On the quasi-Hopf structure of deformed double Yangians

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Abstract

We construct universal twists connecting the centrally extended double Yangian $\mathcal{D}Y(sl(2))_c$ with deformed double Yangians $\mathcal{D}Y_r(sl(2))_c$, thereby establishing the quasi-Hopf structures of the latter.

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1 Introduction

Universal twists connecting (affine) quantum groups to (elliptic) (dynamical) (affine) algebras have been constructed in [1, 2, 3]. They show in particular the quasi-Hopf structure of elliptic and dynamical algebras. These twists transform the universal R-matrix $\mathcal{R}$ of the first object into the universal R-matrix $\mathcal{R}^{\mathcal{F}}$ of the second one as follows:

$$\mathcal{R}_{12}^{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}_{12}\mathcal{F}_{12}^{-1} \ . \tag{1.1}$$

The double degeneracy limits of elliptic R-matrices, whether vertex-type [4, 5, 6] or face-type [7] give rise to algebraic structures which have been variously characterised as scaled elliptic algebras [5, 6], or double Yangian algebras [7, 8, 9]. As pointed out earlier [4, 6][1], although represented by formally identical Yang–Baxter relations $RLL = LLR$ [10], these two classes of objects differ fundamentally in their structures (as is reflected in the very different mode expansions of L defining their individual generators) and must be considered separately.

In our previous paper [7] we have defined, in the quantum inverse scattering or RLL formulation, various algebraic structures of double Yangian type connected by twist-like operators, i.e. such that their evaluated R-matrices were related as:

$$R_{12}^{F} = F_{21}R_{12}F_{12}^{-1} \tag{1.2}$$

for a particular matrix F. It was conjectured that these twist-like operators were indeed evaluation representations of universal twists obeying a shifted cocycle condition thereby raising the relation (1.2) to the status of a genuine twist connection (1.1) between quasi-Hopf algebras.

We shall be concerned here only with algebraic structures related to the algebra $\widehat{sl(2)}_c$, and henceforth dispense with indicating it explicitly: for instance $\mathcal{D}Y$ is thus to be understood as $\mathcal{D}Y(\widehat{sl(2)}_c)$.

It is our purpose here to establish such connections, at the level of universal R-matrices, between the double Yangian structures respectively known as $\mathcal{D}Y$, $\mathcal{D}Y_r^{V6}$, $\mathcal{D}Y_r^{V8}$ and $\mathcal{D}Y_r^{F}$. $\mathcal{D}Y$ is the double Yangian defined in [9, 11]. $\mathcal{D}Y_r^{V6}$ is characterised by a scaled "elliptic" R-matrix defined in [4], $\mathcal{D}Y_r^{V8}$ is characterised by a scaled R-matrix defined in [6, 5]. In connection with our previous caveat, note that these R-matrices are also used to describe respectively the scaled elliptic algebras $\mathcal{A}_{\hbar,0}$, $\mathcal{A}_{\hbar,\eta}$ [5, 6, 4]. $\mathcal{D}Y_r^{F}$ is the deformed double Yangian obtained by a particular limit of the dynamical R-matrix characterising elliptic $\mathcal{B}_{q,p,\lambda}$ algebra [7].

A crucial ingredient for our procedure is a linear (difference) equation obeyed by the twist. This type of equation for twist operators was first written in [12]. It is also present in [2, 3]. Our method consists in *i)* finding a twist-like action in representation *ii)* interpreting this representation as an infinite product *iii)* defining a linear equation obeyed by this infinite product *iv)* promoting this linear equation for the representation to the level of linear equation for universal twist. *v)* The solution of this linear equation is obtained as a infinite product as in [2] which *vi)* is then proved to obey the shifted cocycle condition as in [2, 3] and *vii)* has an evaluation representation identical to the twist-like action found in *i)*.

[1]We wish to thank S. Pakuliak for clarifying this point to us.

This provides us with the universal R-matrix and quasi-Hopf structure of the twisted algebras $\mathcal{D}Y_r^{V6,V8,F}$, thereby realising a fully consistent description of these algebraic structures.

The universal R-matrix and Hopf algebra structure for $\mathcal{D}Y$ were described in [9, 11]. We construct a universal twist between $\mathcal{D}Y$ and $\mathcal{D}Y_r^{V6}$. We then show the existence of a universal coboundary (trivial) twist, the evaluation of which realises the connection between the evaluated R-matrices of $\mathcal{D}Y_r^{V6}$ and $\mathcal{D}Y_r^{V8}$, leading to identification of these two as quasi-Hopf algebras. Finally another universal coboundary-like twist realises, when evaluated, the connection between the R-matrices of $\mathcal{D}Y_r^{V6}$ and $\mathcal{D}Y_r^{F}$.

It follows that the three deformed structures are in fact one single quasi-Hopf algebra described by three different choices of generators, more precisely given in three different gauges.

We shall denote throughout this paper $\mathcal{F}[\mathcal{A}\,;\mathcal{B}]$ the universal or represented twist connecting R-matrices as $\mathcal{R}_\mathcal{B} = \mathcal{F}_{21}[\mathcal{A}\,;\mathcal{B}]\;\mathcal{R}_\mathcal{A}\;\mathcal{F}_{12}^{-1}[\mathcal{A}\,;\mathcal{B}]$.

2 Presentation of the double Yangians $\mathcal{D}Y$ and $\mathcal{D}Y_r$

2.1 Double Yangian $\mathcal{D}Y$

The double Yangian $\mathcal{D}Y$ is defined by the R-matrix

$$R(\beta) = \rho(\beta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{i\beta}{i\beta+\pi} & \dfrac{\pi}{i\beta+\pi} & 0 \\ 0 & \dfrac{\pi}{i\beta+\pi} & \dfrac{i\beta}{i\beta+\pi} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{2.1}$$

with the normalisation factor

$$\rho(\beta) = \frac{\Gamma_1(\frac{i\beta}{\pi} \mid 2)\;\Gamma_1(2+\frac{i\beta}{\pi} \mid 2)}{\Gamma_1(1+\frac{i\beta}{\pi} \mid 2)^2} , \tag{2.2}$$

together with the relations

$$R_{12}(\beta_1-\beta_2)\,L_1^\pm(\beta_1)\,L_2^\pm(\beta_2) \;=\; L_2^\pm(\beta_2)\,L_1^\pm(\beta_1)\,R_{12}(\beta_1-\beta_2) \,. \tag{2.3}$$

$$R_{12}(\beta_1-\beta_2-i\pi c)\,L_1^-(\beta_1)\,L_2^+(\beta_2) \;=\; L_2^+(\beta_2)\,L_1^-(\beta_1)\,R_{12}(\beta_1-\beta_2) \,. \tag{2.4}$$

and the mode expansions

$$L^+(\beta) = \sum_{k\ge 0} L_k^+ \beta^{-k} \qquad \text{and} \qquad L^-(\beta) = \sum_{k\le 0} L_k^- \beta^{-k} \,. \tag{2.5}$$

It is important to point out that L^+ and L^- are independent. There exists in this case a Gauss decomposition of the Lax matrices allowing for an alternative Drinfeld presentation [11].
Indeed, $L^\pm$ are decomposed as

$$L^\pm(x) = \begin{pmatrix} 1 & f^\pm(x^\mp) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_1^\pm(x) & 0 \\ 0 & k_2^\pm(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^\pm(x) & 1 \end{pmatrix} \tag{2.6}$$

with $x^+ \equiv x \equiv \frac{i\beta}{\pi}$ and $x^- \equiv x - c$. Furthermore, $k_1^\pm(x) k_2^\pm(x-1) = 1$ and one defines $h^\pm(x) \equiv k_2^\pm(x)^{-1} k_1^\pm(x)$.

The evaluation representation π_x is then easily defined by its action on a two-dimensional vector space by

$$\pi_x(e_k) = x^k \sigma^+ \,, \qquad \pi_x(f_k) = x^k \sigma^- \,, \qquad \pi_x(h_k) = x^k \sigma^3 \,, \tag{2.7}$$

where

$$e^\pm(u) \equiv \pm \sum_{\substack{k\geq 0 \\ k<0}} e_k u^{-k-1} \,, \qquad f^\pm(u) \equiv \pm \sum_{\substack{k\geq 0 \\ k<0}} f_k u^{-k-1} \,, \qquad h^\pm(u) \equiv 1 \pm \sum_{\substack{k\geq 0 \\ k<0}} h_k u^{-k-1} \,. \tag{2.8}$$

2.2 Deformed double Yangian $\mathcal{D}Y_r^{V6}$

The R-matrix of the deformed double Yangian $\mathcal{D}Y_r^{V6}$ is related to the two-body S matrix of the sine–Gordon theory $S_{SG}(\beta, r)$ and is given by

$$R_{V6}(\beta, r) = \mathrm{cotg}(\frac{i\beta}{2}) S_{SG}(\beta, r) = \rho_r(\beta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sin\frac{i\beta}{r}}{\sin\frac{\pi+i\beta}{r}} & \dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+i\beta}{r}} & 0 \\ 0 & \dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+i\beta}{r}} & \dfrac{\sin\frac{i\beta}{r}}{\sin\frac{\pi+i\beta}{r}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,, \tag{2.9}$$

where the normalisation factor is

$$\rho_r(\beta) = \frac{S_2^2(1 + \frac{i\beta}{\pi} \mid r, 2)}{S_2(\frac{i\beta}{\pi} \mid r, 2)\, S_2(2 + \frac{i\beta}{\pi} \mid r, 2)} \,. \tag{2.10}$$

$S_2(x|\omega_1, \omega_2)$ is Barnes' double sine function of periods ω_1 and ω_2 defined by:

$$S_2(x|\omega_1, \omega_2) = \frac{\Gamma_2(\omega_1 + \omega_2 - x \mid \omega_1, \omega_2)}{\Gamma_2(x \mid \omega_1, \omega_2)} \,, \tag{2.11}$$

where Γ_r is the multiple Gamma function of order r given by

$$\Gamma_r(x|\omega_1, \ldots, \omega_r) = \exp\left(\frac{\partial}{\partial s} \sum_{n_1,\ldots,n_r \geq 0} (x + n_1\omega_1 + \cdots + n_r\omega_r)^{-s} \Big|_{s=0} \right) \,. \tag{2.12}$$

The algebra $\mathcal{D}Y_r^{V6}$ is defined by

$$R_{12}(\beta_1 - \beta_2)\, L_1(\beta_1)\, L_2(\beta_2) = L_2(\beta_2)\, L_1(\beta_1)\, R^*_{12}(\beta_1 - \beta_2) \,, \tag{2.13}$$

where $R^*_{12}(\beta,r) \equiv R_{12}(\beta,r-c)$.
The Lax matrix L must now be expanded on both positive *and* negative powers as

$$L(\beta) = \sum_{k\in\mathbb{Z}} L_k \beta^{-k} \, . \tag{2.14}$$

A presentation similar to the double Yangian case is achieved by introducing the following two matrices:

$$L^+(\beta) \equiv L(\beta - i\pi c)\,, \tag{2.15}$$

$$L^-(\beta) \equiv \sigma_3\, L(\beta - i\pi r)\, \sigma_3 \, . \tag{2.16}$$

They obey coupled exchange relations following from (2.13) and periodicity/unitarity properties of the matrices R_{12} and R^*_{12}:

$$R_{12}(\beta_1-\beta_2)\, L^\pm_1(\beta_1)\, L^\pm_2(\beta_2) = L^\pm_2(\beta_2)\, L^\pm_1(\beta_1)\, R^*_{12}(\beta_1-\beta_2)\,, \tag{2.17}$$

$$R_{12}(\beta_1-\beta_2-\, i\pi c)\, L^+_1(\beta_1)\, L^-_2(\beta_2) = L^-_2(\beta_2)\, L^+_1(\beta_1)\, R^*_{12}(\beta_1-\beta_2)\, . \tag{2.18}$$

Contrary to the case of the double Yangian, the matrices L^+ and L^- are *not* independent. Note also that, due to conflicting conventions, the $r\to\infty$ limit of $L^\pm$ in $\mathcal{D}Y^{V6}_r$ corresponds to $L^\mp$ in $\mathcal{D}Y$.

2.3 Deformed double Yangian $\mathcal{D}Y^{V8}_r$

The R-matrix of the deformed double Yangian $\mathcal{D}Y^{V8}_r$ is obtained as the scaling limit of the R-matrix of the elliptic algebra $\mathcal{A}_{q,p}$ [4, 5]. It reads

$$R_{V8}(\beta,r) = \rho_r(\beta) \begin{pmatrix} \dfrac{\cos\frac{i\beta}{2r}\,\cos\frac{\pi}{2r}}{\cos\frac{\pi+i\beta}{2r}} & 0 & 0 & -\dfrac{\sin\frac{i\beta}{2r}\,\sin\frac{\pi}{2r}}{\cos\frac{\pi+i\beta}{2r}} \\ 0 & \dfrac{\sin\frac{i\beta}{2r}\,\cos\frac{\pi}{2r}}{\sin\frac{\pi+i\beta}{2r}} & \dfrac{\cos\frac{i\beta}{2r}\,\sin\frac{\pi}{2r}}{\sin\frac{\pi+i\beta}{2r}} & 0 \\ 0 & \dfrac{\cos\frac{i\beta}{2r}\,\sin\frac{\pi}{2r}}{\sin\frac{\pi+i\beta}{2r}} & \dfrac{\sin\frac{i\beta}{2r}\,\cos\frac{\pi}{2r}}{\sin\frac{\pi+i\beta}{2r}} & 0 \\ -\dfrac{\sin\frac{i\beta}{2r}\,\sin\frac{\pi}{2r}}{\cos\frac{\pi+i\beta}{2r}} & 0 & 0 & \dfrac{\cos\frac{i\beta}{2r}\,\cos\frac{\pi}{2r}}{\cos\frac{\pi+i\beta}{2r}} \end{pmatrix} . \tag{2.19}$$

It is also obtained from the R-matrix of $\mathcal{D}Y^{V6}_r$ by a gauge transformation [4]. The algebra $\mathcal{D}Y^{V8}_r$ is defined by the same relation as $\mathcal{D}Y^{V6}_r$, albeit with the matrix R_{V8}, and the same type of Lax matrix with positive and negative modes.

2.4 Deformed double Yangian $\mathcal{D}Y_r^F$

The R-matrix of $\mathcal{D}Y_r^F$ is given by

$$R(\beta;r)=\rho_r(\beta)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \dfrac{\sin\frac{i\beta}{r}}{\sin\frac{\pi+i\beta}{r}} & e^{\beta/r}\dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+i\beta}{r}} & 0\\ 0 & e^{-\beta/r}\dfrac{\sin\frac{\pi}{r}}{\sin\frac{\pi+i\beta}{r}} & \dfrac{\sin\frac{i\beta}{r}}{\sin\frac{\pi+i\beta}{r}} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{2.20}$$

The normalisation factor is the same as for $\mathcal{D}Y_r^{V6}$. The definition of the algebra and the Lax operator are again formally identical.

3 Twist from $\mathcal{D}Y$ to $\mathcal{D}Y_r$: representation formula

3.1 A notation for P_{12} invariant matrices

Let us denote by $M(b^+,b^-)$ the 4×4 matrix given by

$$M(b^+,b^-)\equiv\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & \frac{1}{2}(b^++b^-) & \frac{1}{2}(b^+-b^-) & 0\\ 0 & \frac{1}{2}(b^+-b^-) & \frac{1}{2}(b^++b^-) & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{3.1}$$

With this definition, we have $M(a,b)M(a',b')=M(aa',bb')$ and $M(a,b)^{-1}=M(a^{-1},b^{-1})$.

Now,

$$R[\mathcal{D}Y](\beta)=\rho(\beta)M\left(1,\frac{i\beta-\pi}{i\beta+\pi}\right). \tag{3.2}$$

We have $R[\mathcal{D}Y_r^{V6}](\beta)=\rho_r(\beta)M(b_r^+,b_r^-)$, with

$$b_r^+ = \frac{\cos\frac{i\beta-\pi}{2r}}{\cos\frac{i\beta+\pi}{2r}}=\frac{\Gamma_1(r+\frac{i\beta}{\pi}+1|2r)\Gamma_1(r-\frac{i\beta}{\pi}-1|2r)}{\Gamma_1(r+\frac{i\beta}{\pi}-1|2r)\Gamma_1(r-\frac{i\beta}{\pi}+1|2r)}, \tag{3.3}$$

$$b_r^- = \frac{\sin\frac{i\beta-\pi}{2r}}{\sin\frac{i\beta+\pi}{2r}}=\frac{\Gamma_1(\frac{i\beta}{\pi}+1|2r)\Gamma_1(2r-\frac{i\beta}{\pi}-1|2r)}{\Gamma_1(\frac{i\beta}{\pi}-1|2r)\Gamma_1(2r-\frac{i\beta}{\pi}+1|2r)} \tag{3.4}$$

$$= \frac{\Gamma_1(2r+\frac{i\beta}{\pi}+1|2r)\Gamma_1(2r-\frac{i\beta}{\pi}-1|2r)}{\Gamma_1(2r+\frac{i\beta}{\pi}-1|2r)\Gamma_1(2r-\frac{i\beta}{\pi}+1|2r)}\cdot\frac{i\beta-\pi}{i\beta+\pi}. \tag{3.5}$$

3.2 The linear equation in representation

We remark that the normalisation factor of $\mathcal{D}Y_r^{V6}$ can be rewritten as:

$$\rho_r(\beta) = \rho_F(-\beta;r)\rho(\beta)\rho_F(\beta;r)^{-1} \tag{3.6}$$

with

$$\rho_F(\beta) = \frac{\Gamma_2(\frac{i\beta}{\pi}+1+r \mid 2,r)^2}{\Gamma_2(\frac{i\beta}{\pi}+r \mid 2,r)\Gamma_2(\frac{i\beta}{\pi}+2+r \mid 2,r)} \,. \tag{3.7}$$

Equations (3.2-3.6) allow us to write:

$$R[\mathcal{D}Y_r^{V6}] = F_{21}(-\beta)R[\mathcal{D}Y]F_{12}(\beta)^{-1} \,. \tag{3.8}$$

Using the notation (3.1), $F_{12}(\beta)$ is given by

$$F_{12}(\beta) = \rho_F(\beta)\cdot M\left(\frac{\Gamma_1(\frac{i\beta}{\pi}+r-1|2r)}{\Gamma_1(\frac{i\beta}{\pi}+r+1|2r)},\frac{\Gamma_1(\frac{i\beta}{\pi}+2r-1|2r)}{\Gamma_1(\frac{i\beta}{\pi}+2r+1|2r)}\right) \,. \tag{3.9}$$

This twist-like matrix reads

$$F_{12}(\beta) = \rho_F(\beta)\prod_{n=1}^{\infty} M\left(1,\frac{i\beta+\pi+2n\pi r}{i\beta-\pi+2n\pi r}\right) M\left(\frac{i\beta+\pi+(2n-1)\pi r}{i\beta-\pi+(2n-1)\pi r},1\right) \tag{3.10}$$

$$= \prod_{n=1}^{\infty} R(\beta-i(2n)\pi r)^{-1}\ \tau(R(\beta-i(2n-1)\pi r)^{-1}) \tag{3.11}$$

with

$$\tau(M(a,b)) = M(b,a) \tag{3.12}$$

and where, unless differently specified, R is the R-matrix of $\mathcal{D}Y$. One uses here the representation of $\rho_F(\beta)$ as an infinite product

$$\rho_F(\beta) = \prod_{n=1}^{\infty}\rho(\beta-in\pi r)^{-1} \,. \tag{3.13}$$

The automorphism τ may be interpreted as the adjoint action of $(-1)^{\frac{1}{2}h_0^{(1)}}$, so that

$$F_{12}(\beta) = \prod_{n=1}^{\infty} R(\beta-i(2n)\pi r)^{-1}\,\mathrm{Ad}\left((-1)^{\frac{1}{2}h_0^{(1)}}\right) R(\beta-i(2n-1)\pi r)^{-1} \tag{3.14}$$

$$= \prod_{n=1}^{\infty}\mathrm{Ad}\left((-1)^{\frac{n}{2}h_0^{(1)}}\right) R(\beta-in\pi r)^{-1} \,. \tag{3.15}$$

Hence F is solution of the difference equation

$$F(\beta-i\pi r) = (-1)^{-\frac{1}{2}h_0^{(1)}}F(\beta)(-1)^{\frac{1}{2}h_0^{(1)}}\cdot R(\beta-i\pi r) \,. \tag{3.16}$$

It would be tempting to relate the automorphism τ to the one used in [3], although the naive scaling of the latter does not give back the former. For instance our τ is inner not outer.
All the infinite products are logarithmically divergent. They are consistently regularised by the Γ_1 and Γ_2 functions. In particular, $\lim_{r\to\infty} F = M(1,1) = \mathbb{1}_4$.

4 The universal form of $\mathcal{F}[\mathcal{D}Y\,;\mathcal{D}Y_r^{V6}]$

We construct a universal twist $\mathcal{F}$ from $\mathcal{D}Y$ to $\mathcal{D}Y_r^{V6}$, such that

$$F(\beta_1-\beta_2)=\pi_{\beta_1}\otimes\pi_{\beta_2}(\mathcal{F})\ . \tag{4.1}$$

The form of the difference equation (3.16) obeyed by the conjectural representation of the twist, together with the known generic structures of linear equations obeyed by universal twists [12, 2, 3] lead us to postulate the following linear equation for $\mathcal{F}$:

$$\mathcal{F}\equiv\mathcal{F}(r)=\mathrm{Ad}(\phi^{-1}\otimes\mathbb{1})(\mathcal{F})\cdot\mathcal{C} \tag{4.2}$$

with

$$\phi \;=\; (-1)^{\frac{1}{2}h_0}e^{(r+c)d}\ , \tag{4.3}$$
$$\mathcal{C} \;\equiv\; e^{-\alpha c\otimes d-\gamma d\otimes c}\mathcal{R}\ . \tag{4.4}$$

We now prove the consistency of these postulates. We will use the following preliminary properties:

- The operator d in the double Yangian $\mathcal{D}Y$ is defined by $[d,e(u)]=\frac{d}{du}e(u)$ (see [11]). The evaluation representations are related through $\pi_{\beta+\beta'}=\pi_\beta\circ\mathrm{Ad}(\exp(\frac{i\beta'}{\pi}d))$.
- The operator d satisfies $\Delta(d)=d\otimes 1+1\otimes d$.
- The generator h_0 of $\mathcal{D}Y$ is such that

$$h_0e(u)=e(u)(h_0+2)\ ,\qquad h_0f(u)=f(u)(h_0-2)\ ,\qquad [h_0,h(u)]=0\ , \tag{4.5}$$

  and hence $\tau=Ad\left((-1)^{\frac{1}{2}h_0^{(1)}}\right)$ satisfies $\tau^2=1$.

The equation (4.2) can be solved by

$$\mathcal{F}(r)=\prod_k^{\longleftarrow}\mathcal{F}_k(r)\ ,\qquad \mathcal{F}_k(r)=\phi_1^k\mathcal{C}_{12}^{-1}\phi_1^{-k}\ . \tag{4.6}$$

It is easily seen that equation (3.15) is the evaluation representation of this universal formula.
As in [3], $\mathcal{F}_k$ satisfy the following properties:

$$(\Delta\otimes\mathrm{id})(\mathcal{F}_k(r)) \;=\; \mathcal{F}_k^{(23)}(r+c_1)\mathcal{F}_k^{(13)}\left(r+c_2+\frac{\alpha}{k}c_2\right)\ , \tag{4.7}$$
$$(\mathrm{id}\otimes\Delta)(\mathcal{F}_k(r)) \;=\; \mathcal{F}_k^{(12)}(r)\mathcal{F}_k^{(13)}\left(r-\frac{\gamma}{k}c_2\right)\ , \tag{4.8}$$

and

$$\mathcal{F}_k^{(12)}(r)\mathcal{F}_{k+l}^{(13)}\left(r+\frac{l-\gamma}{k+l}c_2\right)\mathcal{F}_l^{(23)}(r+c_1)=\mathcal{F}_l^{(23)}(r+c_1)\mathcal{F}_{k+l}^{(13)}\left(r+\frac{l+\alpha}{k+l}c_2\right)\mathcal{F}_k^{(12)}(r)\ . \tag{4.9}$$

It is then straightforward to follow [3] to prove the shifted cocycle relation, provided that $\alpha+\gamma=-1$. We then have

$$\mathcal{F}^{(12)}(r)(\Delta\otimes \mathrm{id})(\mathcal{F}(r)) = \mathcal{F}^{(23)}\left(r+c^{(1)}\right)(\mathrm{id}\otimes\Delta)(\mathcal{F}(r)) \,. \tag{4.10}$$

It follows that $\mathcal{R}_{\mathcal{D}Y_r^{V6}}{}_{12} = \mathcal{F}_{21}\mathcal{R}_{12}\mathcal{F}_{12}^{-1}$ satisfies a shifted Yang–Baxter equation

$$\mathcal{R}_{12}(r+c^{(3)})\,\mathcal{R}_{13}(r)\,\mathcal{R}_{23}(r+c^{(1)}) = \mathcal{R}_{23}(r)\,\mathcal{R}_{13}(r+c^{(2)})\,\mathcal{R}_{12}(r)\,, \tag{4.11}$$

and that $\mathcal{D}Y_r^{V6}$ is a quasi-Hopf algebra with $\Delta^{\mathcal{F}}(x)=\mathcal{F}\Delta(x)\mathcal{F}^{-1}$ and $\Phi_{123}=\mathcal{F}_{23}(r)\mathcal{F}_{23}(r+c^{(1)})^{-1}$.

5 Twist to $\mathcal{D}Y_r^{V8}$

5.1 In representation

The R-matrix of $\mathcal{D}Y_r^{V6}$ and $\mathcal{D}Y_r^{V8}$ are related by

$$R[\mathcal{D}Y_r^{V8}] = K_{21}\; R[\mathcal{D}Y_r^{V6}]\; K_{12}^{-1}\,, \tag{5.1}$$

where

$$K = V\otimes V \qquad \text{with} \qquad V=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}\,. \tag{5.2}$$

This implies an isomorphism between $\mathcal{D}Y_r^{V8}$ and $\mathcal{D}Y_r^{V6}$ where the Lax operators are connected by $L_{V8}=VL_{V6}V^{-1}$.

5.2 Universal form

We identify V with an evaluation representation of an element g

$$V\equiv \pi_x(g) \qquad \text{with} \qquad g=\exp\left(\frac{\pi}{2}(f_0-e_0)\right)\,. \tag{5.3}$$

Since e_0 and f_0 lie in the undeformed Hopf subalgebra $sl(2)$ of $\mathcal{D}Y$ [11], the coproduct of g reads

$$\Delta(g)=g\otimes g \tag{5.4}$$

so that

$$g_1g_2\,\Delta^{\mathcal{F}}(g^{-1})\mathcal{F} = g_1g_2\,\mathcal{F}g_1^{-1}g_2^{-1}\,. \tag{5.5}$$

The two-cocycle $g_1g_2\Delta^{\mathcal{F}}(g^{-1})$ is a coboundary (with respect to the coproduct $\Delta^{\mathcal{F}}$). In representation, (5.5) is equal to the scaling limit of the represented twist from $\mathcal{U}_q$ to $\mathcal{A}_{q,p}$ [3, 7].
Note that this case is similar to the gauge transformation used in [15] although g is not purely Cartan.
It follows that

$$\mathcal{R}[\mathcal{D}Y_r^{V8}] \equiv g_1g_2\,\Delta_{21}^{\mathcal{F}}(g^{-1})\;\mathcal{R}[\mathcal{D}Y_r^{V6}]\;\Delta_{12}^{\mathcal{F}}(g)\,g_1^{-1}g_2^{-1} \tag{5.6}$$

satisfies the shifted Yang–Baxter equation (4.11).
To recover (5.1), use (5.5) and remark that $\pi_x\otimes\pi_x(g\otimes g)$ commutes with $R[\mathcal{D}Y]$.

6 Twist to $\mathcal{D}Y_r^F$

6.1 Twist in representation

The R-matrices of $\mathcal{D}Y_r^{V6}$ and $\mathcal{D}Y_r^F$ are related by:

$$R[\mathcal{D}Y_r^F](\beta_1-\beta_2) = K_{21}^{(6)}(\beta_2,\beta_1)\; R[\mathcal{D}Y_r^{V6}](\beta_1-\beta_2)\; (K_{12}^{(6)})^{-1}(\beta_1,\beta_2)\;, \tag{6.1}$$

where

$$K^{(6)}(\beta_1,\beta_2) = V'(\beta_1)\otimes V'(\beta_2) \qquad \text{with} \qquad V'(\beta) = \begin{pmatrix} e^{\frac{\beta}{2r}} & 0 \\ 0 & e^{-\frac{\beta}{2r}} \end{pmatrix}\;. \tag{6.2}$$

6.2 Universal twist

Again, one identifies $V'(\beta)$ as the evaluation representation of an algebra element

$$V'(\beta) = \pi_\beta\,(g')\;, \tag{6.3}$$

where

$$g' = \exp\left(\frac{h_1}{2r}\right)\;. \tag{6.4}$$

One then defines the following shifted coboundary

$$\mathcal{K}_{12}(r) = g'(r)\otimes g'(r+c^{(1)})\;\Delta^{\mathcal{F}}(g'^{-1})\;. \tag{6.5}$$

It obeys a shifted cocycle condition

$$\mathcal{K}_{12}(r)\;(\Delta^{\mathcal{F}}\otimes \mathrm{id})\mathcal{K}(r) = \mathcal{K}_{23}(r+c^{(1)})\;(\mathrm{id}\otimes\Delta^{\mathcal{F}'})\mathcal{K}(r)\;, \tag{6.6}$$

with $\mathcal{F}'_{23}(r) = \mathcal{F}_{23}(r+c^{(1)})$, as a consequence of

$$(\Delta^{\mathcal{F}}\otimes \mathrm{id})\;\Delta^{\mathcal{F}}(g'^{-1}) = (\mathrm{id}\otimes\Delta^{\mathcal{F}'})\;\Delta^{\mathcal{F}}(g'^{-1})\;, \tag{6.7}$$

which is the coassociativity property for $\Delta^{\mathcal{F}}$.
Finally

$$\mathcal{R}[\mathcal{D}Y_r^F] \equiv \mathcal{K}_{21}(r)\mathcal{R}[\mathcal{D}Y_r^{V6}]\;\mathcal{K}_{12}^{-1}(r) \tag{6.8}$$

satisfies the shifted Yang–Baxter equation (4.11). Moreover, (6.8) together with (6.5) show that $\mathcal{D}Y_r^F$ and $\mathcal{D}Y_r^{V6}$ are the same quasi-Hopf algebra.

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