LU QI-KENG'S PROBLEM

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ABSTRACT. This expository article, intended to be accessible to students, surveys results about the presence or absence of zeroes of the Bergman kernel function of a bounded domain in $\mathbb{C}^n$. Six open problems are stated.

CONTENTS

1. INTRODUCTION

When does a convergent infinite series of holomorphic functions have zeroes? This question is a fundamental, difficult problem in mathematics.

For example, the series $\sum_{n=0}^{\infty} z^n/n!$ is zero-free, but how can one tell this without a priori knowledge that the series represents the exponential function? Changing the initial term of this series produces a new series that does have zeroes, since the range of the exponential function is all non-zero complex numbers. The problem of determining

when a series has zeroes is essentially equivalent to the hard problem of determining the range of a holomorphic function that is presented as a series.

A famous instance of the problem of locating zeroes of infinite series is the Riemann hypothesis about the zeta function: namely, the conjecture that when $0 < \operatorname{Re} z < 1/2$, the convergent series $\sum_{n=1}^{\infty} (-1)^n/n^z$ has no zeroes. This formulation of the Riemann hypothesis is equivalent to the usual statement that the zeroes of $\zeta(z)$ in the critical strip where $0 < \operatorname{Re} z < 1$ all lie on the line where $\operatorname{Re} z = 1/2$. Indeed, when $\operatorname{Re} z > 1$, absolute convergence justifies writing that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^z} = -\sum_{n=1}^{\infty} \frac{1}{n^z} + 2 \sum_{\text{even } n} \frac{1}{n^z} = \zeta(z)(2^{1-z} - 1).$$

By the principle of persistence of functional relationships, the expressions on the outer ends of this equality still agree when $0 < \operatorname{Re} z < 1$, so $\zeta(z)$ and $\sum_{n=1}^{\infty} (-1)^n/n^z$ have the same zeroes in the interior of the critical strip. Moreover, the well known functional equation for the zeta function implies that the zeroes of ζ in the critical strip are symmetric about the point $1/2$, so it suffices to examine the left half of the critical strip for zeroes.

In this article, I shall discuss a different instance of the general problem of locating zeroes of infinite series. The Bergman kernel function is most conveniently expressed as the sum of a convergent infinite series. In 1966, Lu Qi-Keng [17] asked: for which domains is the Bergman kernel function $K(z, w)$ zero-free? I shall address three general methods for approaching this problem, and I shall give examples both of domains whose Bergman kernel functions are zero-free and of domains whose Bergman kernel functions have zeroes.

2. The Bergman kernel function

2.1. **Definition.** The Bergman kernel function $K(z, w)$ of a domain D in $\mathbb{C}^n$ is the unique sesqui-holomorphic[1] function satisfying the skew-symmetry property that $K(z, w) = \overline{K(w, z)}$ and the reproducing property that

$$f(z) = \int_D K(z, w) f(w) \, d\text{Volume}_w \qquad \text{for all } z \text{ in } D$$

for every square-integrable holomorphic function f on D. Equivalently, $K(z, w) = \sum_j \varphi_j(z)\overline{\varphi_j(w)}$, where $\{\varphi_j\}$ is an orthonormal basis for the Hilbert space of square-integrable holomorphic functions on D. To

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[1]Sesqui-holomorphic means holomorphic in the first variable and conjugate holomorphic in the second variable.

compute the Bergman kernel function, one typically chooses an orthogonal basis, calculates the normalizing factors, and sums the series.

For example, the monomials 1, z, z^2, ... , are orthogonal on the unit disk in the complex plane, and the norm of z^k is $(\int_0^{2\pi} \int_0^1 r^{2k+1} \, dr \, d\theta)^{1/2}$, or $\sqrt{2\pi/(2k+2)}$. Therefore the Bergman kernel function $K(z, w)$ of the unit disk equals

$$
(1) \qquad \sum_{k=0}^{\infty} \frac{k+1}{\pi} \cdot z^k \bar{w}^k = \frac{1}{\pi} \cdot \frac{1}{(1 - z\bar{w})^2}.
$$

This result is compatible with the Cauchy integral formula

$$
f(z) = \frac{1}{2\pi i} \oint_{|w|=1} \frac{f(w)}{w - z} \, dw,
$$

which can be rewritten by Green's formula as

$$
f(z) = \frac{1}{\pi} \iint_{|w|<1} \frac{f(w)}{(1 - z\bar{w})^2} \, d\mathrm{Area}_w,
$$

thus confirming that the kernel function (1) does have the required reproducing property.

2.2. **Transformation rule.** If $f : D_1 \to D_2$ is a biholomorphic[2] mapping, and if K_1 and K_2 denote the Bergman kernel functions of the domains D_1 and D_2 in $\mathbb{C}^n$, then

$$
(2) \qquad K_1(z, w) = (\det f'(z)) K_2(f(z), f(w)) (\overline{\det f'(w)}).
$$

This relationship holds because if $\{\varphi_j\}$ is an orthonormal basis for the square-integrable holomorphic functions on D_2, then $\{(\det f') \varphi_j \circ f\}$ is an orthonormal basis for the square-integrable holomorphic functions on D_1. For example, scaling (1) shows that the Bergman kernel function of the disk of radius r is equal to $\dfrac{1}{\pi r^2} \cdot \dfrac{1}{(1 - z\bar{w}/r^2)^2}$.

It is an observation of Steven R. Bell [3, 4] that a similar transformation rule holds even when $f : D_1 \to D_2$ is a branched m-fold covering (a proper holomorphic mapping): namely,

$$
\sum_{k=1}^{m} K_1(z, f_k^{-1}(w)) (\overline{\det (f_k^{-1})'(w)}) = (\det f'(z)) K_2(f(z), w),
$$

where the f_k^{-1} are the m holomorphic local inverses of f. This formula is not valid on the branching set, where local inverses are not defined.

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[2]Biholomorphic means holomorphic with a holomorphic inverse.

The Bergman kernel function $K(z,w)$ of a simply-connected planar domain D is related to the Riemann mapping function f that maps D onto the unit disk, taking the point a to 0: namely,

$$(3) \qquad f'(z) = K(z,a)\sqrt{\frac{\pi}{K(a,a)}}.$$

Indeed, the transformation rule (2) implies that

$$K(z,w) = f'(z) \cdot \frac{1}{\pi} \cdot \frac{1}{(1 - f(z)\overline{f(w)})^2} \cdot \overline{f'(w)}.$$

Since $f(a) = 0$, and $f'(a)$ is real and positive, setting $w = a$ implies that $\pi K(z,a) = f'(z)f'(a)$, and then setting $z = a$ makes it possible to eliminate $f'(a)$ to obtain (3).

Since the Riemann mapping function solves a certain extremal problem, the connection in one dimension with the Bergman kernel function suggests studying the Bergman kernel function in higher dimensions in connection with extremal problems.

2.3. **Extremal properties.** One can use extremal characterizations of the Bergman kernel function to help prove theorems about convergence of the Bergman kernel functions of a convergent sequence of domains. In the following two properties, the point w is fixed in a domain D in $\mathbb{C}^n$, and $\{\varphi_j\}$ is an orthonormal basis for the Hilbert space $A^2(D)$ of square-integrable holomorphic functions on D.

1. In the class of holomorphic functions f on D such that $\int_D |f|^2 \leq 1$, the maximal value of $|f(w)|^2$ is $K(w,w)$. In other words, $K(w,w)$ is the square of the norm of the functional from $A^2(D)$ to $\mathbb{C}$ that evaluates a function at the point w.

   Indeed, if $f(z) = \sum_j c_j \varphi_j(z)$, and if $\sum_j |c_j|^2 \leq 1$, then the Cauchy-Schwarz inequality implies that $|\sum_j c_j \varphi_j(w)|^2$ is bounded above by $\sum_j |\varphi_j(w)|^2$, which equals $K(w,w)$; and the upper bound is attained if c_j is taken equal to $\overline{\varphi_j(w)}/(\sum_j |\varphi_j(w)|^2)^{1/2}$.

2. In the class of holomorphic functions f on D satisfying the nonlinear constraint that $f(w) \geq \int_D |f|^2$, the function with the maximal value at w is $K(\cdot,w)$.

   Indeed, it is evident that the function $K(\cdot,w) = \sum_j \overline{\varphi_j(w)}\,\varphi_j$ is in the class. On the other hand, if $f = \sum_j c_j \varphi_j$ is an arbitrary member of the class, then the preceding extremal property implies that $f(w)^2 \leq K(w,w)\sum_j |c_j|^2$; and since the defining property of the class implies that $(\sum_j |c_j|^2)^2 \leq f(w)^2$, it follows that $\sum_j |c_j|^2 \leq K(w,w)$, and hence $f(w) \leq K(w,w)$.

3. Motivation for Lu Qi-Keng's problem

The Riemann mapping theorem characterizes the planar domains that are biholomorphically equivalent to the unit disk. In higher dimensions, there is no Riemann mapping theorem,[3] and two natural problems arise.

1. Are there canonical representatives of biholomorphic equivalence classes of domains?
2. How can one tell that two particular domains are biholomorphically inequivalent?

As an approach to the first question, Stefan Bergman introduced the notion of a "representative domain" to which a given domain may be mapped by "representative coordinates". If g_{jk} denotes the Bergman metric $\dfrac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z,z)$, where K is the Bergman kernel function, then the local representative coordinates based at the point a are

$$\sum_{k=1}^{n} g_{kj}^{-1}(a) \left. \frac{\partial}{\partial \bar{w}_k} \log \frac{K(z,w)}{K(w,w)} \right|_{w=a}, \qquad j = 1, \ldots, n.$$

These coordinates take a to 0 and have complex Jacobian matrix at a equal to the identity.

Zeroes of the Bergman kernel function $K(z,w)$ evidently pose an obstruction to the global definition of Bergman representative coordinates. This observation was Lu Qi-Keng's motivation for asking which domains have zero-free Bergman kernel functions.

On the other hand, if the Bergman kernel function of a domain does have zeroes, then the transformation rule (2) shows that the zero set is a biholomorphically invariant object. Therefore zero sets of Bergman kernel functions could be a tool for addressing the second question stated above. This idea has not yet been exploited in the literature.

4. First examples

The Bergman kernel function (1) of the unit disk is evidently zero-free. Consequently, the Bergman kernel function of every bounded, simply connected, planar domain is zero-free: apply either the transformation rule (2) or the explicit formula (3) relating the Riemann mapping function to the Bergman kernel function.

On the other hand, the Bergman kernel function of every annulus does have zeroes [22, 24]; more generally, the Bergman kernel function

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[3]More precisely, in order to obtain a generalized Riemann mapping theorem, one needs either new hypotheses [10] or new definitions [23].

of every bounded, multiply connected, planar domain with smooth boundary has zeroes [26].

Isolated singularities of square-integrable holomorphic functions are removable, and therefore the Bergman kernel function does not see isolated punctures in a domain. For example, the Bergman kernel function of a punctured disk is zero-free. On the other hand, a finitely connected planar domain with no singleton boundary component can be mapped biholomorphically to a smoothly bounded domain. Consequently, if a bounded planar domain is finitely connected and has at least two non-singleton boundary components, then its Bergman kernel function has zeroes.

I do not know if a corresponding statement holds for infinitely connected planar domains. For example, delete from the open unit disk a countable sequence of pairwise disjoint closed disks that accumulate only at the boundary of the unit disk. Does the Bergman kernel function of the resulting domain have zeroes?

Problem 1. Give necessary and sufficient conditions on an infinitely connected planar domain for its Bergman kernel function to have zeroes.

It is easy to see that in higher dimensions, the Bergman kernel function of a product domain is the product of the Bergman kernel functions of the lower dimensional domains. Consequently, the Bergman kernel function of a polydisc is zero-free, while the Bergman kernel function of the Cartesian product of a disc with an annulus does have zeroes. The Bergman kernel function $K(z, w)$ of the unit ball in $\mathbb{C}^n$ is the zero-free function $\dfrac{n!}{\pi^n} \cdot \dfrac{1}{(1 - \langle z, w \rangle)^{n+1}}$, where $\langle z, w \rangle$ denotes the scalar product $z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n$.

Even without knowing this explicit formula, one can see that the Bergman kernel function of the unit ball is zero-free. Since the ball is a complete circular domain,[4] its space of square-integrable holomorphic functions has an orthonormal basis whose first element is a constant (namely, the reciprocal of the square root of the volume of the domain) and whose other elements are functions that vanish at the origin. Consequently, the Bergman kernel function K has the property that $K(z, 0)$ is a non-zero constant function of z. Since the ball is homogeneous,[5]

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[4]A domain is called complete circular if whenever it contains a point z, it also contains the one-dimensional disk $\{ \lambda z : |\lambda| \leq 1 \}$.

[5]A domain is called homogeneous if it has a transitive automorphism group, that is, if any point of the domain can be mapped to any other point by a biholomorphic self-mapping of the domain.

the transformation rule (2) implies that the Bergman kernel function is nowhere zero. The same argument shows that every bounded, homogeneous, complete circular domain has a zero-free Bergman kernel function [2].

For many years it was thought that sufficiently nice, topologically trivial, bounded domains in $\mathbb{C}^n$ should have zero-free Bergman kernel functions. For example, all strongly convex,[6] sufficiently small perturbations of the ball have zero-free Bergman kernel functions if "small" is interpreted in the C^∞ topology on domains [12]. It turns out, however, to be the generic situation for the Bergman kernel function of a domain of holomorphy to have zeroes, if "generic" is interpreted in the very flexible Hausdorff topology on domains [8].

Problem 2. Do there exist arbitrarily small class C^1 perturbations of the ball whose Bergman kernel functions have zeroes?

I shall now discuss three techniques that can be used to show the existence of interesting domains whose Bergman kernel functions have zeroes.

5. Variation of domains

If reasonable domains Ω_j converge in a reasonable way to a limiting domain Ω, then the Bergman kernel functions $K_{\Omega_j}(z, w)$ converge to $K_\Omega(z, w)$ uniformly on compact subsets of $\Omega \times \Omega$. The word "reasonable" can be made precise [8], but here I will simply mention two examples of reasonable behavior. The first example is a fundamental theorem of I. P. Ramadanov [21] which started the whole theory.

- The Ω_j form an increasing sequence whose union is Ω.
- The Ω_j are bounded pseudoconvex[7] domains whose complements converge in the Hausdorff metric to the complement of an Ω whose boundary is locally a graph.

An example of unreasonable convergence is a sequence of disks shrinking down to a disk with a slit [25].

The proof of the convergence theorem exploits the extremal characterization of the Bergman kernel function from section 2.3. The application to Lu Qi-Keng's problem is that by Hurwitz's theorem, if the Bergman kernel function of the limiting domain Ω has zeroes, then

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[6] The statement is also true for strongly pseudoconvex domains, that is, domains that locally can be mapped biholomorphically to strongly convex domains.

[7] A domain is pseudoconvex if it is the union of an increasing sequence of strongly pseudoconvex domains, as defined in the preceding footnote. According to the solution of the Levi problem (see, for example, [16]), pseudoconvex domains are the same as domains of holomorphy.

so does the Bergman kernel function of the approximating domain Ω_j when j is sufficiently large.

Consequently, to construct a nice domain whose Bergman kernel function has zeroes, it suffices to construct a degenerate domain whose Bergman kernel function has zeroes, and then to approximate the degenerate domain by nice ones. For example, an easy calculation shows that constant functions are not square-integrable on the domain

$$(4) \qquad \left\{ (z_1, z_2) \in \mathbf{C}^2 : |z_2| < \frac{1}{1 + |z_1|} \right\},$$

although the domain does support some non-constant square-integrable holomorphic functions. Therefore the Bergman kernel function of this domain has a zero at the origin, but is not identically zero. By approximating this domain from inside, one sees that there exists a bounded, smooth, logarithmically convex, complete Reinhardt domain[8] whose Bergman kernel function has zeroes [7]. This example was surprising when it was first discovered; indeed, it contradicts a theorem previously published by two different authors [18, Theorem 1], [14, Corollary] and applied by a third [15].

Nguyên Viêt Anh, a student in Marseille, recently showed [1] how to approximate the domain (4) from inside by *concrete* domains that are smooth, algebraic, logarithmically convex, complete Reinhardt domains. Namely, the domain defined by the inequality

$$(5) \quad |z_2|^{2k}(1 + |z_1|)^{2k} + |z_2|^{2k}(1 - |z_1|)^{2k} + \left(\frac{|z_1|^2 + |z_2|^2}{k} \right)^k < 1$$

has the indicated properties when k is a positive integer, and so the Bergman kernel function of this domain must have zeroes when k is sufficiently large.

Problem 3. How large must k be in order for the Bergman kernel function of the domain (5) to have zeroes?

The technique of variation of domains can be used to prove the statement at the end of section 4 that every nice domain can be arbitrarily closely approximated in the Hausdorff metric by a nice domain whose Bergman kernel function has zeroes. View the starting domain as the Earth, and place in orbit around the Earth a small copy of one of the bounded domains just discussed. The Bergman kernel function of this

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[8]A domain is called complete Reinhardt if whenever it contains a point $(z_1, \ldots, z_n)$, it also contains the polydisc $\{ (\lambda_1 z_1, \ldots, \lambda_n z_n) : |\lambda_1| \leq 1, \ldots, |\lambda_n| \leq 1 \}$. The logarithmically convex complete Reinhardt domains are the convergence domains of power series.

disconnected region has zeroes because the Bergman kernel function of the satellite has zeroes. Now attach the satellite to the Earth by a thin tether. If the tether is sufficiently thin, then the principle of variation of domains implies that the Bergman kernel function of the joined domain has zeroes. According to the barbell lemma [16, Chapter 5, Exercise 21] in a suitable formulation [8], the joined domain can be made strongly pseudoconvex if the Earth is.

6. Variation of weights

In the representation of the Bergman kernel function as an infinite series $\sum_j \varphi_j(z)\overline{\varphi_j(w)}$, the basis elements φ_j are supposed to be orthonormal for integration with respect to Lebesgue measure. It is natural to consider the analogous construction when Lebesgue measure is multiplied by a positive weight function.

For instance, if G is a domain in $\mathbb{C}^n$, and Ω is a Hartogs domain in $\mathbb{C}^{n+1}$ with base G, which means that $\Omega = \{\, (z, z_{n+1}) \in \mathbb{C}^{n+1} : z \in G \text{ and } |z_{n+1}| < r(z) \,\}$, where r is a positive function on G, then the Bergman kernel function of Ω restricted to the base G equals the weighted Bergman kernel function of G corresponding to the weight πr^2. This follows because the Bergman kernel function is uniquely determined by its reproducing property, and holomorphic functions on G correspond to holomorphic functions on Ω that are independent of the extra variable. Consequently, zeroes of weighted Bergman kernel functions give rise to zeroes of ordinary Bergman kernel functions of higher-dimensional domains.

As a concrete example, consider on a bounded domain G in $\mathbb{C}^n$ containing the origin the weight function $\exp(-t\|z\|)$, where t is a real parameter, and $\|z\|$ denotes the Euclidean length $\sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $K_t(z, w)$ denote the Bergman kernel function of G with respect to this weight. It is a special case of a recent theorem of Miroslav Engliš [11] that this weighted Bergman kernel function must have zeroes near the origin when t is sufficiently large.

Remarkably, the proof depends on the non-smoothness of the weight function. The idea is to show that $\lim_{t\to\infty} K_t(z, z)^{1/t} = e^{\|z\|}$. It follows that the function $K_t(z, w)$ cannot be zero-free for every large t, for if it were, then a sesqui-holomorphic branch of $K_t(z, w)^{1/t}$ could be defined near the origin. When $t \to \infty$, there would be a limiting sesqui-holomorphic function $L(z, w)$ such that $L(z, z) = e^{\|z\|}$; but this is impossible because the function $e^{\|z\|}$ is not real analytic at the origin.

The verification that $\lim_{t\to\infty} K_t(z, z)^{1/t} = e^{\|z\|}$ is carried out via an upper estimate and a lower estimate. Let $\|f\|_t$ denote the weighted

norm $(\int_G |f(w)|^2 e^{-t\|w\|}\, dV_w)^{1/2}$. If f is a holomorphic function, and B_z is a small ball centered at z with volume $|B_z|$, then the mean-value property of holomorphic functions and the Cauchy-Schwarz inequality imply that $|f(z)|$ is bounded above by $\|f\|_t |B_z|^{-1}(\int_{B_z} e^{t\|w\|}\, dV_w)^{1/2}$. Therefore $K_t(z, z)$, which is the square of the norm of the point evaluation functional, is bounded above by $|B_z|^{-1} \sup_{w \in B_z} e^{t\|w\|}$, and so $\limsup_{t\to\infty} K_t(z, z)^{1/t} \leq \sup_{w \in B_z} e^{\|w\|}$. Now let the radius of the ball B_z shrink to zero to conclude that $\limsup_{t\to\infty} K_t(z, z)^{1/t} \leq e^{\|z\|}$.

For the lower bound, use the convexity of the Euclidean norm and the representation of a supporting hyperplane as the zero set of the real part of a linear holomorphic function. For each point z in the domain, there is a holomorphic function g such that $\operatorname{Re} g(w) \leq \|w\|$ for all w, and $\operatorname{Re} g(z) = \|z\|$. Since $K_t(z, z)$ is the square of the norm of the point evaluation functional, it is no smaller than $|e^{tg(z)}|/\|e^{tg/2}\|_t^2$, which in turn is no smaller than $e^{t\|z\|}$ divided by the volume of the domain. Consequently, $\liminf_{t\to\infty} K_t(z, z)^{1/t} \geq e^{\|z\|}$.

Problem 4. For concrete examples, determine how large t must be taken in Engliš's theorem to guarantee that the weighted Bergman kernel function has zeroes.

7. WEIGHTED DISK KERNELS AND CONVEX DOMAINS

In the preceding section, I remarked that the Bergman kernel function of a Hartogs domain in $\mathbb{C}^{n+1}$ is related to a weighted Bergman kernel function on the base domain in $\mathbb{C}^n$. For the same reason, a multi-dimensional domain that is fibered over a one-dimensional base has a Bergman kernel function that is related to a weighted Bergman kernel function on the base. In this section, I shall discuss an interesting example of this general principle.

The domain in $\mathbb{C}^n$ defined by the inequality

$$(6) \qquad |z_1| + |z_2|^{2/p_2} + \cdots + |z_n|^{2/p_n} < 1$$

has a Bergman kernel function whose restriction to the z_1-axis is proportional to the weighted Bergman kernel function for the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ with weight $(1 - |z|)^{p_2 + \cdots + p_n}$. Here the p_j can be arbitrary positive real numbers, and the proportionality constant is the volume of the $(n-1)$-dimensional domain defined by the inequality

$$|z_2|^{2/p_2} + \cdots + |z_n|^{2/p_n} < 1.$$

Accordingly, it is useful to compute explicitly the weighted Bergman kernel function K_q for the unit disk with weight $(1 - |z|)^q$, where $q > 0$.

The square of the norm of the monomial z^k with weight factor $(1-|z|)^q$ is

$$\int_0^{2\pi} \int_0^1 r^{2k+1}(1-r)^q \, dr \, d\theta = 2\pi B(2k+2, q+1),$$

where B is the Beta function defined in terms of the Gamma function by $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$. Consequently, the weighted Bergman kernel function $K_q(z,w)$ equals $(2\pi)^{-1}\sum_{k=0}^{\infty}(z\bar{w})^k/B(2k+2, q+1)$. A closed form expression for this series is most conveniently written in terms of the squares of the variables:

$$(7) \qquad K_q(z^2, w^2) = \frac{(q+1)}{4\pi z\bar{w}}\left[\frac{1}{(1-z\bar{w})^{q+2}} - \frac{1}{(1+z\bar{w})^{q+2}}\right].$$

The powers of $(1\pm z\bar{w})$ are to be understood as principal branches. The validity of this closed form expression can be verified by the binomial series expansion.

The explicit expression (7) implies that the weighted Bergman kernel function K_q has zeroes in the interior of the unit disk if and only if $q > 2$. Indeed, taking limits in (7) shows that K_q is not equal to 0 when either coordinate is equal to 0, so it is only necessary to decide if $(1-t)^{q+2} = (1+t)^{q+2}$ for some non-zero t in the unit disk. Since the mapping $t \mapsto (1+t)/(1-t)$ takes the unit disk bijectively to the right half-plane, with the origin going to the point 1, following this mapping by the mapping $u \mapsto u^{q+2}$ produces a composite mapping that takes some non-zero point of the open unit disk to the point 1 if and only if $q > 2$.

Consequently, the Bergman kernel function of the domain (6) is guaranteed to have zeroes if $p_2 + \cdots + p_n > 2$. Moreover, this domain is geometrically convex if no p_j exceeds 2. Therefore, one can exhibit many concrete examples of convex domains whose Bergman kernel functions have zeroes [9]. Here are some:

$$(8) \qquad \begin{aligned} &\{\, z \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3| < 1 \,\}, \\ &\{\, z \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3|^2 < 1 \,\}, \\ &\{\, z \in \mathbb{C}^4 : |z_1| + |z_2|^2 + |z_3|^2 + |z_4|^4 < 1 \,\}. \end{aligned}$$

Using a different method, Peter Pflug and E. H. Youssfi found some other interesting examples of convex domains whose Bergman kernel functions have zeroes. Even though the "minimal ball" in $\mathbb{C}^n$ defined by the inequality

$$(9) \qquad |z_1|^2 + \cdots + |z_n|^2 + |z_1^2 + \cdots + z_n^2| < 1$$

lacks multi-circular symmetry, its Bergman kernel function is known explicitly [19], and in [20] the authors analyzed the explicit formula to see that the Bergman kernel function of this domain has zeroes when $n \geq 4$.

Although the convex domains defined by (8) and (9) do not have smooth boundaries, they can be approximated from inside by smoothly bounded, strongly convex domains. From the method of variation of domains in section 5, it follows that when $n \geq 3$, there exist smoothly bounded, strongly convex domains in $\mathbb{C}^n$ whose Bergman kernel functions have zeroes. Pflug and Youssfi even showed in [20] how to write down concrete examples of bounded, smooth, algebraic, strongly convex domains that approximate (9) from inside.

Using the same idea, Nguyên Viêt Anh [1] gave concrete examples of bounded, smooth, algebraic, strongly convex, Reinhardt domains in $\mathbb{C}^n$ whose Bergman kernel functions have zeroes when $n \geq 3$. For example, when k is a sufficiently large positive integer, the inequality

$$(10) \qquad (|z_1|^2 + |z_2|^2 + |z_3|^2)^{2k} + \sum_{\substack{\pm \\ (8 \text{ terms})}} (\pm|z_1| \pm |z_2| \pm |z_3|)^{2k} < 1$$

defines such a domain in $\mathbb{C}^3$.

To see that (10) has the required properties, first observe that the ℓ_{2k} norm decreases to the ℓ_∞ norm as $k \to \infty$, so these domains are interior approximations to the domain $\{\, z \in \mathbb{C}^3 : |z_1| + |z_2| + |z_3| < 1 \,\}$, which is one of the domains (8) whose Bergman kernel functions have zeroes. The odd powers of the $|z_j|$ in the expansion of (10) cancel out by symmetry, so the defining function is equivalent to a polynomial. It would be obvious that the defining function (10) is convex if it had $\operatorname{Re} z_j$ in place of $|z_j|$, for a convex function of a linear function is convex; now observe that positive combinations of even powers are increasing, and the composite of a convex increasing function with the convex function $|z_j|$ is convex.

The two-dimensional domain defined by the inequality $|z_1| + |z_2| < 1$ is a borderline case for the preceding considerations. It turns out [9] that the Bergman kernel function of this domain has no zeroes in the interior of the domain, although it does have zeroes on the boundary.

Problem 5. Exhibit a bounded convex domain in $\mathbb{C}^2$ whose Bergman kernel function has zeroes in the interior of the domain.

8. Conclusion

It is a difficult problem to determine whether the Bergman kernel function of a specific domain has zeroes or not. If the kernel function

is presented as an infinite series, then locating the zeroes may be of the same order of difficulty as proving the Riemann hypothesis; and even if the series can be summed in closed form, determining whether or not 0 is in the range may be hard.

In this article, I have emphasized examples in which the Bergman kernel function does have zeroes. As the subject developed historically, such examples were considered surprising. From our current perspective, it would be more surprising to find some simple geometric condition guaranteeing that the Bergman kernel function is zero-free.

Students planning further investigation of the Bergman kernel function might consult, in addition to the journal articles I have cited, Bell's book [5] about the one-dimensional theory, the book of Jarnicki and Pflug [13], and Stefan Bergman's own book [6]. I offer the following problem as an illustration of how much remains to be discovered about the zeroes of the Bergman kernel function.

Problem 6. Characterize the vectors $(p_1, p_2, \ldots, p_n)$ of positive numbers for which the Bergman kernel function of the domain in $\mathbb{C}^n$ defined by the inequality

$$|z_1|^{2/p_1} + |z_2|^{2/p_2} + \cdots + |z_n|^{2/p_n} < 1$$

is zero-free.

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