

Comparing coefficients of E_γ leads to the desired result. □

11.6. Let $F^W\subset F$ be the W-invariant functions in F, i.e. the functions $f\in F$ satisfying $f=s_1f$. Equivalently, F^W consists of the functions $f\in F$ satisfying $\widetilde{C}_+f=f$, where $\widetilde{C}_+=(1+k_1^2)^{-1}(1+k_1\widetilde{T}_1)$.

Let $\mathcal{F}_+$ be the restriction of the non-symmetric Askey-Wilson transform $\mathcal{F}$ to $\mathcal{A}^W\subset\mathcal{A}$ and let $\mathcal{G}_+$ be the restriction of $\mathcal{G}$ to $F^W\subset F$.

Proposition. *$\mathcal{F}_+$ is a linear bijection from $\mathcal{A}^W$ to F^W with inverse $c^{-1}\mathcal{G}_+$, where $c = c_{\underline{t},q}$ is the constant defined in theorem 11.5. Furthermore, the constant c can be rewritten as $c = \frac{1}{2}(1+k_1^2)^2\, w_+(\gamma_0^{-1}; \underline{\tilde{t}}; q)\,\big(1,1\big)_{\underline{t},q}$, and*

$$\mathcal{F}_+(f)(\gamma) = \frac{1}{2}(1+k_1^2)\,\big(f, E^+_{s(\gamma)}(\,\cdot\,; \underline{t}; q)\big)_{\underline{t},q},$$
$$\mathcal{G}_+(g)(x) = (1+k_1^2)\sum_{m\in\mathbb{Z}_+} g(\gamma'_m)E^+_{s(\gamma_m)}(x; \underline{t}; q)\, w_+(\gamma'_m; \underline{\tilde{t}}; q)$$

for all $f \in \mathcal{A}^W$ and all $g \in F^W$.

Proof. By lemma 6.4 we have $\langle 1,1\rangle_{\underline{t},q} = \frac{1}{2}(1+k_1^2)\big(1,1\big)_{\underline{t},q}$. Furthermore, we have $w(\gamma_0^{-1}; \underline{\tilde{t}}; q) = (1+k_1^2)w_+(\gamma_0^{-1}; \underline{\tilde{t}}; q)$ since $\alpha(\gamma_0^{-1}; k_1, k_0) = 1 + k_1^2$. This gives the alternative formula for the constant $c_{\underline{t},q}$.

Observe that $E^+_{s(\gamma)}(x; \underline{t}^{-1}; q^{-1}) = E^+_{s(\gamma)}(x; \underline{t}; q)$ by proposition 6.9 and by the W-invariance of $E^+_{s(\gamma)}$. By lemma 6.4, 10.6 and the fact that $C_+^* = C'_+$ (see the proof of proposition 6.8), we then derive for $f \in \mathcal{A}^W$ and $\gamma \in \sigma'$ that

$$\begin{aligned}\mathcal{F}_+(f)(\gamma) &= \langle f, E'_\gamma\rangle_{\underline{t},q} = \langle C_+f, E'_\gamma\rangle_{\underline{t},q} = \langle f, C'_+E'_\gamma\rangle_{\underline{t},q}\\ &= \langle f, E^+_{s(\gamma)}\rangle_{\underline{t},q} = \frac{1}{2}(1+k_1^2)\big(f, E^+_{s(\gamma)}\big)_{\underline{t},q}.\end{aligned}$$

In particular, we have $\mathcal{F}(\mathcal{A}^W) \subset F^W$. For $0 \neq m \in \mathbb{Z}$ we have

$$w(\gamma'_m; \underline{\tilde{t}}; q) + w(\gamma'_{-m}; \underline{\tilde{t}}; q) = (1+k_1^2)w_+(\gamma'_m; \underline{\tilde{t}}; q).$$

Indeed, we use here that $w_+(\gamma; \underline{\tilde{t}}; q) = w_+(\gamma^{-1}; \underline{\tilde{t}}; q)$ by the W-invariance of the weight function $\Delta_+(\,\cdot\,; \underline{\tilde{t}}; q)$, and that $\alpha(\gamma; k_1, k_0) + \alpha(\gamma^{-1}; k_1, k_0) = 1 + k_1^2$, see 6.4. Hence we obtain for $g \in F^W$,

$$\begin{aligned}\mathcal{G}_+(g)(x) = \mathcal{G}(\widetilde{C}_+g)(x) = \big(C_+\mathcal{G}(g)\big)(x) &= \sum_{\gamma\in\sigma'} g(\gamma)E^+_{s(\gamma)}(x; \underline{t}; q)\, w(\gamma; \underline{\tilde{t}}; q)\\ &= (1+k_1^2)\sum_{m\in\mathbb{Z}_+} g(\gamma'_m)E^+_{s(\gamma_m)}(x; \underline{t}; q)\, w_+(\gamma'_m; \underline{\tilde{t}}; q).\end{aligned}$$

In particular, $\mathcal{G}(F^W) \subset \mathcal{A}^W$. Combined with proposition 11.5, this completes the proof of the proposition. □

Definition. *The bijection $\mathcal{F}_+ : \mathcal{A}^W \to F^W$ is called the symmetric Askey-Wilson transform.*

11.7. We can repeat now the proof of theorem 11.5**(ii)** for the symmetric Askey-Wilson transform $\mathcal{F}_+$, using the alternative descriptions for $\mathcal{F}_+$ and $\mathcal{G}_+$ as given in proposition 11.6. This gives the following result on the quadratic norms of the symmetric Askey-Wilson polynomials.

Corollary. *For all $\gamma \in \sigma$ we have*

$$\frac{\big(E^+_{s(\gamma)}, E^+_{s(\gamma)}\big)_{\underline{t},q}}{\big(1,1\big)_{\underline{t},q}} = \frac{w_+(\gamma_0^{-1}; \underline{\tilde{t}}; q)}{w_+(\gamma^{-1}; \underline{\tilde{t}}; q)}.$$

12. The fundamental shift operator and the constant term

12.1. In theorem 11.5 and corollary 11.7 we have obtained explicit expressions of $\langle E_\gamma, E'_{\gamma^{-1}}\rangle_{\underline{t};q}$ and of $\big(E^+_{s(\gamma)}, E^+_{s(\gamma)}\big)_{\underline{t},q}$ in terms of the constant term $\langle 1,1\rangle_{\underline{t},q} = \frac{1}{2}(1+k_1^2)\big(1,1\big)_{\underline{t},q}$ for all $\gamma\in\sigma$. The constant term $\big(1,1\big)_{\underline{t},q}$ is the well-known Askey-Wilson integral, which has been evaluated in many different ways, see for instance [1], [9], [12] and [18]. We give in this section yet another proof for the evaluation of $\big(1,1\big)$ using shift operators.

12.2. In the following lemma we define explicit linear maps from symmetric Laurent polynomials to anti-symmetric Laurent polynomials and conversely in terms of the Cherednik-Dunkl operator Y. Recall the definition of the idempotents $C_\pm\in H_0\subset H = H(R;k_0,k_1)$, see 5.1.

Lemma. *Let $h_\pm(Y) = h_\pm(Y;k_0,k_1)\in H(R;k_0,k_1)$ be defined by*

$$h_\pm(Y) = Y^{\mp 1}(Y^{\pm 1}-k_0k_1)(Y^{\pm 1}+k_0^{-1}k_1).$$

(i) *We have $h_+(Y)C_+ = C_-h_+(Y)C_+$ and $h_-(Y)C_- = C_+h_-(Y)C_-$ in H, i.e. $h_\pm(Y)\mathcal{A}_\pm\subseteq\mathcal{A}_\mp$ under the action of the fundamental representation $\pi_{\underline{t},q}$.*

(ii) *We have $C_\pm h_\pm(Y)C_\mp = -h_\mp(Y)C_\mp$ in H.*

Proof. This follows by a straightforward computation from Lusztig's formula 2.16, together with the fact that $(T_1\mp k_1^{\pm 1})C_\pm = 0$ in $H_0\subset H$. □

12.3. By lemma 12.2 and lemma 7.1 we have well-defined linear endomorphisms $G_\pm(\underline{t};q):\mathcal{A}^W\to\mathcal{A}^W$ defined by

$$\big(G_+f\big)(x) = \delta(x)^{-1}\big(h_+(Y)f\big)(x),\qquad \big(G_-f\big) = \big(h_-(Y)(\delta.f)\big)(x),\qquad f\in\mathcal{A}^W,$$

where $h_\pm(Y)$ act under the fundamental representation $\pi_{\underline{t},q}$. Observe that G_- can be realized as the element $h_-(Y)\delta(z)$ in $\mathcal{H}(S;\underline{t};q)$.

Proposition. *For $m\in\mathbb{N}$, we have*

$$\begin{aligned} G_+(\underline{t};q)P_m^+(\cdot;\underline{t};q) &= h_+(\gamma_m;k_0,k_1)P^+_{m-1}(x;k_0,qk_1,u_0,u_1;q),\\ G_-(\underline{t};q)P^+_{m-1}(\cdot;k_0,qk_1,u_0,u_1;q) &= h_-(\gamma_m;k_0,k_1)P_m^+(x;\underline{t};q).\end{aligned}$$

Proof. Let $m\in\mathbb{N}$. Since $\mathcal{A}(m) = \operatorname{span}\{P_m^+,P_m^-\}$ is an H-module with $\mathcal{A}_-(m) = \mathcal{A}(m)\cap\mathcal{A}_- = \operatorname{span}\{P_m^-\}$, we have $h_+(Y;k_0,k_1)P_m^+(\cdot;\underline{t};q) = c_+(m)P_m^-(\cdot;\underline{t};q)$ for some constant $c_+(m)$, see theorem 4.4, proposition 5.3 and lemma 12.2. Comparing leading terms using proposition 3.4, we see that $c_+(m) = h_+(\gamma_m)$. By the generalized Weyl character formula, see proposition 7.3, we obtain

$$G_+(\underline{t};q)P_m^+(\cdot;\underline{t};q) = h_+(\gamma_m)P^+_{m-1}(\cdot;k_0,qk_1,u_0,u_1;q).$$

The shift property of G_- is proved in a similar manner. □

12.4. In the remainder of this section we assume that the parameters satisfy the additional conditions 6.1. We write

$$\big(G_+(\underline{t}^{-1};q^{-1})f\big)(x) = \delta'(x)^{-1}\big(h_+(Y';k_0^{-1},k_1^{-1})f\big)(x),\qquad f\in\mathcal{A}$$

and $G_-(\underline{t}^{-1};q^{-1}) = h_-(Y';k_0^{-1},k_1^{-1})\delta'(z')\in\mathcal{H}(S;\underline{t}^{-1};q^{-1})$ for the shift-operators with respect to inverse parameters, where $\delta'(x) = \delta(x;k_0^{-1},k_1^{-1})$ (see 7.2). The two shift operators G_+ and G_- are each-others adjoint in the following sense.

Proposition. *For all $f,g\in\mathcal{A}^W$, we have*

$$\begin{aligned}\big(G_-(\underline{t};q)f,g\big)_{\underline{t},q}&=\big(f,G_+(\underline{t}^{-1};q^{-1})g\big)_{k_0,qk_1,u_0,u_1,q},\\ \big(G_+(\underline{t};q)f,g\big)_{k_0,qk_1,u_0,u_1,q}&=k_1^4\big(f,G_-(\underline{t}^{-1};q^{-1})g\big)_{\underline{t},q}.\end{aligned}$$

Proof. Let $f,g\in\mathcal{A}^W$. By lemma 6.4, proposition 6.6 and lemma 7.1 we have

$$\begin{aligned}\frac{1}{2}(1+k_1^2)\big(G_-(\underline{t};q)f,g\big)_{\underline{t},q}&=\langle G_-(\underline{t};q)f,g\rangle_{\underline{t},q}\\ &=\langle\delta(z)f,h_-((Y')^{-1})g\rangle_{\underline{t},q}=\langle\delta(z)f,C'_-h_-((Y')^{-1})C'_+g\rangle_{\underline{t},q},\end{aligned}$$

where $C'_\pm\in\mathcal{H}(S;\underline{t}^{-1};q^{-1})$ are the images of the primitive idempotents of $H_0(k_1^{-1})$ under $\pi_{\underline{t}^{-1},q^{-1}}$, see 5.1 (compare with the proof of proposition 6.8). Now observe that $h_-((Y')^{-1};k_0,k_1)=-k_1^2h_-(Y';k_0^{-1},k_1^{-1})$, hence

$$C'_-h_-((Y')^{-1})C'_+=-k_1^2C'_-h_-(Y';k_0^{-1},k_1^{-1})C'_+=k_1^2h_+(Y';k_0^{-1},k_1^{-1})C'_+$$

by lemma 12.2. Consequently, we obtain

$$\begin{aligned}\frac{1}{2}(1+k_1^2)\big(G_-(\underline{t};q)f,g\big)_{\underline{t},q}&=k_1^2\langle\delta(z)f,h_+(Y';k_0^{-1},k_1^{-1})g\rangle_{\underline{t},q}\\ &=k_1^2\langle\delta(z)f,\delta'(z')G_+(\underline{t}^{-1};q^{-1})g\rangle_{\underline{t},q}\\ &=\frac{1}{2}(1+k_1^2)\big(f,G_+(\underline{t}^{-1};q^{-1})g\big)_{k_0,qk_1,u_0,u_1,q}\end{aligned}$$

where the last equality follows from lemma 7.2. The second formula is proved in a similar manner. □

12.5. We write $\nu(f)=\nu_{a,b,c,d}(f)=\big(f,f\big)_{\underline{t},q}$ for the "quadratic norm" of f with respect to the bilinear form $\big(.,.\big)_{\underline{t},q}$, where (a,b,c,d) is the reparametrized multiplicity function, see 5.7. We use the short-hand notation

$$\nu_{a,b,c,d}(P_m^+)=\nu_{a,b,c,d}(P_m^+(\cdot;a,b,c,d)),\qquad m\in\mathbb{Z}_+.$$

Corollary. *For $m\in\mathbb{N}$, we have*

$$\nu_{a,b,c,d}(P_m^+)=\frac{(1-q^m)(1-q^{m-1}cd)}{(1-q^mab)(1-q^{m-1}abcd)}\nu_{qa,qb,c,d}(P_{m-1}^+).$$

Proof. This is an immediate consequence of proposition 12.3, proposition 12.4 and proposition 6.9. □

12.6. Observe that the Askey-Wilson second order q-difference operator L (see 5.8) and its eigenvalues $\gamma_m+\gamma_m^{-1}$ ($m\in\mathbb{Z}_+$) are symmetric in a,b,c,d. It follows that the symmetric Askey-Wilson polynomials $P_m^+(x;a,b,c,d)$ ($m\in\mathbb{Z}_+$) are symmetric in the four parameters a,b,c,d. Hence corollary 12.5 can be reformulated with the special role of (a,b) replaced by an arbitrary pair of the four parameters a, b, c, d. This leads to the following result.

Corollary. *Let* $k,l,m,n\in\mathbb{Z}_+$ *and set* $t=k+l+m+n$. *Then*

$$\frac{\nu_{a,b,c,d}(P_t^+)}{\nu_{q^{2k}a,q^{2l}b,q^{2m}c,q^{2n}d}(1)}=\frac{\big(q,q^{2t-1}abcd,q^{2k+2l}ab,q^{2k+2m}ac,q^{2k+2n}ad,q^{2l+2m}bc,q^{2l+2n}bd,q^{2m+2n}cd;q\big)_\infty}{\big(q^{t+1},q^{t-1}abcd,q^tab,q^tac,q^tad,q^tbc,q^tbd,q^tcd;q\big)_\infty}.$$

Proof. We write

$$\frac{\nu_{a,b,c,d}(P_k^+)}{\nu_{q^2a,b,c,d}(P_{k-1}^+)}=\frac{\nu_{a,b,c,d,q}(P_k^+)}{\nu_{qa,qb,c,d}(P_{k-1}^+)}\frac{\nu_{qa,qb,c,d}(P_{k-1}^+)}{\nu_{qa,b,q^{-1}c,d}(P_k^+)}\frac{\nu_{qa,b,q^{-1}c,d}(P_k^+)}{\nu_{q^2a,b,c,d}(P_{k-1}^+)}$$

for $k\in\mathbb{N}$ and use the symmetry in the parameters a,b,c,d and corollary 12.5 to obtain

$$\frac{\nu_{a,b,c,d}(P_k^+)}{\nu_{q^2a,b,c,d}(P_{k-1}^+)}=\frac{(1-q^k)(1-q^{k-1}bc)(1-q^{k-1}bd)(1-q^{k-1}cd)}{(1-q^{k-1}abcd)(1-q^kab)(1-q^kac)(1-q^kad)}$$

for $k\in\mathbb{N}$. Now use again the symmetry in the parameters a,b,c,d and complete induction with respect to k,l,m and n to obtain the desired result. □

12.7. Corollary 12.6 relates the quadratic norm $\nu(P_m^+)$ to the constant term $\nu(1)$, but it can also be used to evaluate the Askey-Wilson integral $\nu(1)$ itself. The Askey-Wilson integral was evaluated for the first time by Askey and Wilson [1, theorem 2.1] (see also e.g. [9], [12] and [18] for alternative proofs).

Theorem (Constant term evaluation)**.** *We have*

$$\nu_{a,b,c,d}(1)=\frac{2\big(abcd;q\big)_\infty}{\big(q,ab,ac,ad,bc,bd,cd;q\big)_\infty}.$$

Proof. Let $(a,b,c,d)=(1,-1,q^{\frac12},-q^{\frac12})$, then we have $\Delta_+(x)\equiv1$ for the corresponding weight function of the bilinear form $\big(.,.\big)$. Hence $P_k^+(x)=x^k+x^{-k}$ ($k\in\mathbb{N}$) for the corresponding symmetric Askey-Wilson polynomial. Furthermore,

$$\nu_{1,-1,q^{\frac12},-q^{\frac12}}(P_k)=2,\quad k\in\mathbb{N}.$$

Combined with corollary 12.6 this implies that the theorem is correct for all parameter values $(a,b,c,d)=\big(q^{2k},-q^{2l},q^{\frac12+2m},-q^{\frac12+2n}\big)$ with $k,l,m,n\in\mathbb{Z}_+$ and $k+l+m+n\in\mathbb{N}$ (we use here that the formula in corollary 12.6 extends to this particular choice of parameter values by continuity). The proof is now completed by analytic continuation. □

12.8. The constant term evaluation (theorem 12.7) and corollary 12.5 yield an explicit evaluation of $\nu_{a,b,c,d}(P_m^+)$ for $m\in\mathbb{Z}_+$ which is in accordance with Askey and Wilson's result [1, theorem 2.2]. As remarked in 7.5, this then yields explicit evaluations for all the diagonal terms in the bi-orthogonality relations of proposition 6.7 and proposition 6.8.

Another way to obtain the diagonal terms explicitly is by using corollary 11.7 and 10.3 (respectively theorem 11.5 and proposition 10.2) to reduce the diagonal terms for the (non-)symmetric Askey-Wilson polynomials to the constant term evaluation (theorem 12.7). Explicitly, we obtain the following formulas for the diagonal terms:

$$\big(P_m^+,P_m^+\big)=\frac{2\big(q^{2m-1}abcd,q^{2m}abcd;q\big)_\infty}{\big(q^{m+1},q^mab,q^mac,q^mad,q^mbc,q^mbd,q^mcd,q^{m-1}abcd;q\big)_\infty}$$

for $m \in \mathbb{Z}_+$,

$$\langle P_m, P_m' \rangle = \frac{\big(q^{2m}abcd, q^{2m}abcd; q\big)_\infty}{\big(q^{m+1}, q^{m+1}ab, q^m ac, q^m ad, q^m bc, q^m bd, q^m cd, q^m abcd; q\big)_\infty}$$

for $m \in \mathbb{Z}_+$,

$$\langle P_{-m}, P_{-m}' \rangle = \frac{\big(q^{2m-1}abcd, q^{2m-1}abcd; q\big)_\infty}{\big(q^m, q^m ab, q^m ac, q^m ad, q^m bc, q^m bd, q^{m-1}cd, q^{m-1}abcd; q\big)_\infty}$$

for $m \in \mathbb{N}$ and finally

$$\langle P_m^-, P_m^{-\prime} \rangle = \frac{ab-1}{ab} \frac{\big(q^{2m-1}abcd, q^{2m}abcd; q\big)_\infty}{\big(q^m, q^{m+1}ab, q^m ac, q^m ad, q^m bc, q^m bd, q^{m-1}cd, q^m abcd; q\big)_\infty}$$

for $m \in \mathbb{N}$.

12.9. There is yet another way to relate the diagonal terms of the non-symmetric Askey-Wilson polynomials to the constant term $\langle 1, 1\rangle$. This method is based on the Rodrigues type formula for the non-symmetric Askey-Wilson polynomial (see proposition 9.3), which allows us to compute the diagonal terms by induction with respect to the degree of the non-symmetric Askey-Wilson polynomial. For the induction step, one needs the following two additional properties of the intertwiners. The first property is that $S_0^* = q^{-1}S_0'$ and $S_1^* = S_1'$, where $S_0', S_1' \in \mathcal{H}(S; \underline{t}^{-1}; q^{-1})$ are the intertwiners with respect to inverse parameters, cf. proposition 8.7. The second property is

$$\begin{aligned} S_0^2 &= q^{-1}u_1^2 \prod_{\xi=\pm 1} (1 - u_0^{-1}u_1^{-1}q^{\xi/2}Y^\xi)(1 + u_0 u_1^{-1} q^{\xi/2} Y^\xi), \\ S_1^2 &= k_1^2 \prod_{\xi=\pm 1} (1 - k_0^{-1}k_1^{-1}Y^\xi)(1 + k_0 k_1^{-1} Y^\xi) \end{aligned}$$

which are most easily proved in the image of the duality isomorphism μ, see Sahi [19, corollary 5.2] in the higher rank setting. We leave the details to the reader.

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