State-dependent utility maximization in Lévy markets

José E. Figueroa-López and Jin Ma

Department of Statistics
Purdue University
West Lafayette, IN 47906
e-mail: figueroa@stat.purdue.edu

Department of Mathematics
University of Southern California
Los Angeles, CA 90089
e-mail: jinma@usc.edu

Abstract: We revisit Merton's portfolio optimization problem under bounded state-dependent utility functions, in a market driven by a Lévy process Z extending results by Karatzas et. al. [8] and Kunita [11]. The problem is solved using a *dual variational problem* as it is customarily done for non-Markovian models. One of the main features here is that the domain of the *dual problem* enjoys an explicit "parametrization", built on a *multiplicative optional decomposition* for nonnegative supermartingales due to Föllmer and Kramkov [2]. As a key step in obtaining the representation result we prove a *closure property* for integrals with respect to Poisson random measures, a result of interest on its own that extends the analog property for integrals with respect to a fixed semimartingale due to Mémin [13]. In the case that (i) the Lévy measure ν of Z is atomic with a finite number of atoms or that (ii) $\Delta S_t/S_{t^-} = \zeta_t \vartheta(\Delta Z_t)$ for a process ζ and a deterministic function ϑ, we explicitly characterize the admissible trading strategies and show that the dual solution is a risk-neutral local martingale.

AMS 2000 subject classifications: Primary: 93E20, 60G51, secondary: 62P05.
Keywords and phrases: Portfolio optimization, Lévy market, duality method, utility maximization, shortfall risk minimization.

1. Introduction

The task of determining good trading strategies is a fundamental problem in mathematical finance. A typical approach to this problem aims at finding the trading strategy that maximizes, for example, the final expected utility, which is defined as a deterministic, concave, and increasing function $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ of the final wealth. There are, however, many applications where a utility function has to change with the underlying securities, or more generally, with the source of randomness (say a Brownian motion). For example, in the so-called *optimal partial replication* of a contingent claim, introduced by Föllmer and Leukert [3], one tries to find the trading strategy that best replicates the claim H under a budget constraint. In particular, when the market is incomplete, it is often more beneficial to allow certain degree of "shortfall" in order to reduce the "super hedging cost", a threshold for the minimum initial wealth so that super-hedging is feasible (see, e.g., [1] and [9] for more details). Mathematically, such a shortfall risk could be measured by the expected loss

$$\mathbb{E}\left[L\left((H - V_{T})^{+}\right)\right],$$

where L is the "loss function", a convex increasing function that incorporates the investor's attitude towards the shortfall $(H - V_{T})^{+}$, and the value process V is subject to the constraint $V_0 \leq z$. Such a problem can then be formulated as a utility maximization problem with a bounded *state-dependent utility*, in which the utility function is defined by (cf. [3]):

$$U(v;\omega) := L(H(\omega)) - L((H(\omega) - v)^{+}), \qquad \omega \in \Omega. \tag{1.1}$$

In general, we can define a *state-dependent utility* as a function $U : \mathbb{R}_{+} \times \Omega \to \mathbb{R}_{+}$ such that $U(\cdot;\omega)$ is a utility function for each $\omega \in \Omega$. The utility maximization problem is then defined as

$$u(z) := \sup\left\{\mathbb{E}\left[U(V_{T}(\cdot), \cdot)\right] : \ V \text{ is admissible and } V_0 \leq z\right\}, \tag{1.2}$$

where the supreme is taken over all wealth processes $\{V_t\}_{t \leq T}$ generated by admissible trading strategies (see Section 2 for a precise definition).

The existence and essential uniqueness of the solution to the problem (1.2) was proved in [3] for a general semimartingale price model using a convex duality method, built on a celebrated bipolar theorem by Kramkov and Schachermayer [10]. However, this approach does not seem to shed any light on how to compute, in a feasible manner, the optimal trading strategy. This is partly due to the generality of the problem considered there. In this paper we shall consider the market model in which the price is driven by a Lévy process, and we propose a more manageable dual problem with a specific domain. We should note that our method can be extended to handle more general jump-diffusion models driven by even additive processes.

The problem of utility maximization can be traced back to Merton [14]-[15]. In a Brownian-driven market model, Karatzas et. al. [8] developed a program, known as the *convex duality method*, that has become one of the most powerful methods, yet relatively explicit and simple, to analyze optimal portfolio problems in non-Markovian markets. They prove that the marginal utility of the optimum final wealth is proportional to the risk-neutral local martingale that minimizes a "dual" problem, defined as another optimization problem with the objective function being the Legendre-Fenchel-type transformation of the original utility function. To be more precise, consider a minimization problem

$$v_{\Gamma}(y) = \inf_{\xi \in \Gamma} \mathbb{E}\left[\widetilde{U}\left(y\xi_{T}\right)\right], \qquad y > 0, \tag{1.3}$$

where Γ, the so-called *dual domain or class*, consists of (risk-neutral) exponential local martingales, and $\widetilde{U}(\cdot)$ stands for the *convex dual function* of $U(\cdot)$. The idea is first to find, for any $y > 0$, a minimizer $\xi_y^* \in \Gamma$ of (1.3), which in turn induces a "potential" optimal terminal wealth V_y^* in the sense that the so-called weak duality relation

$$u(z) \leq \mathbb{E}\left[U(V_y^*)\right] \tag{1.4}$$

holds. If one can further show that for some $y^* > 0$, there exists an admissible portfolio β^* such that $V_T^{\beta^*} \geq V_{y^*}^*$, then clearly equality holds in (1.4) (a property typically called *strong duality*), and β^* solves the original problem. Customarily, finding the optimal portfolio β^* relies on a variational problem for the dual value function and the existence of the minimizer $\xi_{y^*}^*$ utilizes the particular form of the market model and some general properties of the utility function.

More recently, the convex duality method was further extended to a general "jump-diffusion" market by Kunita [11] building on an exponential representation for *positive* local supermartingales as well as a variational equality for the dual problem. To ensure the attainability of the dual problem, it is required that the utility function satisfies the same conditions as [8] (one of which is unboundedness), and that the dual domain Γ contain all positive "risk-neutral" local supermartingales.

The main purpose of this paper is to further extend the seminal approach of [8] to the case of *state-dependent, bounded* utility functions. For simplicity, we will be contented with a market with only one stock, whose jumps are driven by a Lévy process $Z := \{Z_t\}_{t\geq 0}$, but our analysis can be readily extended to more general jump-diffusion multidimensional models such as the one considered in [11]. We should emphasize that the boundedness and potential non-differentiability of the utility function causes some technical subtleties. For example, the dual optimal process can be 0 with positive probability, thus the representation theorem of Kunita does not apply anymore. To get around these difficulties we shall reconsider the dual problem over an arbitrary subclass. Using an exponential representation for nonnegative supermartingales due to Föllmer and Kramkov [2], we show how to construct suitable explicit dual classes associated with certain classes of semimartingales that are closed under Émery's topology. To work with this last condition, we prove a closure property for integrals with respect to Poisson random measures, a result of interest on its own that extends the analog property for integrals with respect to a fixed semimartingale due to Mémin [13]. It is also worth mentioning that part of our approach relies on the fundamental characterization of contingent claims that are super-replicable (see [1] and [9]), while that of Föllmer and Leukert [3] (see also Xu [18]) was based on the bipolar theorem of [10]. We feel that the convex duality approach of [8] that we develop in this paper offers several advantages. The proofs are more direct and the dual problem might be more suitable for computational purposes since the dual class enjoys an "explicit" description and "parametrization". In the case that (i) the jumps of the price process S are driven by the superposition of finitely-many shot-noise Poisson processes or that (ii) $\Delta S_t / S_{t^-} = \zeta_t \vartheta(\Delta Z_t)$, we are even able to show that the dual solution is a risk-neutral local martingale.

We would like to remark that some of our results in Section 3 below may look similar to those in [18, Chapter 3], but there are essential differences. For example, the model in [18] exhibits only finite-jump activity, while our model allows general jumps. Also, [18] allows only downward price jumps, an assumption that seems to be crucial for the approach there, which was based on the existence of the solutions to certain stochastic differential equations (see, e.g., [18, Lemma 3.3, Proposition 3.4]). We should point out that our approach is also valid for general additive processes, including the time-inhomogeneous cases considered in [18]. We present an argument in (ii) of Section 6 to justify this point.

The paper is organized as follows. In Section 2 we introduce the financial model, along with some basic terminology that will be used throughout the paper. The convex duality method is revised in Section 3, where a potential optimal final wealth satisfying (1.4) is constructed. An explicit description of a dual class for which equality in (1.4) holds is presented in Section 4, along with some interesting simple characterizations of the dual optimum. In particular, as it was mentioned earlier, we prove that under certain conditions in the structure of the jumps, the dual optimum is actually a local martingale and we also provide an explicit characterization of the admissible trading strategies. In section 5 we show that the potential optimal final wealth is attained by an admissible trading strategy, as the last step for proving the existence of an optimal portfolio. Finally, we give some concluding remarks in Section 6. Some necessary fundamental theoretical results behind our approach are collected in Appendix A, such as the exponential representation for nonnegative supermartingales of Föllmer and Kramkov [2] and the closure property for integrals with respect to Poisson random measures that was previously mentioned.

2. Notation and problem formulation

Throughout this paper we assume that all the randomness comes from a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which there is defined a Lévy process Z with Lévy triplet $(\sigma^2, \nu, 0)$ (see Sato [17] for the terminology). By the Lévy-Itô decomposition, there exist a standard Brownian motion W and an independent Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}\backslash\{0\}$ with mean measure $\mathbb{E}\, N(dt, dz) = \nu(dz)dt$, such that

$$Z_t = \sigma W_t + \int_0^t \int_{|z|\leq 1} z\, \widetilde{N}(ds, dz) + \int_0^t \int_{|z|>1} z N(ds, dz), \tag{2.1}$$

where $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Let $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ be the natural filtration generated by W and N, augmented by all the null sets in $\mathcal{F}$ so that it satisfies the *usual conditions* (see e.g. [16]).

The market model

We assume that there are two assets in the market: a risk free bond (or money market account), and a risky asset, say, a stock. The case of multiple stocks, such as the one studied in [11], can be treated in a similar way without substantial difficulties (see section 6 for more details). As it is customary all the processes are taken to be discounted to the present value so that the value B_t of the risk-free asset can be assumed to be identically equal to 1. The (discounted) price of the stock follows the stochastic differential equation

$$d\,S_t = S_{t^-}\left\{b_t\,dt + \sigma_t\,dW_t + \int_{\mathbb{R}_0} v(t,z)\widetilde{N}(dt,dz)\right\}, \tag{2.2}$$

where $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$, $b \in L^1_{loc}$, $\sigma \in L^2_{loc}(W)$, and $v \in G_{loc}(N)$ (see [6] for the terminology). More precisely, b, σ, and v are predictable processes such that $v(\cdot,\cdot) > -1$ a.s. (hence, $S_\cdot > 0$ a.s.), and that

$$\int_0^{\cdot} |b_t|dt, \quad \int_0^{\cdot} |\sigma_t|^2 dt, \quad \text{and} \quad (\sum_{s\leq\cdot} v^2(s,\Delta Z_s))^{1/2}$$

are locally integrable. Finally, we assume that the market is free of arbitrage so that there exists a risk-neutral probability measures $\mathbb{Q}$ such that the (discounted) process S_t, $0 \leq t \leq T$, is an $\mathbb{F}$-local martingale under $\mathbb{Q}$. Throughout, $\mathcal{M}$ will stand for the class of all equivalent risk neutral measures $\mathbb{Q}$.

Admissible trading strategies and the utility maximization problem.

A trading strategy is determined by a predictable locally bounded process β representing the *proportion of total wealth invested in the stock*. Then, the resulting wealth process is governed by the stochastic differential equation:

$$V_t = w + \int_0^t V_{s^-}\,\frac{\beta_s}{S_{s^-}}\,dS_s, \quad 0 < t \leq T, \tag{2.3}$$

where w stands for the initial endowment. For future reference, we give a precise definition of "*admissible strategies*".

Definition 2.1. *The process $V^{w,\beta} := V$ solving (2.3) is called the value process corresponding to the self-financing portfolio with initial endowment w and trading strategy β. We say that a value process $V^{w,\beta}$ is "admissible" or that the process β is "admissible" for w if $V^{w,\beta}_t \geq 0$, $\forall\, t \in [0,T]$.*

For a given initial endowment w, we denote the set all admissible strategies for w by $\mathcal{U}^w_{ad}$, and the set of all admissible value processes by $\mathcal{V}^w_{ad}$. In light of the Doléans-Dade stochastic exponential of semimartingales (see e.g. Section I.4f in [6]), one can easily obtain necessary and sufficient conditions for admissibility.

Proposition 2.2. *A predictable locally bounded process β is admissible if and only if*

$$\mathbb{P}\left[\{\omega\in\Omega : \beta_t v(t,\Delta Z_t) \geq\, -1, \quad \textit{for a.e. } t\leq T\}\right] = 1.$$

To define our utility maximization problem, we begin by introducing the *bounded state-dependent utility function.*

Definition 2.3. *A random function $U : \mathbb{R}_+ \times \Omega \mapsto \mathbb{R}_+$ is called a "bounded and state-dependent utility function" if*

1. *$U(\cdot, \omega)$ is nonnegative, non-decreasing, and continuous on $[0, \infty)$;*
2. *For each fixed w, the mapping $\omega \mapsto U(w, \omega)$ is $\mathcal{F}_T$-measurable;*
3. *There is an $\mathcal{F}_T$-measurable, positive random variable H such that for all $\omega \in \Omega$, $U(\cdot, \omega)$ is a strictly concave differentiable function on $(0, H(\omega))$, and it holds that*

$$U(w, \omega) \equiv U(w \wedge H(\omega), \omega), \qquad w \in \mathbb{R}_+; \tag{2.4}$$

$$\mathbb{E}\left[U\left(H; \cdot\right)\right] < \infty; \tag{2.5}$$

Notice that the $\mathcal{F}_T$-measurability of the random variable $\omega \to U(V_T(\omega), \omega)$ is automatic because $U(w, \omega)$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}_T$-measurable in light of the above conditions 1 and 2. We remark that while the assumption (2.5) is merely technical, the assumption (2.4) is motivated by the shortfall risk measure (1.1). Our utility optimization problem is thus defined as

$$u(z) := \sup\left\{\mathbb{E}\left[U(V_T(\cdot), \cdot)\right] : \ V \in \mathcal{V}_{ad}^{w} \ \text{ with } \ w \leq z\right\}. \tag{2.6}$$

for any $z > 0$. We should note that the above problem is relevant only for those initial wealths z that are smaller than $\bar{w} := \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}\left\{H\right\}$, the super-hedging cost of H. Indeed, if $z \geq \bar{w}$, then there exists an admissible trading strategy β^* for z such that $V_T^{z,\beta^*} \geq H$ almost surely, and consequently, $u(z) = \mathbb{E}\left[U(H, \cdot)\right]$ (see [1] and [9] for this *super-hedging* result).

Our main objectives in the rest of the paper are the following: (1) Define the dual problem and identify the relation between the value functions of the primal and the dual problems; (2) By suitably defining the dual domain, prove the attainability of the associated dual problem; and (3) Show that the potential optimum final wealth induced by the minimizer of the dual problem can be realized by an admissible portfolio. We shall carry out these tasks in the remaining sections.

3. The duality method and the dual problems

In this section we introduce the *dual problems* corresponding to the primal problem (1.2). We begin by defining the so-called *convex dual function* of $U(\cdot; \omega)$:

$$\widetilde{U}(y, \omega) := \sup_{0 \leq z \leq H(\omega)} \left\{U\left(z, \omega\right) - yz\right\}. \tag{3.1}$$

We note that the function $\widetilde{U}$ is closely related to the Legendre-Fenchel transformation of the convex function $-U(-z)$. It can be easily checked that $\widetilde{U}(\cdot; \omega)$ is convex and differentiable everywhere, for each ω. Furthermore, if we denote the *generalized inverse* function of $U'(\cdot, \omega)$ by

$$I(y, \omega) := \inf\left\{z \in (0, H(\omega)) | U'(z, \omega) < y\right\}, \tag{3.2}$$

with the convention that $\inf \emptyset = \infty$, then it holds that

$$\widetilde{U}'(y,\omega) = -\left(I(y;\omega) \wedge H\right), \quad \forall\, y > 0, \tag{3.3}$$

and the function $\tilde{U}$ has the following representation

$$\widetilde{U}(y,\omega) = U\left(I(y,\omega) \wedge H(\omega), \omega\right) - y\left(I(y,\omega) \wedge H(\omega)\right). \tag{3.4}$$

Remark 3.1. *We point out that the random fields defined in (3.1)-(3.2) are $\mathcal{B}([0,\infty)) \times \mathcal{F}_T$-measurable. For instance, in the case of $\widetilde{U}$, we can write*

$$\widetilde{U}(y,\omega) = \sup_{z \geq 0}\{U(z,\omega) - yz\}\mathbf{1}_{\{z \leq H(\omega)\}},$$

and we will only need to check that $(y,\omega) \to \{U(z,\omega) - yz\}\mathbf{1}_{\{z \leq H(\omega)\}}$ is jointly measurable for each fixed z. This last fact follows because the random field in question is continuous in the spatial variable y for each ω, and is $\mathcal{F}_T$-measurable for each y. In light of (3.3), it transpires that the random field $I(y,\omega) \wedge H(\omega)$ is jointly measurable. Given that the subsequent dual problems and corresponding solutions are given in terms of the fields $\widetilde{U}(y,\omega)$ and $I(y,\omega) \wedge H(\omega)$ (see Definition 3.6 and Theorem 3.5 below), the measurability of several key random variable below is guaranteed.

Next, we introduce the so-called "dual class".

Definition 3.2. *Let $\widetilde{\Gamma}$ be the class of nonnegative supermartingales ξ such that*

(i) $0 \leq \xi(0) \leq 1$, *and*

(ii) for each locally bounded admissible trading strategy β, $\{\xi(t)V_t^{\beta}\}_{t \leq T}$ is a supermartingale.

To motivate the construction of the dual problems below we note that if $\xi \in \widetilde{\Gamma}$ and V is the value process of a self-financing admissible portfolio with initial endowment $V_0 \leq z$, then $\mathbb{E}\left[\xi(T)\left(V_T \wedge H\right)\right] \leq z$, and it follows that

$$\begin{aligned}
\mathbb{E}\left[U\left(V_T, \cdot\right)\right] &\leq \mathbb{E}\left[U\left(V_T \wedge H, \cdot\right)\right] - y\left(\mathbb{E}\left[\xi(T)\left(V_T \wedge H\right)\right] - z\right) \\
&\leq \mathbb{E}\{\sup_{0 \leq z' \leq H(\cdot)}\{U\left(z', \cdot\right) - y\xi(T)z'\}\} + zy \\
&= \mathbb{E}\{\widetilde{U}(y\xi(T), \cdot)\} + zy.
\end{aligned} \tag{3.5}$$

for any $y \geq 0$. The dual problem is defined as follows.

Definition 3.3. *Given a subclass $\Gamma \subset \widetilde{\Gamma}$, the minimization problem*

$$v_{\Gamma}(y) := \inf_{\xi \in \Gamma} \mathbb{E}\left[\widetilde{U}(y\xi(T), \omega)\right], \quad y > 0, \tag{3.6}$$

is called the "dual problem induced by Γ". The class Γ is referred to as a dual domain (or class) and $v_{\Gamma}(\cdot)$ is called its dual value function.

Notice that, by (3.5) and (3.6), we have the following *weak duality* relation between the primal and dual value functions:

$$u(z) \leq v_{\Gamma}(y) + zy, \tag{3.7}$$

valid for all $z, y \geq 0$. The effectiveness of the dual problem depends on the attainability of the lower bound in (3.7) for some $y^* = y^*(z) > 0$ (in which case, we say that *strong duality* holds), and the attainability of its corresponding dual problem (3.6). The following important properties will be needed for future reference.

Proposition 3.4. *The dual value function v_{Γ} corresponding to a subclass Γ of $\widetilde{\Gamma}$ satisfies the following properties:*

(1) v_{Γ} is non-increasing on $(0, \infty)$ and $\mathbb{E}\left[U(0; \cdot)\right] \leq v_{\Gamma}(y) \leq \mathbb{E}\left[U(H; \cdot)\right]$.
(2) If

$$0 < w_{\Gamma} := \sup_{\xi \in \Gamma} \mathbb{E}\left[\xi(T) H\right] < \infty, \tag{3.8}$$

then v_{Γ} is uniformly continuous on $(0, \infty)$, and

$$\lim_{y \downarrow 0} \frac{\mathbb{E}\left[U(H; \cdot)\right] - v_{\Gamma}(y)}{y} = \sup_{\xi \in \Gamma} \mathbb{E}\left[\xi(T) H\right]. \tag{3.9}$$

(3) There exists a process $\widetilde{\xi} \in \widetilde{\Gamma}$ such that $\mathbb{E}\left[\widetilde{U}(y\, \widetilde{\xi}(T), \cdot)\right] \leq v_{\Gamma}(y)$.
(4) If Γ is a convex set, then (i) v_{Γ} is convex, and (ii) there exists a $\xi^* \in \widetilde{\Gamma}$ attaining the minimum $v_{\Gamma}(y)$. Furthermore, the optimum ξ^* can be "approximated" by elements of Γ in the sense that there exists a sequence $\{\xi^n\}_n \subset \Gamma$ for which $\xi^n(T) \to \xi^*(T)$, a.s.

Proof. For simplicity, we write $v(y) = v_{\Gamma}(y)$. The monotonicity and range of values of v are straightforward. To prove (2), notice that since $\widetilde{U}(\cdot; \omega)$ is convex, non-increasing, and $\widetilde{U}'(0^+; \omega) = -H(\omega)$, we have

$$\frac{\mathbb{E}\left[U(H; \cdot)\right] - \inf_{\xi} \mathbb{E}\left[\widetilde{U}(y \xi(T))\right]}{y} \leq \sup_{\xi \in \Gamma} \mathbb{E}\left[\xi(T) H\right].$$

On the other hand, by the mean value theorem, dominated convergence theorem, (3.3), and Assumption 2.3,

$$\mathbb{E}\left[H \hat{\xi}(T)\right] \leq \liminf_{y \downarrow 0} \frac{\mathbb{E}\left[U(H; \cdot)\right] - v(y)}{y} \leq \sup_{\xi \in \Gamma} \mathbb{E}\left[\xi(T) H\right],$$

for every $\hat{\xi} \in \Gamma$. Then, (2) is evident. Uniform continuity is straightforward since for any h small enough it holds that

$$|v_{\Gamma}(y + h) - v_{\Gamma}(y)| \leq w_{\Gamma} |h|,$$

The part (i) of (4) is well-known. Let us turn out to prove (3) and part (ii) in (4). Let $\{\xi^n\}_{n\geq 1} \subset \Gamma \subset \widetilde{\Gamma}$ be such that

$$\lim_{n\to\infty} \mathbb{E}\left[\widetilde{U}(y\xi_T^n, \omega)\right] = v_{\Gamma}(y). \tag{3.10}$$

Without loss of generality, one can assume that each process ξ^n is constant on $[T, \infty)$. By Lemma 5.2 in [2], there exist $\bar{\xi}^n \in \operatorname{conv}(\xi^n, \xi^{n+1}, \dots)$, $n \geq 1$, and a nonnegative supermartingale $\{\widetilde{\xi}_t\}_{t\geq 0}$ with $\widetilde{\xi}_0 \leq 1$ such that $\{\bar{\xi}^n\}_{n\geq 1}$ is *Fatou convergent to $\widetilde{\xi}$ on the rational numbers* π; namely,

$$\widetilde{\xi}_t = \limsup_{s\downarrow t\,:\,s\in\pi} \limsup_{n\to\infty} \bar{\xi}_s^n = \liminf_{s\downarrow t\,:\,s\in\pi} \liminf_{n\to\infty} \bar{\xi}_s^n, \quad a.s. \tag{3.11}$$

for all $t \geq 0$. By Fatou's Lemma, it is not hard to check that $\left\{\tilde{\xi}(t)V_t\right\}_{t\leq T}$ is a supermartingale for every admissible portfolio with value process V, and hence, $\widetilde{\xi} \in \widetilde{\Gamma}$. Next, since the ξ^n's are constant on $[T, \infty)$ and $\widetilde{U}(\cdot;\omega)$ is convex, Fatou's Lemma implies that

$$\mathbb{E}\left[\widetilde{U}(y\tilde{\xi}_T, \omega)\right] \leq v_{\Gamma}(y),$$

Finally, we need to verify that, when Γ is convex, equality above is attained and and that $\widetilde{\xi}$ can be approximated by elements of Γ. Both facts are clear since $\{\bar{\xi}^n\} \subset \Gamma$ and $\lim_{n\to\infty} \bar{\xi}_T^n = \widetilde{\xi}_T$ a.s. Then, by the continuity and boundedness of $\widetilde{U}$,

$$v_{\Gamma}(y) \leq \lim_{n\to\infty} \mathbb{E}\left[\widetilde{U}(y\bar{\xi}_T^n B_T^{-1})\right] = \mathbb{E}\left[\widetilde{U}(y\widetilde{\xi}_T B_T^{-1}, \omega)\right].$$

□

We now give a result that is crucial for proving the strong duality in (3.7).

Theorem 3.5. *Suppose that (3.8) is satisfied and Γ is convex. Then, for any $z \in (0, w_{\Gamma})$, there exist $y(z) > 0$ and $\xi_{y(z)}^* \in \widetilde{\Gamma}$ such that*

(i) $\mathbb{E}\left[\widetilde{U}\left(y(z)\xi_{y(z)}^*(T), \omega\right)\right] \leq \mathbb{E}\left[\widetilde{U}\left(y(z)\xi(T), \omega\right)\right], \quad \forall \quad \xi \in \Gamma$;

(ii) $\mathbb{E}\left[V_z^{\Gamma}\, \xi_{y(z)}^*(T)\right] = z$, *where*

$$V_z^{\Gamma} := I\left(y(z)\xi_{y(z)}^*(T)\right) \wedge H;$$

(iii) $u(z) \leq \mathbb{E}\left[U\left(V_z^{\Gamma}; \omega\right)\right]$.

Proof. We borrow the idea of [8]. For simplicity let us write $v(y)$ instead of $v_{\Gamma}(y)$. Recall that $w_{\Gamma} := \sup_{\xi\in\Gamma} \mathbb{E}\left[\xi(T)H\right]$ and define $v(0) := \mathbb{E}\left[U(H;\omega)\right]$. In light of Lemma 3.4, the continuous function $f_z(y) := v(y) + zy$ satisfies

$$\lim_{y\downarrow 0} \frac{f_z(y) - f_z(0)}{y} = -w_{\Gamma} + z < 0, \quad \text{and} \quad f_z(\infty) = \infty,$$

for all $z < w_\Gamma$. Thus, $f_z(\cdot)$ attains its minimum at some $y(z) \in (0,\infty)$. By Proposition 3.4, we can find a $\xi_{y(z)} \in \widetilde{\Gamma}$ such that

$$v(y(z)) = \mathbb{E}\left[\widetilde{U}(y(z)\xi_{y(z)}(T), \omega)\right],$$

proving the (i) above. Now, consider the function

$$F(u) := uy(z)z + \mathbb{E}\left[\widetilde{U}\left(uy(z)\xi_{y(z)}(T)\right)\right], \quad u > 0.$$

Since $\xi_{y(z)}$ can be approximated by elements in Γ, for each $\varepsilon > 0$ there exists a $\xi^{y,\varepsilon}_{y(z)} \in \Gamma$ such that

$$\mathbb{E}\left[\widetilde{U}\left(y\xi_{y(z)}(T)\right)\right] > \mathbb{E}\left[\widetilde{U}\left(y\xi^{y,\varepsilon}_{y(z)}(T)\right)\right] - \varepsilon.$$

It follows that for each $\varepsilon > 0$,

$$\begin{aligned}\inf_{u>0} F(u) &\geq \inf_{y>0}\left\{yz + \mathbb{E}\left[\widetilde{U}\left(y\xi^{y,\varepsilon}_{y(z)}(T)\right)\right]\right\} - \varepsilon \\ &\geq \inf_{y>0}\{yz + v(y)\} - \varepsilon \\ &= y(z)z + \mathbb{E}\left[\widetilde{U}\left(y(z)\xi_{y(z)}(T)\right)\right] - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the function $F(u)$ attains its minimum at $u = 1$. On the other hand, $\frac{F(1+h)-F(1)}{h}$ equals

$$y(z)z + \mathbb{E}\left[\frac{\widetilde{U}((1+h)y(z)\xi_{y(z)}(T)) - \widetilde{U}(y(z)\xi_{y(z)}(T))}{h}\right],$$

which converges to

$$y(z)z - y(z)\mathbb{E}\left[\left(I\left(y(z)\xi_{y(z)}(T)\right) \wedge H\right)\xi_{y(z)}(T)\right]$$

as $h \to 0$. Here, we use (3.3) and the dominated convergence theorem. Then,

$$\mathbb{E}\left[\left(I\left(y(z)\xi_{y(z)}(T)\right) \wedge H\right)\xi_{y(z)}(T)\right] = z.$$

This proves (ii) of the theorem, and also, (iii) in light of (3.4) and (3.5). □

We note that Theorem 3.5 essentially provides an upper bound for the optimal final utility of the form $\mathbb{E}\left[U\left(V^\Gamma_z; \omega\right)\right]$, for certain "reduced" contingent claim $V^\Gamma_z \leq H$. By suitably choosing the dual class Γ, we shall prove in the next two sections that this reduced contingent claim is (super-) replicable with an initial endowment z.

4. Characterization of the optimal dual

We now give a full description of a dual class Γ for which *strong duality*, i.e., $u(z) = v_{\Gamma}(y) + zy$, holds. Denote $\mathcal{V}^{+}$ to be the class of all real-valued càdlàg, non-decreasing, adapted processes A null at zero. We will call such a process "increasing". In what follows we let $\mathcal{E}(X)$ be the Doléans-Dade stochastic exponential of the semimartingale X (see e.g. [6] for their properties). Let

$$\mathcal{S} := \left\{ X_t := \int_0^t G(s)dW_s + \int_0^t \int_{\mathbb{R}_0} F(s,z)\widetilde{N}(ds,dz) : F \geq -1 \right\}, \qquad (4.1)$$

and consider the associated class of exponential local supermartingales:

$$\Gamma(\mathcal{S}) := \{\xi := \xi_0 \mathcal{E}(X - A) : X \in \mathcal{S}, A \text{ increasing, and } \xi \geq 0\}. \qquad (4.2)$$

In (4.1), we assume that $G \in L^2_{loc}(W)$, $F \in G_{loc}(N)$, and that $F(t,\cdot) = G(t) = 0$, for all $t \geq T$. The following result shows not only that the class

$$\Gamma := \widetilde{\Gamma} \cap \Gamma(\mathcal{S}), \qquad (4.3)$$

is convex, but also that the dual optimum, whose existence is deduced from Theorem 3.5, remains in Γ. The proof of this result is based on a powerful representation for nonnegative supermartingales due to Föllmer and Kramkov [2] (see Theorem A.1 in the appendix), and a technical result about the closedness of the class of integrals with respect to Poisson random measures, under Émery's topology. We shall differ the presentation of these two fundamental results to Appendix A in order to continue with our discussion of the dual problem.

Theorem 4.1. *The class* Γ *is convex, and if (3.8) is satisfied, the dual optimum* $\xi^*_{y(z)}$ *of Theorem 3.5 belongs to* Γ*, for any* $0 < z < w_{\Gamma}$.

Proof. Let us check that $\mathcal{S}$ meets with the conditions in Theorem A.1. Indeed, each X in $\mathcal{S}$ is locally bounded from below since, defining $\tau_n := \inf\{t \geq 0 : X_t < -n\}$,

$$X_t^{\tau_n} \geq X_{\tau_n^-} - (\Delta X_{\tau_n})^- \mathbf{1}_{\tau_n < \infty} \geq -n - 1,$$

where $(x)^- = -x\mathbf{1}_{x<0}$. Condition (i) of Theorem A.1 is straightforward, while condition (ii) follows from Theorem A.3. Finally, condition (iii) holds because the processes in $\mathcal{S}$ are already local martingales with respect to $\mathbb{P}$ and hence $\mathbb{P} \in \mathcal{P}(\mathcal{S})$ with $A^{\mathcal{S}}(\mathbb{P}) \equiv 0$. By the Corollary A.2, we conclude that $\Gamma(\mathcal{S})$ is convex and closed under Fatou convergence on dense countable sets. On the other hand, $\widetilde{\Gamma}$ is also convex and closed under Fatou convergence, and thus so is the class $\Gamma := \widetilde{\Gamma} \cap \Gamma(\mathcal{S})$. To check the second statement, recall that the existence of the dual minimizer $\xi^*_{y(z)}$ in Theorem 3.5 is guaranteed from Proposition 3.4, where it is seen that $\xi^*_{y(z)}$ is the Fatou limit of a sequence in Γ (see the proof of Proposition 3.4). This suffices to conclude that $\xi^*_{y(z)} \in \Gamma$ since Γ is closed under under Fatou convergence. ∎

In the rest of this section, we present some properties of the elements in Γ and of the dual optimum $\xi^* \in \Gamma$. In particular, conditions on the "parameters" (G, F, A) so that $\xi \in \Gamma(\mathcal{S})$ is in $\widetilde{\Gamma}$ are established. First, we note that without lose of generality, A can be assumed predictable.

Lemma 4.2. *Let*

$$\xi := \xi_0 \mathcal{E}(X - A) \in \Gamma(\mathcal{S}). \tag{4.4}$$

Then, there exist a predictable process $A^p \in \mathcal{V}^+$ and a process $\widehat{X} \in \mathcal{S}$ such that $\xi = \xi_0 \mathcal{E}(\widehat{X} - A^p)$.

Proof. Let $X_t := \int_0^t G(s) dW_s + \int_0^t \int_{\mathbb{R}_0} F(s,z) \widetilde{N}(ds, dz) \in \mathcal{S}$. Since $F \in G_{loc}(N)$, there are stopping times $\tau_n' \nearrow \infty$ such that

$$\mathbb{E} \int_0^{\tau_n'} \int_{\mathbb{R}} |F(s,z)| \mathbf{1}_{|F|>1} \nu(dz) ds < \infty;$$

cf. Theorem II.1.33 in [6]. Now, define

$$\tau_n'' := \inf\{t \geq 0 : A_t > n\},$$

and $\tau_n := \tau_n' \wedge \tau_n''$. Then,

$$\begin{aligned} \mathbb{E} A_\infty^{\tau_n} &= \mathbb{E}[A_{\tau_n^-}] + \mathbb{E}\left[\Delta A_{\tau_n}\right] \leq n + 1 + \mathbb{E}\left[|F(\tau_n, Z_{\tau_n})|\right] \\ &\leq n + 2 + \mathbb{E} \int_0^{\tau_n} \int_{\mathbb{R}} |F(s,z)| \mathbf{1}_{|F|>1} \nu(dz) ds < \infty, \end{aligned}$$

where we used that $\Delta X_t - \Delta A_t \geq -1$. Therefore, A is locally integrable, increasing, and thus, its predictable compensator A^p exists. Now, by the representation theorem for local martingales (see Theorem III.4.34 [6]), the local martingale $X' := A - A^p$ admit the representation

$$X'_\cdot := \int_0^\cdot G'(s) dW_s + \int_0^\cdot \int_{\mathbb{R}_0} F'(s,z) \widetilde{N}(ds, dz).$$

Finally, $\xi = \xi_0 \mathcal{E}(X - A) = \xi_0 \mathcal{E}(X - X' - A^p)$. The conclusion of the proposition follows since $\widehat{X} := X - X'$ is necessarily in $\mathcal{S}$. □

The following result gives necessary conditions for a process $\xi \in \Gamma(\mathcal{S})$ to belong to $\widetilde{\Gamma}$. Recall that a predictable increasing process A can be uniquely decomposed as the sum of three predictable increasing processes,

$$A = A^c + A^s + A^d, \tag{4.5}$$

where A^c is the *absolutely continuous* part, A^s is the *singular continuous* part, and $A_t^d = \sum_{s \leq t} \Delta A_s$ is the jump part (cf. Theorem 19.61 in [4]).

Proposition 4.3. *Let $\xi := \xi_0 \mathcal{E}(X - A) \geq 0$, where $\xi_0 > 0$,*

$$X_t := \int_0^t G(s) dW_s + \int_0^t \int_{\mathbb{R}_0} F(s,z) \widetilde{N}(ds, dz) \in \mathcal{S},$$

and A is an increasing predictable process. Let τ be the "sinking time" of the supermartingale ξ:

$$\tau := \sup_n \inf\{t : \xi_t < \frac{1}{n}\} = \inf\{t : \Delta X_t = -1 \quad \text{or} \quad \Delta A_t = 1\}.$$

Also, let $a_t = \frac{dA_t^c}{dt}$. Then, $\{\xi_t S_t\}_{t\leq T}$ is a supermartingale if and only if the following two conditions are satisfied:

(i) There exist stopping times $\tau_n \nearrow \tau$ such that

$$\mathbb{E}\int_0^{\tau_n}\int_{\mathbb{R}} |v(s,z)F(s,z)|\nu(dz)ds < \infty. \tag{4.6}$$

(ii) For $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$h_t \leq a_t, \tag{4.7}$$

for almost every $t \in [0, \tau(\omega)]$, where

$$h_t := b_t + \sigma_t G(t) + \int_{\mathbb{R}} v(t,z)F(t,z)\nu(dz).$$

Proof. Recall that ξ and S satisfies the SDE's

$$d\xi_t = \xi_{t^-}\left(G(t)\,dW_t + \int_{\mathbb{R}_0} F(t,z)\widetilde{N}(dt,dz) - dA_t\right),$$
$$dS_t = S_{t^-}\left(b_t\,dt + \sigma_t\,dW_t + \int_{\mathbb{R}_0} v(t,z)\,\widetilde{N}(dt,dz)\right).$$

Integration by parts and the predictability of A yield that

$$\begin{aligned}\xi_t S_t = \text{local martingale} &+ \int_0^t b_s\,\xi_{s^-}S_{s^-}\,ds + \int_0^t \sigma_s G(s)\,\xi_{s^-}S_{s^-}\,ds \\ &- \int_0^t \xi_{s^-}S_{s^-}\,dA_s + \int_0^t\int_{\mathbb{R}_0} v(s,z)F(s,z)\xi_{s^-}S_{s^-}N(ds,dz).\end{aligned} \tag{4.8}$$

Suppose that $\{\xi_t S_t\}_{t\geq 0}$ is a nonnegative supermartingale. Then, the integral $\int_0^t\int_{\mathbb{R}} v(s,z)F(s,z)\xi_{s^-}S_{s^-}N(ds,dz)$ must have locally integrable variation in light of the Doob-Meyer decomposition for supermartingale (see e.g. Theorem III.13 in [16]). Therefore, there exist stopping times $\tau_n^1 \nearrow \infty$ such that

$$\mathbb{E}\int_0^{\tau_n^1}\int_{\mathbb{R}} |v(s,z)F(s,z)\xi_{s^-}S_{s^-}|\nu(dz)ds < \infty.$$

Then, (i) is satisfied with $\tau_n := \tau_n^1 \wedge \tau_n^2 \wedge \tau_n^3$, where $\tau_n^2 := \inf\{t : \xi_t < \frac{1}{n}\}$ and $\tau_n^3 := \inf\{t : \widetilde{S}_t < \frac{1}{n}\}$. Next, we can write (4.8) as

$$\xi_t S_t = \text{local martingale} - \int_0^t \xi_{s^-}S_{s^-}(dA_s - h_s ds).$$

By the Doob-Meyer representation for supermartingales and the uniqueness of the canonical decomposition for special semimartingales, the last integral must be increasing. Then, $a_t \geq h_t$ for $t \leq \tau$ since $\xi_{t^-} > 0$ and $\xi_t = 0$ for $t \geq \tau$ (see I.4.61 in [6]).

We now turn to the sufficiency of conditions (i)-(ii). Since $\{\xi_{t^-} S_{t^-}\}_{t\geq 0}$ is locally bounded,

$$\int_0^t \int_{\mathbb{R}} |v(s,z)F(s,z)| \xi_{s^-} S_{s^-} \mathbf{1}_{s\leq\tau_n} \nu(dz)ds$$

is locally integrable. Then, from (4.8), we can write

$$\xi_{t\wedge\tau_n} S_{t\wedge\tau_n} = \text{local martingale} - \int_0^t \xi_{s^-} S_{s^-} \mathbf{1}_{s\leq\tau_n} (dA_s - h_s ds).$$

Condition (ii) implies that $\{\xi_{t\wedge\tau_n} S_{t\wedge\tau_n}\}$ is a supermartingale, and by Fatou, $\{\xi_{t\wedge\tau} S_{t\wedge\tau}\}_{t\geq 0}$ will be a supermartingale. This concludes the prove since $\xi_t = 0$ for $t \geq \tau$, and thus, $\xi_{t\wedge\tau} S_{t\wedge\tau} = \xi_t S_t$, for all $t \geq 0$. □

The following result gives sufficient and necessary conditions for $\xi \in \Gamma(\mathcal{S})$ to belong to $\widetilde{\Gamma}$. Its proof is similar to that of Proposition 4.3.

Proposition 4.4. *Under the setting and notation of Proposition 4.3, $\xi \in \Gamma(\mathcal{S})$ belongs to $\widetilde{\Gamma}$ if and only if condition (i) in Proposition 4.3 holds and, for any locally bounded admissible trading strategies β,*

$$\mathbb{P}\left[\{\omega : h_t\beta_t \leq a_t, \text{ for a.e. } t \in [0, \tau(\omega)]\}\right] = 1. \tag{4.9}$$

The previous result can actually be made more explicit under additional information on the structure of the jumps exhibited by the stock price process. We consider two cases: when the jumps come from the superposition of shot-noise Poisson processes, and when the random field v exhibit a multiplicative structure. Let us first extend Proposition 2.2 in these two cases.

Proposition 4.5. *(i) Suppose ν is atomic with finitely many atoms $\{z_i\}_{i=1}^k$. Then, a predictable locally bounded strategy β is admissible if and only if $\mathbb{P}\times dt$-a.e.*

$$-\frac{1}{\max_i v(t,z_i) \vee 0} \leq \beta_t \leq -\frac{1}{\min_i v(t,z_i) \wedge 0}.$$

(ii) Suppose that $v(t,z) = \zeta_t \vartheta(z)$, for a predictable locally bounded process ζ such that $\mathbb{P}\times dt$-a.e. $\zeta_t(\omega) \neq 0$ and ζ_t^{-1} is locally bounded, and a deterministic function ϑ such that $\nu(\{z : \vartheta(z) = 0\}) = 0$. Then, a predictable locally bounded strategy β is admissible if and only if $\mathbb{P}\times dt$-a.e.

$$-\frac{1}{\bar{\vartheta} \vee 0} \leq \beta_t \, \zeta_t \leq -\frac{1}{\underline{\vartheta} \wedge 0},$$

where $\bar{\vartheta} := \sup\{\vartheta(z) : z \in \mathrm{supp}(\nu)\}$ and $\underline{\vartheta} := \inf\{\vartheta(z) : z \in \mathrm{supp}(\nu)\}$.

Proof. From Proposition 2.2, recall that $\mathbb{P}$-a.s.

$$\beta_t v(t, \Delta Z_t) \geq -1,$$

for a.e. $t \leq T$. Then, for any closed set $C \subset \mathbb{R}_0$, $0 \leq s < t$, and $A \in \mathcal{F}_s$,

$$\sum_{s<u\leq t} \chi_A(\omega)\,\chi_C(\Delta Z_u)\,\{\beta_u v(u, \Delta Z_u) + 1\} \geq 0.$$

Taking expectation, we get

$$\mathbb{E}\int_s^t \chi_A \int_C \{\beta_u v(u,z) + 1\}\,\nu(dz)du \geq 0.$$

Since such processes $H_u(\omega) := \chi_{A\times(s,t]}(\omega, u)$ generate the class of predictable processes, we conclude that $\mathbb{P}\times dt$-a.e.

$$-1 \leq \beta_t \frac{\int_C v(t,z)\nu(dz)}{\nu(C)}.$$

Let us prove (ii) (the proof of (i) is similar). Notice that

$$\inf_{z\in U} \vartheta(z) = \inf_{C\subset\mathbb{R}_0} \frac{\int_C \vartheta(z)\nu(dz)}{\nu(C)} \leq \sup_{C\subset\mathbb{R}_0} \frac{\int_C \vartheta(z)\nu(dz)}{\nu(C)} = \sup_{z\in U} \vartheta(z),$$

where U is the support of ν. Suppose that $\inf_{z\in U}\vartheta(z) < 0 < \sup_{z\in U}\vartheta(z)$. Then, by considering closed sets $C_n, C_n^{'} \subset \mathbb{R}_0$ such that

$$\frac{\int_{C_n}\vartheta(z)\nu(dz)}{\nu(C_n)} \nearrow \sup_z \vartheta(z), \quad \text{and} \quad \frac{\int_{C_n'}\vartheta(z)\nu(dz)}{\nu(C_n')} \searrow \inf_z \vartheta(z),$$

as $n\to\infty$, we can prove the necessity. The other two cases (namely, $\inf_z \vartheta(z) \geq 0$ or $0 \geq \sup_z \vartheta(z)$) are proved in a similar way. Sufficiency follows since, $\mathbb{P}$-a.s.,

$$\{t \leq T : \beta_t\zeta_t v\,(\Delta Z_t) < -1\} \subset \{t \leq T : \beta_t\zeta_t \sup_{z\in U} v(z) < -1\} \cup \{t \leq T : \beta_t\zeta_t \inf_{z\in U} v(z) < -1\}.$$

□

Example 4.6. It is worth pointing out some consequences:

(a) In the time homogeneous case, where $v(t,z) = z$, the extreme points of the support of ν (or what accounts to the same, the infimum and supremum of all possible jump sizes) determine completely the admissible strategies. For instance, if the Lévy process can exhibit arbitrarily large or arbitrarily close to -1 jump sizes, then

$$0 \leq \beta_t \leq 1;$$

a constraint that can be interpreted as absence of shortselling and bank borrowing (this fact was already pointed out by Hurd [5]).

(b) In the case that $\underline{\vartheta} \geq 0$, the admissibility condition takes the form $-1/\bar{\vartheta} \leq \beta_t \, \zeta_t$. If in addition $\zeta_. < 0$ (such that the stock prices exhibits only downward sudden movements), then $-1/(\bar{\vartheta}\zeta_t) \geq \beta_t$, and $\beta_. \equiv -c$, with $c > 0$ arbitrary, is admissible. In particular, from Proposition 4.4, if $\xi \in \Gamma(\mathcal{S})$ belongs to $\widetilde{\Gamma}$, then a.s. $h_t \beta_t \leq a_t$, for a.e. t $\leq \tau$. This means that $\xi \in \Gamma(\mathcal{S}) \cap \widetilde{\Gamma}$ if and only if condition (i) in Proposition 4.3 holds and $\mathbb{P}-$a.s. $h_t \geq 0$, for a.e. $t \leq \tau$. For a general ζ and still assuming that $\underline{\vartheta} \geq 0$, it follows that β admissible and $\xi \in \Gamma(\mathcal{S}) \cap \widetilde{\Gamma}$ satisfy that $\mathbb{P}$-a.s.

$$-\frac{1}{\bar{\vartheta}(\zeta_t \vee 0)} \leq \beta_t \leq -\frac{1}{\bar{\vartheta}(\zeta_t \wedge 0)}, \quad \text{and} \quad h_t \zeta_t^{-1} \mathbf{1}_{\{t \leq \tau\}} \leq 0,$$

for a.e. $t \geq 0$. □

We now extend Proposition 4.4 in the two cases introduced in Proposition 4.5. Its proof follows from Propositions 4.4 and 4.5.

Proposition 4.7. *Suppose that either (i) or (ii) in Proposition 4.5 are satisfied, in which case, define:*

$$\widehat{h}_t := \begin{cases} -\frac{h_t}{\max_i v(t,z_i) \vee 0} \mathbf{1}_{\{h_t < 0\}} - \frac{h_t}{\min_i v(t,z_i) \wedge 0} \mathbf{1}_{\{h_t > 0\}}, & \textit{if (i) holds true,} \\ \\ -\frac{h_t \zeta_t^{-1}}{\bar{\vartheta} \vee 0} \mathbf{1}_{\{h_t \zeta_t^{-1} < 0\}} - \frac{h_t \zeta_t^{-1}}{\underline{\vartheta} \wedge 0} \mathbf{1}_{\{h_t \zeta_t^{-1} > 0\}}, & \textit{if (ii) holds true.} \end{cases}$$

Then, a process $\xi \in \Gamma(\mathcal{S})$ belongs to $\widetilde{\Gamma}$ if and only if condition (i) in Proposition 4.3 holds, and for $\mathbb{P}$-a.e. ω, $\widehat{h}_t(\omega) \mathbf{1}_{\{t \leq \tau(\omega)\}} \leq a_t(\omega) \mathbf{1}_{\{t \leq \tau(\omega)\}}$, for a.e. $t \geq 0$.

We remark that the cases $\underline{\vartheta} \geq 0$ and $\bar{\vartheta} \leq 0$ do not lead to any absurd in the definition of $\hat{h}$ above as we are using the convention that $0 \cdot \infty = 0$. Indeed, for instance, if $\underline{\vartheta} \geq 0$, it was seeing that $h_t \zeta_t^{-1} \leq 0$, for a.e. $t \leq \tau$, and thus, we set the second term in the definition of $\hat{h}$ to be zero.

Now we can give a more explicit characterization of the dual solution $\xi^* = \mathcal{E}(X^* - A^*)$ to the problem (3.6), which existence was established in Proposition 4.1. For instance, we will see that A^* is absolutely continuous up to a predictable stopping time. Below, we refer to Proposition 4.3 for the notation.

Proposition 4.8. *Let $\xi := \xi_0 \, \mathcal{E}(X - A) \in \Gamma(\mathcal{S})$, $\tau_{_A} := \inf\{t : \Delta A_t = 1\}$, and $\widetilde{A}_t := \int_0^t a_s \, ds + \mathbf{1}_{\{t \geq \tau_{_A}\}}$. The followings two statements hold true:*

(1) $\widetilde{\xi} := \xi_0 \mathcal{E}\left(X - \widetilde{A}\right) \geq \xi$. Furthermore, $\xi \in \widetilde{\Gamma}$ if and only if $\widetilde{\xi} \in \widetilde{\Gamma}$.

(2) Suppose that either of the two conditions in Proposition 4.7 are satisfied and denote

$$\widehat{A}_t := \int_0^t \widehat{h}_s \mathbf{1}_{s \leq \tau} \, ds + \mathbf{1}_{\{t \geq \tau_{_A}\}},$$

where $\widehat{h}$ is defined accordingly to the assumed case. Then, $\xi_. \leq \widehat{\xi}_.$, and furthermore, the process $\widehat{\xi} := \xi_0 \mathcal{E}(X - \widehat{A})$ belongs to $\widetilde{\Gamma}$ if $\xi \in \widetilde{\Gamma}$.

Proof. Let A^c, A^s, A^d denote the increasing predictable processes in the decomposition (4.5) of A. Since A is predictable, there is no common jump times between X and A. Then,

$$\begin{aligned}\xi_t &= \xi_0 e^{X_t - A_t - \frac{1}{2}<X^c,X^c>_t} \prod_{s\leq t}(1+\Delta X_s)e^{-\Delta X_s} \prod_{s\leq t}(1-\Delta A_s)e^{\Delta A_s} \\ &\leq \xi_0 e^{X_t - A^c_t - \frac{1}{2}<X^c,X^c>_t} \prod_{s\leq t}(1+\Delta X_s)e^{-\Delta X_s}\mathbf{1}_{\{t<\tau_A\}} = \widetilde{\xi}_t,\end{aligned}$$

where we used that $A_t - \sum_{s\leq t}\Delta A_s = A^c_t + A^s_t \geq A^c_t$, and $\prod_{s\leq t}(1-\Delta A_s) \leq \mathbf{1}_{\{t<\tau_A\}}$. Since both processes ξ and $\widetilde{\xi}$ enjoy the same absolutely continuous part, and the same sinking time, the second statement in (1) is straightforward from Proposition 4.4. Part (2) follows from Proposition 4.7 since the process $\widehat{a}_t := \widehat{h}_t \mathbf{1}_{t\leq\tau}$ is nonnegative, predictable (since h is predictable), and locally integrable (since $0 \leq \widehat{h} \leq a$). □

We remark that part (2) in Proposition 4.8 remains true if we take $\widehat{A}_t := \int_0^t \widehat{h}_s \mathbf{1}_{s\leq\tau_A}\, ds + \mathbf{1}_{\{t\geq\tau_A\}}$. The following result is similar to Proposition 3.4 in Xu [18] and implies, in particular, that the optimum dual ξ^* can be taken to be a local martingale.

Proposition 4.9. *Suppose that either condition (i) or (ii) of Proposition 4.5 is satisfied. Moreover, in the case of condition (ii), assume additionally that*

$$\nu\left(\{z\in\operatorname{supp}(\nu)\backslash\{0\} : \vartheta(z) = c\}\right) > 0, \tag{4.10}$$

for $c = \underline{\vartheta}$ if $\bar{\vartheta} > 0$, and for $c = \bar{\vartheta}$ if $\underline{\vartheta} < 0$. Let $\xi \in \widetilde{\Gamma}\cap\Gamma(\mathcal{S})$. Then, there exists $\widetilde{X} \in \mathcal{S}$ such that $\widetilde{\xi} := \xi_0\,\mathcal{E}(\widetilde{X}) \in \widetilde{\Gamma}$ and $\xi_\cdot \leq \widetilde{\xi}_\cdot$. Furthermore, $\{\widetilde{\xi}(t)V^\beta_t\}_{t\leq T}$ is a local martingale for all locally bounded admissible trading strategies β.

Proof. Let us prove the case when condition (i) in Proposition 4.5 is in force. In light of Proposition 4.8, we assume without loss of generality that $A_t = \int_0^t a_t dt + \mathbf{1}_{\{t\geq\tau_A\}}$, with $a_t := \hat{h}_t \mathbf{1}_{\{t\leq\tau\}}$. Assume that $\min_i v(t,z_i) < 0 < \max_i v(t,z_i)$. Otherwise if, for instance, $\max_i v(t,z_i) \leq 0$, then it can be shown that $h_t \geq 0$, a.s. (similarly to case (b) in Example 4.6), and the first term of $\hat{h}$ is 0 under our convention that $\infty\cdot 0 = 0$. Notice that, in any case, one can find a predictable process z taking values on $\{z_i\}_{i=1}^n$, such that

$$\hat{h}_t = -\frac{h_t}{v(t,z(t))}.$$

Write $\widetilde{X}_\cdot := \int_0^\cdot G(s)dW_s + \int_0^\cdot\int_{\mathbb{R}_0}\widetilde{F}(s,z)d\widetilde{N}(s,z)$ for a $\widetilde{F} \in G_{loc}(N)$ to be determined in the sequel. For $\widetilde{\xi} \geq \xi$ it suffices to prove the existence of a field D satisfying both conditions below:

$$(a)\ D \geq 0 \quad \text{and} \quad (b)\ \int_{\mathbb{R}_0} D(t,z)\nu(dz)\mathbf{1}_{\{t\leq\tau\}} \leq \hat{h}_t \mathbf{1}_{\{t\leq\tau\}},$$

(then, $\widetilde{F}$ is defined as $D+F$). Similarly, for $\widehat{\xi}$ to belong to $\widetilde{\Gamma}$ it suffices that

$$(c)\ h_t + \int_{\mathbb{R}_0} v(t,z) D(t,z)\nu(dz) = 0.$$

Taking

$$D(t,z) := -\frac{h_t}{v(t,z(t))\nu(\{z(t)\})}\mathbf{1}_{\{z=z(t)\}},$$

clearly non-negative, (b) and (c) hold with equality. Moreover, the fact that inequalities (c) hold with equality implies that $\{\widehat{\xi}(t)V_t^{\beta}\}_{t\leq T}$ is a local martingale for all locally bounded admissible trading strategy β (this can be proved using the same arguments as in the sufficiency part of Proposition 4.3). Now suppose that condition (ii) in Proposition 4.5 holds. For simplicity, let us assume that $\underline{\vartheta} < 0 < \bar{\vartheta}$ (the other cases can be analyzed following arguments similar to Example 4.6). Notice that (4.10) implies the existence of a Borel $\underline{C}$ (resp. $\bar{C}$) such that $\vartheta(z) \equiv \underline{\vartheta}$ on $\underline{C}$ (resp. $\vartheta(z) \equiv \bar{\vartheta}$ on $\bar{C}$) and $0 < \nu(\underline{C}), \nu(\bar{C}) < \infty$. Taking

$$D(t,z) := -\frac{h_t\zeta_t^{-1}}{\bar{\vartheta}\nu(\bar{C})}\mathbf{1}_{\bar{C}}(z)\mathbf{1}_{\{h_t\zeta_t^{-1}<0\}} - \frac{h_t\zeta_t^{-1}}{\underline{\vartheta}\nu(\underline{C})}\mathbf{1}_{\underline{C}}(z)\mathbf{1}_{\{h_t\zeta_t^{-1}>0\}},$$

(b) and (c) above will hold with equality. □

5. Replicability of the upper bound

We now show that the tentative optimum final wealth V_z^{Γ}, suggested by the inequality (iii) in Proposition 4.1, is (super-) replicable. We will combine the dual optimality of ξ^* with the *super-hedging theorem*, which states that given a contingent claim $\widehat{H}$ satisfying $\bar{w} := \sup_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}\{\widehat{H}\} < \infty$, one can find for any fixed $z \geq \bar{w}$ an admissible trading strategy β^* (depending on z) such that $V_T^{z,\beta^*} \geq \widehat{H}$ almost surely (see Kramkov [9], and also Delbaen and Schachermayer [1]). Recall that $\mathcal{M}$ denotes the class of all equivalent risk neutral probability measures.

Proposition 5.1. *Under the setting and conditions of Proposition 4.1, for any* $0 < z < w_{\Gamma}$, *there is an admissible trading strategy* β^* *for* z *such that*

$$V_T^{z,\beta^*} \geq I\left(y(z)\xi^*_{y(z)}(T)\right) \wedge H,$$

and thus, the optimum of $u(z)$ *is reached at the strategy* β^*. *In particular,*

$$V_T^{z,\beta^*} = I\left(y(z)\xi^*_{y(z)}(T)\right),$$

when $I\left(y(z)\xi^*_{y(z)}(T)\right) < H$.

Proof. For simplicity, we write $\xi^*_t := \xi^*_{y(z)}(t)$, $y = y(z)$, and

$$V^* = I\left(y(z)\xi^*_{y(z)}(T)\right) \wedge H.$$

Fix an equivalent risk neutral probability measure $\mathbb{Q} \in \mathcal{M}$, and let $\xi'_t = \frac{d\,\mathbb{Q}|_{\mathcal{F}_t}}{d\,\mathbb{P}|_{\mathcal{F}_t}}$ be its corresponding density processes. Here, $\mathbb{Q}|_{\mathcal{F}_t}$ (resp. $\mathbb{P}|_{\mathcal{F}_t}$) is the restriction of the measure $\mathbb{Q}$ (resp. $\mathbb{P}$) to the filtration $\mathcal{F}_t$. Under $\mathbb{Q}$, $S_{\cdot}$ is a local martingale, and then, for any locally bounded β, $V^{\beta}_{\cdot}$ is a $\mathbb{Q}$-local martingale. By III.3.8.c in [6], $\xi' V^{\beta}$ is a $\mathbb{P}$-local martingale (necessarily nonnegative by admissibility), and thus, ξ' is in $\widetilde{\Gamma}$. On the other hand, ξ' belongs to $\Gamma(\mathcal{S})$ due to the exponential representation for *positive* local martingales in Kunita [12] (alternatively, by invoking Theorems III.8.3, I.4.34c, and III.4.34 in [6], $\xi' \in \Gamma(\mathcal{S})$ even if Z were just an additive process Z). By the convexity of the dual class $\Gamma = \Gamma(\mathcal{S}) \cap \widetilde{\Gamma}$, $\xi^{(\varepsilon)} := \varepsilon\xi' + (1-\varepsilon)\xi^*$ belongs to Γ, for any $0 \le \varepsilon \le 1$. Moreover, since $\widetilde{U}$ is convex and $\widetilde{U}'(y) = -(I(y) \wedge H)$,

$$\left|\frac{\widetilde{U}\left(y\xi^{(\varepsilon)}_{T}\right) - \widetilde{U}\left(y\xi^*_{T}\right)}{\varepsilon}\right| \le yH\left|\xi'_{T} - \xi^*_{T}\right| \le yH\left(\xi'_{T} + \xi^*_{T}\right).$$

The random variable $yH\left(\xi'_{T} + \xi^*_{T}\right)$ is integrable since by assumption $w_{\Gamma} < \infty$. We can then apply dominated convergence theorem to get

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left\{\mathbb{E}\left[\widetilde{U}\left(y\xi^{(\varepsilon)}_{T}\right)\right] - \mathbb{E}\left[\widetilde{U}\left(y\xi^*_{T}\right)\right]\right\} = -y\mathbb{E}\left[V^*\left(\xi'_{T} - \xi^*_{T}\right)\right],$$

which is nonnegative by condition (i) in Proposition 4.1. Then, using condition (ii) in Proposition 4.1,

$$\mathbb{E}_{\mathbb{Q}}\left[V^*\right] = \mathbb{E}\left[V^*\xi'_{T}\right] \le \mathbb{E}\left[V^*\xi^*_{T}\right] = z.$$

Since $\mathbb{Q} \in \mathcal{M}$ is arbitrary, $\sup_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}_{\mathbb{Q}}\left[V^*\right] \le z$. By *the super-heading theorem*, there is an admissible trading strategy β^* for z such that

$$V^{z,\beta^*}_{T} \ge I\left(y(z)\xi^*_{y(z)}(T)\right) \wedge H.$$

The second statement of the theorem is straightforward since $U(z)$ is strictly increasing on $z < H$. □

6. Concluding remarks

We conclude the paper with the following remarks.

(i) The dual class Γ. The dual domain of the dual problem can be taken to be the more familiar class of equivalent risk-neutral probability measures $\mathcal{M}$. To be more precise, define

$$\bar{\Gamma} := \left\{\xi_t := \frac{d\,\mathbb{Q}|_{\mathcal{F}_t}}{d\,\mathbb{P}|_{\mathcal{F}_t}} : \mathbb{Q} \in \mathcal{M}\right\}.$$

Since $\bar{\Gamma}$ is obviously a convex subclass of $\widetilde{\Gamma}$, Theorem 3.5 implies that, as far as

$$0 < \bar{w} := \sup_{\xi \in \bar{\Gamma}} \mathbb{E}\left[\xi_T H\right] < \infty, \tag{6.1}$$

for each $z \in (0, \bar{w})$, there exist $y := y(z) > 0$ and $\xi^* := \xi^*_{y(z)} \in \widetilde{\Gamma}$ (not necessarily belonging to $\bar{\Gamma}$) such that (i)-(iii) in Proposition 4.1 hold with $\Gamma = \bar{\Gamma}$. Finally, one can slightly modify the proof of Proposition 5.1, to conclude the replicability of

$$V_z^{\bar{\Gamma}} := I\left(y\xi^*_T\right) \wedge H.$$

Indeed, in the notation of the proof of the Proposition 5.1, the only step which needs to be justified in more detail is that

$$\mathbb{E}\left[\widetilde{U}\left(y\xi^*_T\right)\right] \leq \mathbb{E}\left[\widetilde{U}\left(y\xi^{(\varepsilon)}_T\right)\right], \tag{6.2}$$

for all $0 \leq \varepsilon \leq 1$, where $\xi^{(\varepsilon)} = \varepsilon\xi' + (1-\varepsilon)\xi^*$ (here, ξ' is a fixed element in $\bar{\Gamma}$). The last inequality follows from the fact that, by Proposition 5.1 (c), ξ^* can be approximated by elements $\{\xi^{(n)}\}_{n\geq 1}$ in $\bar{\Gamma}$ in the sense that $\xi^{(n)}_T \to \xi^*_T$ a.s. Thus, $\xi^{(\varepsilon)}$ can be approximated by the elements $\xi^{(\varepsilon,n)} := \varepsilon\xi' + (1-\varepsilon)\xi^{(n)}$ in $\bar{\Gamma}$, for which we know that

$$\mathbb{E}\left[\widetilde{U}\left(y\xi^*_T\right)\right] \leq \mathbb{E}\left[\widetilde{U}\left(y\xi^{(\varepsilon,n)}_T\right)\right].$$

Passing to the limit as $n \to \infty$, we obtain (6.2).

In particular we conclude that condition (6.1) is sufficient for both the existence of the solution to the primal problem and its characterization in terms of the dual solution $\xi^* \in \widetilde{\Gamma}$ of the dual problem induced by $\Gamma = \bar{\Gamma}$. We now further know that ξ^* belongs to the class $\widetilde{\Gamma} \cap \Gamma(\mathcal{S})$ defined in (4.3), and hence, enjoys an explicit parametrization of the form

$$\xi^* := \mathcal{E}\left(\int_0^{\cdot} G^*(s)dW_s + \int_0^{\cdot}\int_{\mathbb{R}_0} F^*(s,z)\widetilde{N}(ds,dz) - \int_0^{\cdot} a^*_s ds\right),$$

for some triple (G^*, F^*, a^*).

(ii) Market driven by general additive models. Our analysis can be extended to more general multidimensional models driven by additive processes (that is, processes with independent, possibly non-stationary increments; cf. Sato [17] and Kallenberg [7]). For instance, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined a d-dimensional additive process Z with Lévy-Itô decomposition:

$$Z_t = \alpha t + \Sigma W_t + \int_0^t \int_{\{\|z\|>1\}} zN(ds,dz) + \int_0^t \int_{\{\|z\|\leq 1\}} z\widetilde{N}(ds,dz),$$

where W is a standard $d-$dimensional Brownian motion, $N(dt,dz)$ is an independent Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$, and $\widetilde{N}(dt,dz) = N(dt,dz) -$

$\mathbb{E}N(dt, dz)$. Consider a market model consisting of $n+1$ securities: one risk free bond with price

$$dB_t := r_t B_t dt, \quad B_0 = 1, \quad t \geq 0,$$

and n risky assets with prices determined by the following stochastic differential equations with jumps:

$$dS_t^i = S_{t^-}^i \left\{ b_t^i\, dt + \sum_{j=1}^{d} \sigma_t^{ij} dW_t^j + \int_{\mathbb{R}^d} v^i(t,z)\widetilde{N}(ds,dz) \right\}, \quad i = 1, \ldots, n,$$

where the processes r, b, σ, and v are predictable satisfying usual integrability conditions (cf. Kunita [11]). We assume that $\mathcal{F} := \mathcal{F}_{\infty^-}$, where $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by W and N; namely, $\mathcal{F}_t := \sigma(W_s, N([0,s]\times A) : s \leq t, A \in \mathcal{B}(\mathbb{R}^d))$. The crucial property, particular to this market model, that makes our analysis valid is the representation theorem for local martingales relative to Z (see Theorem III.4.34 in [6]). The definition of the dual class Γ given in Section 4 will remain unchanged, and only very minor details will change in the proof of Theorem A.3. Some of the properties of the results in Section 4 regarding the properties of Γ will also change slightly. We remark that, by taking a real (nonhomogeneous) Poisson process, the model and results of Chapter 3 in Xu [18] will be greatly extended. We do not pursue the details here due to the limitation of the length of this paper.

(iii) Optimal wealth-consumption problem. Another classical portfolio optimization in the literature is that of optimal wealth-consumption strategies under a budget constraint. Namely, we allow the agent to spend money outside the market, while maintaining "solvency" throughout $[0,T]$. In that case the agent aims to maximize the cost functional that contains a "*running cost*":

$$\mathbb{E}\left[U_1(V_T) + \int_0^T U_2(t, c_t)dt \right],$$

where c is the instantaneous rate of consumption. To be more precise, the cumulative consumption at time t is given by $C_t := \int_0^t c_u du$ and the (discounted) wealth at time t is given by

$$V_t = w + \int_0^t \beta_u dS_u - \int_0^t c_u du.$$

Here, U_1 is a (state-dependent) utility function and $U_2(t,\cdot)$ is a utility function for each t. The dual problem can now be defined as follows:

$$v_{\Gamma}(y) = \inf_{\xi\in\Gamma} \mathbb{E}\left[\widetilde{U}_1\left(y\xi_T\right) + \int_0^T \widetilde{U}_2(s, y\xi_s ds \right],$$

over a suitable class of supermartingales Γ. For instance, if the support of ν is $[-1,\infty)$, then Γ can be all supermartingales ξ such that $0 \leq \xi_0 \leq 1$ and

$\{\xi_t S_t\}_{t\leq T}$ is a supermartingale. The dual Theorem 3.5 can be extended for this problem. However, the existence of a wealth-consumption strategy pair (β, c) that attains the potential final wealth induced by the optimal dual solution (as in Section 5) requires further work. We hope to address this problem in a future publication.

Appendix A: Convex classes of exponential supermartingales

The goal of this part is to establish the theoretical foundations behind Theorem 4.1. We begin by recalling an important optional decomposition theorem due to Föllmer and Kramkov [2]. Given a family of supermartingales $\mathcal{S}$ satisfying suitable conditions, the result characterizes the nonnegative exponential local supermartingales $\xi := \xi_0 \mathcal{E}(X - A)$, where $X \in \mathcal{S}$ and $A \in \mathcal{V}^+$, in terms of the so-called *upper variation process* for $\mathcal{S}$. Concretely, let $\mathcal{P}(\mathcal{S})$ be the class of probability measures $\mathbb{Q} \sim \mathbb{P}$ for which there is an increasing predictable process $\{A_t\}_{t\geq 0}$ (depending on $\mathbb{Q}$ and $\mathcal{S}$) such that $\{X_t - A_t\}_{t\geq 0}$ is a local supermartingale under $\mathbb{Q}$, for all $X \in \mathcal{S}$. The *smallest*[1] of such processes A is denoted by $A^{\mathcal{S}}(\mathbb{Q})$ and is called the upper variation process for $\mathcal{S}$ corresponding to $\mathbb{Q}$. For easy reference, we state Föllmer and Kramkov's result (see [2] for a proof).

Theorem A.1. *Let $\mathcal{S}$ be a family of semimartingales that are null at zero, and that are locally bounded from below. Assume that $0 \in \mathcal{S}$, and that the following conditions hold:*

(i) *$\mathcal{S}$ is predictably convex;*

(ii) *$\mathcal{S}$ is closed under the Émery distance;*

(iii) $\mathcal{P}(\mathcal{S}) \neq \emptyset$.

Then, the following two statements are equivalent for a nonnegative process ξ:

1. *ξ is of the form $\xi = \xi_0 \mathcal{E}(X - A)$, for some $X \in \mathcal{S}$ and an increasing process $A \in \mathcal{V}^+$;*
2. *$\frac{\xi}{\mathcal{E}(A^{\mathcal{S}}(\mathbb{Q}))}$ is a supermartingale under $\mathbb{Q}$ for each $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$.*

The next result is a direct consequence of the previous representation. Recall that a sequence of processes $\{\xi^n\}_{n\geq 1}$ is said to be "Fatou convergent on π" to a process ξ if $\{\xi^n\}_{n\geq 1}$ is uniformly bounded from below and it holds that

$$\xi_t = \limsup_{s\downarrow t\,:\, s\in\pi} \limsup_{n\to\infty} \xi_s^n = \liminf_{s\downarrow t\,:\, s\in\pi} \liminf_{n\to\infty} \xi_s^n, \tag{A.1}$$

almost surely for all $t \geq 0$.

Proposition A.2. *If $\mathcal{S}$ is a class of semimartingales satisfying the conditions in Theorem A.1, then*

$$\Gamma^0(\mathcal{S}) := \{\xi := \xi_0 \mathcal{E}(X - A) : X \in \mathcal{S},\ A \textit{ increasing, and } \xi \geq 0\}, \tag{A.2}$$

[1] That is, if A satisfies such a property then $A - A^{\mathcal{S}}(\mathbb{Q})$ is increasing.

is convex and closed under Fatou convergence on any fix dense countable set π of $\mathbb{R}_+$; that is, if $\{\xi^n\}_{n\geq 1}$ is a sequence in $\Gamma^0(\mathcal{S})$ that is Fatou convergent on π to a process ξ, then $\xi \in \Gamma^0(\mathcal{S})$.

Proof. The convexity of $\Gamma^0(\mathcal{S})$ is a direct consequence of Theorem A.1, since the convex combination of supermartingales remains a supermartingale. Let us prove the closure property. Fix a $\mathbb{Q} \in \mathcal{P}(\mathcal{S})$ and denote $C_t := \mathcal{E}\left(A^{\mathcal{S}}(\mathbb{Q})\right)$. Notice that $C_t > 0$ because $A^{\mathcal{S}}(\mathbb{Q})_t$ is increasing and hence, its jumps are nonnegative. Since $\xi^n \in \Gamma^0(\mathcal{S})$, $\{C_t^{-1}\xi_t^n\}_{t\geq 0}$ is a supermartingale under $\mathbb{Q}$. Then, for $0 < s' < t'$,

$$\mathbb{E}^{\mathbb{Q}}\left[C_{t'}^{-1}\xi_{t'}^n | \mathcal{F}_{s'}\right] \leq C_{s'}^{-1}\xi_{s'}^n.$$

By Fatou's Lemma and the right-continuity of process C,

$$\mathbb{E}^{\mathbb{Q}}\left[C_t^{-1}\xi_t | \mathcal{F}_{s'}\right] = \mathbb{E}^{\mathbb{Q}}\left[\liminf_{t'\downarrow t: t'\in\pi} \liminf_{n\to\infty} C_{t'}^{-1}\xi_{t'}^n | \mathcal{F}_{s'}\right] \leq C_{s'}^{-1}\xi_{s'}^n.$$

Finally, using the right-continuity of the filtration,

$$\mathbb{E}^{\mathbb{Q}}\left[C_t^{-1}\xi_t | \mathcal{F}_s\right] \leq \liminf_{s'\downarrow s: s'\in\pi} \liminf_{n\to\infty} C_{s'}^{-1}\xi_{s'}^n = C_s^{-1}\xi_s,$$

where $0 \leq s < t$. Since $\mathbb{Q}$ is arbitrary, the characterization of Theorem A.1 implies that $\xi \in \Gamma^0(\mathcal{S})$. □

The most technical condition in Theorem A.1 is the closure property under Émery distance. The following result is useful to deal with this condition. It shows that the class of integrals with respect to a Poisson random measure is closed with respect to Émery distance, thus extending the analog property for integrals with respect to a fixed semimartingale due to Mémin [13].

Theorem A.3. *Let Θ be a closed convex subset of $\mathbb{R}^2$ containing the origin. Let Π be the set of all predictable processes (F, G), $F \in G_{loc}(N)$ and $G \in L^2_{loc}(W)$, such that $F(t, \cdot) = G(t) = 0$, for all $t \geq T$, and $(F(\omega, t, z), G(\omega, t)) \in \Theta$, for $\mathbb{P} \times dt \times \nu(dz)$-a.e. $(\omega, t, z) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_0$. Then, the class*

$$\mathcal{S} := \left\{ X_t := \int_0^t G(s)dW_s + \int_0^t \int_{\mathbb{R}_0} F(s, z)\widetilde{N}(ds, dz) : (F, G) \in \Pi \right\} \qquad \text{(A.3)}$$

is closed under convergence with respect to Émery's topology.

Proof. Consider a sequence of semimartingales

$$X^n(t) := \int_0^t G^n(s)dW_s + \int_0^t \int_{\mathbb{R}} F^n(s, z)\widetilde{N}(ds, dz), \quad n \geq 1,$$

in the class $\mathcal{S}$. Let X be a semimartingale such that $X^n \to X$ under Émery topology. To prove the result, we will borrow some results in [13].

For some $\mathbb{Q} \sim \mathbb{P}$, we denote $\mathcal{M}^2(\mathbb{Q})$ to be the Banach space of all $\mathbb{Q}$-square integrable martingales on $[0, T]$, endowed with the norm $\|M\|_{\mathcal{M}^2(\mathbb{Q})} :=$

$\left(\mathbb{E}^{\mathbb{Q}}\langle M,M\rangle_T\right)^{1/2} = \left(\mathbb{E}^{\mathbb{Q}}[M,M]_T\right)^{1/2}$, and $\mathcal{A}(\mathbb{Q})$ to be the Banach space of all predictable processes on $[0,T]$ that have $\mathbb{Q}$-integrable total variations, endowed with the norm $\|A\|_{\mathcal{A}(\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\mathrm{Var}(A)$. Below, $\mathcal{A}^+_{loc}(\mathbb{Q})$ stands for the localized class of increasing process in $\mathcal{A}(\mathbb{Q})$. By Theorem II.3 in [13], one can extract a subsequence from $\{X^n\}$, still denote it by $\{X^n\}$, for which one can construct a probability measure $\mathbb{Q}$, defined on $\mathcal{F}_T$ and equivalent to $\mathbb{P}_T$ (the restriction of $\mathbb{P}$ on $\mathcal{F}_T$), such that the following assertions hold:

(i) $\xi := \frac{d\mathbb{Q}}{d\mathbb{P}_T}$ is bounded by a constant;

(ii) $X^n_t = M^n_t + A^n_t$, $t \leq T$, for Cauchy sequences $\{M^n\}_{n\geq 1}$ and $\{A^n\}_{n\geq 1}$ in $\mathcal{M}^2(\mathbb{Q})$ and $\mathcal{A}(\mathbb{Q})$, respectively.

Let us extend M^n and A^n to $[0,\infty)$ by setting $M^n_t = M^n_{t\wedge T}$ and $A^n = A^n_{t\wedge T}$ for all $t \geq 0$. Also, we extend $\mathbb{Q}$ for $A \in \mathcal{F}$ by setting $\mathbb{Q}(A) := \int_A \xi d\mathbb{P}$, so that $\mathbb{Q} \sim \mathbb{P}$ (on $\mathcal{F}$). In that case, it can be proved that $\mathcal{A}^+_{loc}(\mathbb{P}) = \mathcal{A}^+_{loc}(\mathbb{Q})$. This follows essentially from Proposition III.3.5 in [6] and Doob's Theorem. Now, let $\xi_t := \frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \mathbb{E}\left[\xi|\mathcal{F}_t\right]$, denote the density process. Since ξ is bounded, both $\{\xi_t\}_t$ and $\{|\Delta\xi_t|\}_t$ are bounded. By Lemma III.3.14 and Theorem III.3.11 in [6], the $\mathbb{P}-$quadratic covariation $[X^n,\xi]$ has $\mathbb{P}-$locally integrable variation and the unique canonical decomposition $M^n + A^n$ of X^n relative to $\mathbb{Q}$ is given by

$$M^n = X^n - \int_0^t \frac{1}{\xi_{s-}} d\left\langle X^n,\xi\right\rangle_s, \quad A^n = \int_0^t \frac{1}{\xi_{s-}} d\left\langle X^n,\xi\right\rangle_s.$$

Also, the $\mathbb{P}$-quadratic variation of the continuous part $X^{n,c}$ of X^n (relative to $\mathbb{P}$), given by $\langle X^{n,c},X^{n,c}\rangle_\cdot = \int_0^\cdot \left(G^n(s)\right)^2 ds$, is also a version of the $\mathbb{Q}$-quadratic variation of the continuous part of X^n (relative to $\mathbb{Q}$). By the representation theorem for local martingales relative to Z (see e.g. Theorem III.4.34 in [6] or Theorem 2.1 in [12]), ξ has the representation

$$\xi_t = 1 + \int_0^t \xi_{s^-} E(s) dW_s + \int_0^t \int_{\mathbb{R}} \xi_{s^-} D(s,z) \widetilde{N}(ds,dz),$$

for predictable D and E necessarily satisfying that $D > -1$,

$$\mathbb{E}\int_0^T \int_{\mathbb{R}} D^2(s,z)\xi_s^2 \nu(dz)ds < \infty, \quad \text{and} \quad \mathbb{E}\int_0^T E^2(s)\xi_s^2 ds < \infty.$$

Then,

$$\left\langle X^n,\xi\right\rangle_t = \int_0^t G^n(s)E(s)\xi_{s^-} ds + \int_0^t \int_{\mathbb{R}} F^n(s,z)D(s,z)\xi_{s^-}\nu(dz)ds,$$
$$A^n_t = \int_0^t \int_{\mathbb{R}} F^n(s,z)D(s,z)\nu(dz)ds + \int_0^t G^n(s)E(s)ds.$$

We conclude that $\Delta M^n_t = \Delta X^n_t = F^n(t,\Delta Z_t)$. Hence, $\Delta M^n = \Delta\widetilde{M}^n$, where $\widetilde{M}^n$ is the purely discontinuous local martingale (relative to $\mathbb{Q}$) defined by

$$\widetilde{M}^n_t := \int_0^t \int_{\mathbb{R}} F^n(s,z)\left(N(ds,dz) - \nu^{\mathbb{Q}}(ds,dz)\right),$$

where $\nu^{\mathbb{Q}}(ds,dz) := Y(s,z))ds\nu(dz)$ is the compensator of N relative to $\mathbb{Q}$ (see Theorem III.3.17 in [6]). It can be shown that $Y = 1 + D$. Notice that $\widetilde{M}^n$ is well-defined since $\mathcal{A}^+_{loc}(\mathbb{P}) = \mathcal{A}^+_{loc}(\mathbb{Q})$ and the Definition III.1.27 in [6]. Then, the purely discontinuous part of the local martingale M^n (relative to $\mathbb{Q}$) is given by $\widetilde{M}^n$ (see I.4.19 in [6]), and since $M^n \in \mathcal{M}^2(\mathbb{Q})$,

$$\mathbb{E}^{\mathbb{Q}}\left[M^n, M^n\right]_T \mathbb{E}^{\mathbb{Q}} \int_0^T \left(F^n(s,z)\right)^2 Y(s,z)\nu(dz)ds + \mathbb{E}^{\mathbb{Q}} \int_0^T \left(G^n(s)\right)^2 ds < \infty.$$

Similarly, since $\{M^n\}_{n\geq 1}$ is a Cauchy sequences under the norm $\mathbb{E}^{\mathbb{Q}}\left[M, M\right]_T$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left[M^n - M^m, M^n - M^m\right]_T = \mathbb{E}^{\mathbb{Q}} \int_0^T \left(F^n(s,z) - F^m(s,z)\right)^2 Y(s,z)\nu(dz)ds \\ + \mathbb{E}^{\mathbb{Q}} \int_0^T \left(G^n(s) - G^m(s)\right)^2 ds \to 0,\end{aligned}$$

as $n, m \to \infty$. Using the notation $\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times \mathbb{R}$ and $\widetilde{\mathcal{P}} := \mathcal{P} \times \mathcal{B}(\mathbb{R})$, where $\mathcal{P}$ is the predictable $\sigma-$ field, we conclude that $\{F^n\}_{n\geq 1}$ is a Cauchy sequence in the Banach space

$$\mathbb{H}_d := \mathbb{L}^2\left(\widetilde{\Omega}, \widetilde{\mathcal{P}}, Y\, d\mathbb{Q}\, d\nu\, dt\right) \cap \mathbb{L}^1\left(\widetilde{\Omega}, \widetilde{\mathcal{P}}, |D|\ d\mathbb{Q}\, d\nu\, dt\right),$$

and thus, there is $F \in \mathbb{H}_d$ such that $F^n \to F$, as $n \to \infty$. Similarly, there exists a G in the Banach space

$$\mathbb{H}_c := \mathbb{L}^2\left(\Omega \times \mathbb{R}_+, \mathcal{P}, d\mathbb{Q}\, dt\right) \cap \mathbb{L}^1\left(\Omega \times \mathbb{R}_+, \mathcal{P}, |E|\ d\mathbb{Q}\, d\nu\, dt\right),$$

such that $G^n \to G$, as $n \to \infty$. In particular, (F, G) satisfies condition (iv) since $Y = 1 + D$ is strictly positive, and each (F^n, G^n) satisfies (iv). Also, $F \in G_{loc}(N)$ relative to $\mathbb{Q}$ in light of $\mathcal{A}^+_{loc}(\mathbb{P}) = \mathcal{A}^+_{loc}(\mathbb{Q})$. Similarly, $\int_0^\cdot G^2(s)ds$ belongs to $\mathcal{A}^+_{loc}(\mathbb{Q})$, and hence, belongs to $\mathcal{A}^+_{loc}(\mathbb{P})$. It follows that the process

$$\widetilde{X} := \int_0^t G(s)dW_s + \int_0^t \int_{\mathbb{R}} F(s,z)\widetilde{N}(ds,dz), \quad n \geq 1,$$

is a well-defined local martingale relative to $\mathbb{P}$. Applying Girsanov's Theorem to $\widetilde{X}$ relative to $\mathbb{Q}$ and following the same argument as above, the purely discontinuous local martingale and bounded variation parts of $\widetilde{X}$ are respectively

$$\begin{aligned}M^d_t &= \int_0^t \int_{\mathbb{R}} F(s,z)\left(N(ds,dz) - \nu^{\mathbb{Q}}(ds,dz)\right), \\ A_t &= \int_0^t \int_{\mathbb{R}} F(s,z)D(s,z)\nu(dz)ds + \int_0^t G(s)E(s)ds.\end{aligned}$$

The continuous part of $\widetilde{X}$ has quadratic variation $\int_0^\cdot G^2(s)ds$. We conclude that $\widetilde{X} \in \mathcal{M}^2(\mathbb{Q}) \oplus \mathcal{A}(\mathbb{Q})$ and $X^n \to \widetilde{X}$ on $\mathcal{M}^2(\mathbb{Q}) \oplus \mathcal{A}(\mathbb{Q})$. Then, X^n converges under Émery's topology to $\widetilde{X}$ and hence, $X = \widetilde{X}$. □

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