Some properties of index of Lie algebras

Vladimir Dergachev

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Contents

1 Introduction

In paper [1] the author and Alexandre Kirillov have computed index [1] for a family of subalgebras of $\mathfrak{gl}(n)$. The papers [3], [4] and some computations done by the author provide insight into situation with subalgebras of other classical groups. One interesting property of this numeric data is that while subalgebras of $\mathfrak{gl}(n)$ exhibit much variety in the possible values of the index the subalgebras of other simple groups generally do not. We believe that this is due to the fact that Lie bracket on $\mathfrak{gl}(n)$ can be derived from multiplication in the associative algebra of matrices, while groups from other

[1] For the definition of index of Lie algebra see [2].

series do not possess this property. Thus one expects to discover that index will exhibit special characteristics in relation to Lie algebras derived from associative algebras. In this paper we explore this idea. We discover that index possesses certain "convexity" properties with respect to the operation of tensor product of associative algebras. Moreover there is a large family of associative algebras for which the convexity inequalities become precise thus shedding light on the richness of structure observed among subalgebras of $\mathfrak{gl}(n)$.

2 Index of Lie algebras

Definition 1 [Lie algebra]
A Lie algebra $\mathfrak{g}$ over field k is a vector space over k with operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties:

1. *skew-symmetry:*
$$[a, b] = -[b, a]$$

2. *Jacobi identity:*
$$[[a, b], c] + [[c, a], b] + [[b, c], a] = 0$$

Definition 2 [Index of Lie algebra] *Pick a basis $e_i \in \mathfrak{g}$. Let B be the multiplication matrix of the bracket product: $b_{i,j} = [e_i, e_j]$. B is then skew-symmetric and it's elements can be considered as polynomials over $\mathfrak{g}^*$ - dual space to $\mathfrak{g}$.*

For $f \in \mathfrak{g}^*$ we define $B|_f$ to be the result of evaluation of elements of B on f, that is $B|_f = \|f([e_i, e_j])\|$.

By definition the index of Lie algebra $\mathfrak{g}$ is

$$\operatorname{ind} \mathfrak{g} = \min_{f \in \mathfrak{g}^*} \dim \ker B|_f$$

We will use Ω to denote an element of $S(\mathfrak{g}^*) \otimes_k \wedge^2 \mathfrak{g}$ that corresponds to B. Let r be the maximum number such that $\wedge^r \Omega \neq 0$.

Proposition 1 *The following numbers are equal:*

- $\operatorname{ind} \mathfrak{g}$

- $\dim \mathfrak{g} - 2r$
- $\dim \ker B$ *, where* B *is considered as a matrix with coefficients in the field* $k(\mathfrak{g}^*)$ *of rational functions on* $\mathfrak{g}^*$
- $\dim \ker B_f$ *for generic (in Zariski sense)* $f \in \mathfrak{g}^*$

For more extended discussion of B and Ω see [1].

3 Tensor products

Definition 3 [Tensor product of matrices]
Let A *and* B *be two matrices with coefficients in rings* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *respectively. Let ring* $\mathcal{R}$ *have the property that* $\mathcal{R} \subset \mathcal{R}_1$ *and* $\mathcal{R} \subset \mathcal{R}_2$*. The tensor product* $A \otimes_{\mathcal{R}} B$ *is defined as a block matrix with each block* (i, j) *having dimensions of matrix* B *and equal to* $A_{i,j} \otimes_{\mathcal{R}} B$*, that is the matrix obtained from* B *by taking tensor products of a certain element of* A *with entrees of* B*. Thus* $A \otimes B$ *has coefficients in* $\mathcal{R}_1 \otimes_{\mathcal{R}} \mathcal{R}_2$*.*

Proposition 2 *The tensor product of matrices has the following properties:*

1. *distributive w.r.t. addition*
2. $(A \otimes_{\mathcal{R}} B) \cdot (C \otimes_{\mathcal{R}} D) = (AC) \otimes_{\mathcal{R}} (BD)$*, where (A,C) and (B,D) are pairs of matrices of the same shape*
3. $(A \otimes_{\mathcal{R}} B)^{-1} = (A^{-1}) \otimes_{\mathcal{R}} (B^{-1})$

Theorem 3 *Let* A *and* B *be square matrices of dimensions* k *and* n *respectively, with coefficients in commutative rings* $\mathcal{R}_1$ *and* $\mathcal{R}_2$*. Let ring* $\mathcal{R}$ *have the property that* $\mathcal{R} \subset \mathcal{R}_1$ *and* $\mathcal{R} \subset \mathcal{R}_2$*. Then*

$$\det (A \otimes_{\mathcal{R}} B) = (\det A)^n \otimes_{\mathcal{R}} (\det B)^k$$

Proof 1. If A and B are diagonal the statement is proved by a simple computation.

2. Let $R_1 = R_2 = R = \mathbb{C}$. Let $A = C_1 D_1 C_1^{-1}$ and $B = C_2 D_2 C_2^{-1}$ where D_1 and D_2 are diagonal. Then

$$\begin{aligned} A \otimes_{\mathcal{R}} B = &\left(C_1 D_1 C_1^{-1}\right) \otimes_{\mathcal{R}} \left(C_2 D_2 C_2^{-1}\right) = \\ &= \left(C_1 \otimes_{\mathcal{R}} C_2\right)\left(D_1 \otimes_{\mathcal{R}} D_2\right)\left(C_1^{-1} \otimes_{\mathcal{R}} C_2^{-1}\right) = \\ &= \left(C_1 \otimes_{\mathcal{R}} C_2\right)\left(D_1 \otimes_{\mathcal{R}} D_2\right)\left(C_1 \otimes_{\mathcal{R}} C_2\right)^{-1} \end{aligned}$$

and

$$\begin{aligned} \det\left(A \otimes_{\mathcal{R}} B\right) = & \\ &= \det\left(\left(C_1 \otimes_{\mathcal{R}} C_2\right)\left(D_1 \otimes_{\mathcal{R}} D_2\right)\left(C_1 \otimes_{\mathcal{R}} C_2\right)^{-1}\right) = \\ &= \det\left(D_1 \otimes_{\mathcal{R}} D_2\right) = \\ &= \left(\det D_1\right)^n \left(\det D_2\right)^k = \\ &= \left(\det A\right)^n \left(\det B\right)^k \end{aligned}$$

3. Since both sides of the equation $\det\left(A \otimes_{\mathcal{R}} B\right) = \left(\det A\right)^n \otimes_{\mathcal{R}} \left(\det B\right)^k$ are polynomials in elements of A and B with integral coefficients and we know that over $\mathbb{C}$ all generic A and B satisfy the equation we must have that the polynomials are identical. This concludes the proof of the theorem.

■

Theorem 4 *Let $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ be three fields, such that $\mathcal{F} \subset \mathcal{F}_1$ and $\mathcal{F} \subset \mathcal{F}_2$. Let A_1 and B_1 be two matrices with coefficients in $\mathcal{F}_1$ and A_2 and B_2 be two matrices with coefficients in $\mathcal{F}_2$.*

Then

1. $\dim\ker\left(B_1 \otimes_{\mathcal{F}} A_2 + A_1 \otimes_{\mathcal{F}} B_2\right) \geq \dim\ker B_1 \cdot \dim\ker B_2$
2. *For generic A_1 and A_2 (in Zariski topology) the inequality above is precise.*

Proof 1. Using elementary linear algebra one obtains matrices C_1 and C_2, such that $\operatorname{rank} C_1 = \dim\ker B_1$ and $\operatorname{rank} C_2 = \dim\ker B_2$ and also $B_1 C_1 = 0$ and $B_2 C_2 = 0$.

Because of properties of tensor product we have

$$\operatorname{rank}\left(C_1 \otimes_{\mathcal{F}} C_2\right) = \operatorname{rank} C_1 \cdot \operatorname{rank} C_2$$

And since

$$(B_1 \otimes_{\mathcal{F}} A_2 + A_1 \otimes_{\mathcal{F}} B_2) \cdot (C_1 \otimes_{\mathcal{F}} C_2) = \\ = (B_1 C_1) \otimes_{\mathcal{F}} (A_2 C_2) + (A_1 C_1) \otimes_{\mathcal{F}} (B_2 C_2) = 0$$

we have $\dim \ker (B_1 \otimes_{\mathcal{F}} A_2 + A_1 \otimes_{\mathcal{F}} B_2) \geq \text{rank } (C_1 \otimes_{\mathcal{F}} C_2)$ which concludes the first part of the proof.

2. Let $\hat{A}_1 = \left\| a^1_{i,j} \right\|$ and $\hat{A}_2 = \left\| a^2_{i,j} \right\|$ where $\left\{ a^1_{i,j} \right\}$ and $\left\{ a^2_{i,j} \right\}$ are two families of independent variables. Let $\hat{\mathcal{F}}_1 = \mathcal{F}_1 \left(\left\{ a^1_{i,j} \right\} \right)$ and $\hat{\mathcal{F}}_2 = \mathcal{F}_2 \left(\left\{ a^2_{i,j} \right\} \right)$ be fields of rational functions over $\left\{ a^1_{i,j} \right\}$ and $\left\{ a^2_{i,j} \right\}$ correspondingly.

   Let
   $$\vec{W} = v_1 \otimes_{\mathcal{F}} w_1 + ... + v_n \otimes_{\mathcal{F}} w_n$$
   be a vector from $\ker \left(B_1 \otimes_{\mathcal{F}} \hat{A}_2 + \hat{A}_1 \otimes_{\mathcal{F}} B_2 \right)$ (v_i have coefficients in $\hat{\mathcal{F}}_1$ and w_j have coefficients in $\hat{\mathcal{F}}_2$). Suppose that decomposition of $\vec{W}$ above is simple in the sense that neither two of v_i (w_j) are proportional and n is the minimum possible number for such a decomposition. Since $\vec{W}$ is defined over $\hat{\mathcal{F}}_1 \otimes_{\mathcal{F}} \hat{\mathcal{F}}_2$ we can find a polynomial p from

   $$\mathcal{Q} = \mathcal{F}_1 \left[\left\{ a^1_{i,j} \right\} \right] \otimes_{\mathcal{F}} \mathcal{F}_2 \left[\left\{ a^2_{i,j} \right\} \right]$$

   such that all elements of $p\vec{W}$ are from $\mathcal{Q}$. We now introduce a bigrading in $\mathcal{Q}$:

   $$\begin{aligned} \deg \mathcal{F} &= (0,0) \\ \deg a^1_{i,j} &= (1,0) \\ \deg a^2_{i,j} &= (0,1) \end{aligned}$$

   The matrix $B_1 \otimes_{\mathcal{F}} \hat{A}_2 + \hat{A}_1 \otimes_{\mathcal{F}} B_2$ has thus two parts - of degree $(0,1)$ and $(1,0)$. Therefore, every element of it's kernel is the tensor product of an element of $\ker B_1$ and an element of $\ker B_2$ - with possible coefficient from $\hat{\mathcal{F}}_1 \otimes_{\mathcal{F}} \hat{\mathcal{F}}_2$. This proves the statement for the particular case of $\hat{A}_1$ and $\hat{A}_2$, thus implying that the equality holds for almost all A_1 and A_2 (in Zariski sense).

   ■

4 Associative algebras

Definition 4 *Let $\mathfrak{A}$ be an associative algebra. Then $\mathfrak{A}^L$ denotes a Lie algebra with the bracket defined as $[a,b] = ab - ba$.*

Theorem 5 [Convexity property of index] *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite dimensional associative algebras over field $\mathcal{F}$. Then*

$$\mathrm{ind}\ (\mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{B})^L \geq \mathrm{ind}\,\mathfrak{A}^L \cdot \mathrm{ind}\,\mathfrak{B}^L \tag{1}$$

Proof Let $\{e_i\}$ be a basis in $\mathfrak{A}$ and $\{g_j\}$ be a basis in $\mathfrak{B}$. Denote by A_1 the multiplication matrix of $\mathfrak{A}$, that is the matrix with coefficients in (first degree) polynomials over $\mathfrak{A}^*$, element (i,j) of A_1 is equal to $e_i \cdot e_j$. Let A_2 denote the multiplication matrix of $\mathfrak{B}$.

The multiplication matrix B_1 of *Lie algebra* $\mathfrak{A}^L$ is given by the formula $B_1 = A_1 - A_1^T$ (here $(\cdot)^T$ denotes transposition). $B_2 = A_2 - A_2^T$ gives the multiplication matrix of $\mathfrak{B}^L$.

The multiplication matrix of $\mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{B}$ in the basis $\{e_i \otimes_{\mathcal{F}} g_j\}$ is $A_1 \otimes_{\mathcal{F}} A_2$. The multiplication matrix B for $(\mathfrak{A} \otimes_{\mathcal{F}} \mathfrak{B})^L$ is computed as follows:

$$\begin{aligned} B = {} & A_1 \otimes_{\mathcal{F}} A_2 - (A_1 \otimes_{\mathcal{F}} A_2)^T = \\ & = A_1 \otimes_{\mathcal{F}} A_2 - A_1^T \otimes_{\mathcal{F}} A_2^T = \\ & = A_1 \otimes_{\mathcal{F}} A_2 - A_1^T \otimes_{\mathcal{F}} A_2 + A_1^T \otimes_{\mathcal{F}} A_2 - A_1^T \otimes_{\mathcal{F}} A_2^T = \\ & = B_1 \otimes_{\mathcal{F}} A_2 + A_1^T \otimes_{\mathcal{F}} B_2 \end{aligned}$$

Theorem 4 implies that $\dim\ker B \geq \dim\ker B_1 \cdot \dim\ker B_2$. One concludes the proof by applying proposition 1. ■

Example 1 Let $\mathfrak{A} = a\mathbb{C} + b\mathbb{C}$ be a 2-dimensional algebra over $\mathbb{C}$ (you can replace $\mathbb{C}$ with you favorite field) with the following multiplication table:

$\times_{\mathfrak{A}}$	a	b
a	a	b
b	a	b

Let $\mathfrak{B}$ be an arbitrary associative algebra over $\mathbb{C}$ with multiplication table A. The the multiplication table of $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$ is:

$\times_{\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B}}$	$a\otimes_{\mathbb{C}}\mathfrak{B}$	$b\otimes_{\mathbb{C}}\mathfrak{B}$
$a\otimes_{\mathbb{C}}\mathfrak{B}$	aA	bA
$b\otimes_{\mathbb{C}}\mathfrak{B}$	aA	bA

The index of algebra $\mathfrak{A}$ is equal to 0. The index of algebra $\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B}$ can be computed as follows:

$$B_{\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B}} = \begin{pmatrix} aA - aA^T & bA - aA^T \\ aA - bA^T & bA - bA^T \end{pmatrix}$$

By definition $\operatorname{ind}\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B} = \ker B_{\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B}}$. Thus we want to find all solutions (v_1, v_2) (in rational functions on $(\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B})^*$) of the following equations:

$$\begin{cases} \left(aA - aA^T\right) v_1 + \left(bA - aA^T\right) v_2 & = & 0 \\ \left(aA - bA^T\right) v_1 + \left(bA - bA^T\right) v_2 & = & 0 \end{cases}$$

A few straightforward transformations lead us to the following system:

$$\begin{cases} (a-b)A^T\left(v_1 + v_2\right) & = & 0 \\ (a-b)A\left(av_1 + bv_2\right) & = & 0 \end{cases}$$

Since a and b are invertible and independent the index of $\mathfrak{A}\otimes_{\mathbb{C}}\mathfrak{B}$ is equal to twice the dimension of the kernel of A. ■

In the particular case considered above the inequality 1 is precise only when the algebra $\mathfrak{B}$ has non-degenerate multiplication table.

5 Characteristic polynomial of associative algebra

Definition 5 [Multiplicative group] *Let $\mathfrak{A}$ be an associative algebra with unity. Then $\mathfrak{A}^*$ - the set of all invertible elements of $\mathfrak{A}$ - can be regarded as a group under the operation of multiplication.*

The Lie algebra of $\mathfrak{A}^*$ is the algebra $\mathfrak{A}$ itself with commutator

$$[a, b] = ab - ba$$

Definition 6 [**Adjoint action**] *Let $g \in \mathfrak{A}^*$ and $Y \in \mathfrak{A} = (\mathfrak{A}^*)^L$. By definition adjoint action of g on X is*

$$\operatorname{Ad}_g Y = gYg^{-1}$$

Let $X \in \mathfrak{A}$. The adjoint action of X on Y is defined as

$$\operatorname{ad}_X Y = XY - YX$$

Definition 7 [**Coadjoint action**] *Let $g \in \mathfrak{A}^*$ and $F \in \mathfrak{A}'$ (here $\mathfrak{A}'$ is the space of linear functionals on $\mathfrak{A}$). Let $Y \in \mathfrak{A}$. By definition coadjoint action of g on F is*

$$\left(\operatorname{Ad}_g^* F\right)(Y) = F\left(g^{-1}Yg\right)$$

Let $X \in \mathfrak{A}$. The coadjoint action of X on F is defined as

$$\left(\operatorname{ad}_X^* F\right)(Y) = F\left(-(XY - YX)\right)$$

Definition 8 [**Characteristic polynomial of associative algebra**] *Let $\{e_i\}$ be some basis of $\mathfrak{A}$. Let A be the multiplication table of $\mathfrak{A}$ considered as matrix which coefficients are polynomials over $\mathfrak{A}'$. Then*

$$\chi(\lambda, \mu, F) = \det\left(\lambda A + \mu A^T\right)$$

is the characteristic polynomial of algebra $\mathfrak{A}$. Here F is an element of $\mathfrak{A}'$.

Lemma 6 *Characteristic polynomial is defined up to a scalar multiple.*

Proof Indeed, change of basis $\{e_i\}$ replaces A with CAC^T. And

$$\det\left(\lambda CAC^T + \mu CA^TC^T\right) = (\det C)^2 \det\left(\lambda A + \mu A^T\right)$$

■

Theorem 7 *The characteristic polynomial is quasi-invariant under coadjoint action. That is*

$$\chi\left(\lambda, \mu, \operatorname{Ad}_g^* F\right) = \left(\det \operatorname{Ad}_g\right)^{-2} \chi(\lambda, \mu, F)$$

Proof Indeed, the matrix element (i, j) of A evaluated in point F is given by the expression $F(e_i e_j)$. Since

$$\left(\mathrm{Ad}_g^* F\right)(e_i e_j) = F(g^{-1} e_i e_j g) = F((g^{-1} e_i g)(g^{-1} e_j g))$$

the substitution $F \to \mathrm{Ad}_g^* F$ is equivalent to change of basis induced by the matrix Ad_g^{-1}. Applying the result of previous lemma one easily obtains the statement of the theorem. ∎

Theorem 8 **[Extended Cayley theorem]** *Let A and B be two $n \times n$ matrices over an algebraically closed field k and C and D be two $m \times m$ matrices over the same field k. Define $\chi(\lambda, \mu) = \det(\lambda A + \mu B)$. Then*

$$\det(\lambda A \otimes C + \mu B \otimes D) = det(\chi(\lambda C, \mu D))$$

Before proceeding with the proof we must explain in what sense we consider $\chi(\lambda C, \mu D)$. Indeed, matrices C and D might not commute making $\chi(\lambda C, \mu D)$ ambiguous. In our situation the right definition is as follows:

First, we notice that $\chi(\lambda, \mu)$ is homogeneous, thus it can be decomposed into a product of linear forms (k is algebraicly closed):

$$\chi(\lambda, \mu) = \prod_i (\lambda \alpha_i + \mu \beta_j)$$

Then we define

$$\chi(\lambda C, \mu D) = \prod_i (\lambda \alpha_i C + \mu \beta_j D)$$

There is still some ambiguity about the order in which we multiply linear combinations of C and D but it does not affect the value of $\det(\chi(\lambda C, \mu D))$.

Proof **Step 1.** Let A and C be identity matrices of sizes $n \times n$ and $m \times m$ respectively.

Then

$$\chi(\lambda, \mu) = \prod_i (\lambda + \mu \gamma_i)$$

where γ_i are eigenvalues of B.

$$\det(\chi(\lambda, \mu D)) = \det\left(\prod_i (\lambda + \mu \gamma_i D)\right)$$

$$= \prod_i \det(\lambda + \mu\gamma_i D) = \prod_{i,j} (\lambda + \mu\gamma_i\epsilon_j) \tag{2}$$

where ϵ_j are eigenvalues of D.

On the other hand one easily derives that eigenvalues of $B \otimes D$ are $\gamma_i\epsilon_j$ and thus

$$\det(\lambda + \mu B \otimes D) = \prod_{i,j} (\lambda + \mu\gamma_i\epsilon_j) = \det(\chi(\lambda, \mu D))$$

Step 2. Let us assume now only that matrices A and C are invertible. We have

$$\chi(\lambda, \mu) = \det(\lambda A + \mu B) = \det(A)\det(\lambda + \mu A^{-1}B)$$

Denote $\chi'(\lambda, \mu) = \det(\lambda + \mu A^{-1}B) = \prod_i(\lambda + \mu\gamma_i)$, where γ_i are eigenvalues of $A^{-1}B$.

$$\begin{aligned}
\det(\chi(\lambda C, \mu D)) = \det\left(\det(A)\prod_i(\lambda C + \mu\gamma_i D)\right) &= \\
= \det(A)^m \prod_i \det(\lambda C + \mu\gamma_i D) &= \\
= \det(A)^m \prod_i \big(\det(C)\det(\lambda + \mu\gamma_i C^{-1}D)\big) &= \\
= \det(A)^m \det(C)^n \det(\chi'(\lambda, \mu C^{-1}D))
\end{aligned}$$

By *step 1* we have

$$\det(\lambda + \mu(A^{-1}B) \otimes (C^{-1}D)) = \det(\chi'(\lambda, \mu C^{-1}D))$$

Observing also that $\det(A \otimes C) = \det(A)^m \det(C)^n$ we obtain

$$\begin{aligned}
\det(\chi(\lambda C, \mu D)) = \det(A \otimes C)\det(\lambda + \mu(A^{-1}B) \otimes (C^{-1}D)) &= \\
= \det(\lambda A \otimes C + \mu B \otimes D)
\end{aligned}$$

which is the desired formula.

Step 3. To prove the formula in general we observe that the both parts involve only polynomials in entries of matrices A,B,C and D. Since the restriction that A and C be invertible selects a Zariski open subset, the formula should hold for all A,B,C,D by continuity. ∎

Proposition 9 *Let A and B be two $n \times n$ matrices. Let $K = \dim\ker A$ and $M = \dim\ker B$. Then $\det(\lambda A + \mu B)$ is divisible by $\lambda^M \mu^K$.*

Theorem 10 *Let $\mathfrak{A}$ be an associative algebra. Then it's characteristic polynomial is divisible by*

$$(\lambda\mu)^{\dim\ker A}(\lambda + \mu)^{\operatorname{ind}\mathfrak{A}}$$

Proof Indeed, using proposition 9 and the fact that $\dim\ker A = \dim\ker A^T$ the characteristic polynomial $\det(\lambda A + \mu A^T)$ is divisible by $(\lambda\mu)^{\dim\ker A}$.

And in view of

$$\det(\lambda A + \mu A^T) = \det((\lambda + \mu)A - \mu(A - A^T))$$

and

$$\dim\ker(A - A^T) = \operatorname{ind}\mathfrak{A}$$

it is divisible by $(\lambda + \mu)^{\operatorname{ind}\mathfrak{A}}$. ■

Example 2 [**GL**(2)] Let a,b,c,d denote the matrix units $E_{1,1}$,$E_{1,2}$,$E_{2,1}$ and $E_{2,2}$ correspondingly. Then the multiplication table A is

	a	b	c	d
a	a	b	0	0
b	0	0	a	b
c	c	d	0	0
d	0	0	c	d

The characteristic polynomial is equal to

$$\begin{aligned}\chi(\lambda, \mu) &= \det(\lambda A + \mu A^T) = \\ &= -(\lambda + \mu)^2(ad - bc)((\lambda - \mu)^2(ad - bc) + \lambda\mu(a + d)^2))\end{aligned} \qquad (3)$$

We see the absence of factors $\lambda\mu$ which corresponds to the fact that multiplication table A is non-degenerate ($\det A = -(ad - bc)^2$) and the degree of the factor $\lambda + \mu$ is exactly equal to the index of GL(2).

Example 3 Let $\mathfrak{A}$ be the following subalgebra of $\mathrm{Mat}_3(\mathbb{C})$:

$$\begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{pmatrix}$$

The multiplication table A of $\mathfrak{A}$ is

	a	b	c	d	e
a	a	b	0	0	0
b	0	0	b	0	0
c	0	0	c	0	0
d	0	0	d	0	0
e	0	0	0	d	e

The characteristic polynomial is equal to

$$\chi(\lambda, \mu) = \lambda^2 \mu^2 (\lambda + \mu) b^2 d^2 (a + c + e)$$

The degree of the factor $\lambda + \mu$ is equal to the index of $\mathfrak{A}$. The presence of factor $\lambda^2\mu^2$ corresponds to the fact that multiplication table is degenerate.

Algebra $\mathfrak{A}$ from the example 3 facilitates construction of the algebra with null characteristic polynomial. Indeed, by theorem 8 characteristic polynomial of algebra $\mathfrak{A} \otimes \mathfrak{A}$ is equal to

$$\chi_{\mathfrak{A}\otimes\mathfrak{A}}(\lambda, \mu) = \det((\lambda A)^2 (\mu A^T)^2 (\lambda A + \mu A^T) b^2 d^2 (a + c + e)) = 0$$

6 Characteristic polynomial of Mat_n

Definition 9 **[Generalized resultant]** *Let $p(x)$ and $q(x)$ be two polynomials over an algebraicly closed field. We define* generalized resultant *of $p(x)$ and $q(x)$ to be*

$$R(\lambda, \mu) = \prod_{i,j} (\lambda \alpha_i + \mu \beta_j)$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are roots of polynomials $p(x)$ and $q(x)$ respectively.

Generalized resultant is polynomial in two variables. It is easy to show that its coefficients are polynomials in coefficients of $p(x)$ and $q(x)$ so the condition on base field to be algebraicly closed can be omitted.

Theorem 11 *The characteristic polynomial $\chi(\lambda, \mu, F)$ for algebra Mat_n in point $F \in \mathrm{Mat}_n^*$ is equal to the generalized resultant of characteristic polynomial of F (as a matrix) with itself times $(-1)^{\frac{n(n-1)}{2}}$. That is*

$$\chi(\lambda, \mu, F) = (-1)^{\frac{n(n-1)}{2}} \det(F)(\lambda + \mu)^n \prod_{i \neq j} (\lambda\alpha_i + \mu\alpha_j)$$

where α_i are eigenvalues of F (this formulation assumes that the base field is algebraicly closed).

Proof We will make use of theorem 7. The coadjoint action on Mat_n^* is simply conjugation by invertible matrices. The generic orbit consists of diagonalizable matrices. Thus we can compute $\chi(\lambda, \mu)$ by assuming first that F is diagonal and then extrapolating the resulting polynomial to the case of all F.

Assume the base field to be $\mathbb{C}$. Let $F = \mathrm{diag}(\alpha_1, ..., \alpha_n)$. We choose a basis $\{E_{i.j}\}$ of matrix units in the algebra Mat_n. The only case when $F(E_{i,j}E_{k,l})$ is non-zero is when $i = l$ and $j = k$. Thus the multiplication table A (restricted to subspace of diagonal matrices in Mat_n^*) is

	$E_{i,i}$	$E_{i,j}^+$	$E_{i,j}^-$
$E_{i,i}$	$\begin{matrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{matrix}$	0	0
$E_{i,j}^+$	0	0	$\begin{matrix} \alpha_{j'} & & 0 \\ & \ddots & \\ 0 & & \alpha_{j''} \end{matrix}$
$E_{i,j}^-$	0	$\begin{matrix} \alpha_{i'} & & 0 \\ & \ddots & \\ 0 & & \alpha_{i''} \end{matrix}$	0

here $E_{i,j}^+$ denotes elements $E_{i,j}$ with $i > j$ and $E_{i,j}^-$ denotes elements $E_{i,j}$ with $i < j$. The matrix $\lambda A + \mu A^T$ will have $(\lambda + \mu)\alpha_i$ in the $E_{i,i} \times E_{i,i}$ block, and the pair $(E_{i,j}^+, E_{i,j}^-)$ will produce a 2×2 matrix

$$\begin{pmatrix} 0 & \lambda\alpha_j + \mu\alpha_i \\ \lambda\alpha_i + \mu\alpha_j & 0 \end{pmatrix}$$

Computing the determinant yields

$$(-1)^{\frac{n(n-1)}{2}}(\lambda+\mu)^n\prod_i \alpha_i \prod_{i\neq j}(\lambda\alpha_i+\mu\alpha_j) = (-1)^{\frac{n(n-1)}{2}}\prod_{i,j}(\lambda\alpha_i+\mu\alpha_j)$$

thus proving the theorem for the case when F is diagonal. But characteristic polynomial $\det(F-x)$ is invariant under coadjoint action. Thus the expression is true for all F up to a possibly missing factor depending only on F (but not λ or μ) which is quasi-invariant under coadjoint action. However, in view of the fact that this multiple must be polynomial in F and that the degree of the expression above in F is exactly n^2 this multiple must be trivial.

The case of arbitrary field is proved by observing that both sides of the equality are polynomials with integral coefficients and thus if equality holds over $\mathbb{C}$ it should hold over any field. ■

We observe that the maximal degree of factor $(\lambda+\mu)$ in characteristic polynomial of Mat_n is equal exactly to the index of Mat_n. Moreover, in the factorization of $\chi(\lambda,\mu)$ all factors except $(\lambda+\mu)$ depend upon F non-trivially. This allows us to prove the following theorem:

Theorem 12 *Let $\mathfrak{A}$ be a an algebra with the property that the maximal n for which $(\lambda+\mu)^n$ divides $\chi_{\mathfrak{A}}$ is equal to* $\mathrm{ind}\,\mathfrak{A}$*. Then*

$$\mathrm{ind}\,(\mathrm{Mat}_N\otimes\mathfrak{A}) = N\mathrm{ind}\,\mathfrak{A}$$

Proof Indeed, from the theorem 1 we know that $\mathrm{ind}\,(\mathrm{Mat}_N\otimes\mathfrak{A})$ is at least $N\mathrm{ind}\,\mathfrak{A}$. On the other hand theorem 10 states that $\mathrm{ind}\,(\mathrm{Mat}_N\otimes\mathfrak{A})$ cannot exceed the power of the factor $\lambda+\mu$ in characteristic polynomial of $\mathrm{Mat}_N\otimes\mathfrak{A}$. This polynomial can be computed using theorem 8:

$$\begin{aligned}\chi_{\mathrm{Mat}_N\otimes\mathfrak{A}} &= \det\left((\lambda A+\mu A^T)^N \det(F)\prod_{i\neq j}(\lambda\alpha_i A+\mu\alpha_j A^T)\right) = \\ &= \left(\det(\lambda A+\mu A^T)\right)^N \det(F)^{\dim\mathfrak{A}}\prod_{i\neq j}\det(\lambda\alpha_i A+\mu\alpha_j A^T) = \\ &= (\chi_{\mathfrak{A}})^N \det(F)^{\dim\mathfrak{A}}\prod_{i\neq j}\det(\lambda\alpha_i A+\mu\alpha_j A^T)\end{aligned}$$

Since we know that $\mathrm{ind}\,\mathfrak{A}$ is equal to the power of the factor $\lambda+\mu$ the term $(\chi_{\mathfrak{A}})^N$ is divisible by exactly $\lambda+\mu$ to the power $N\mathrm{ind}\,\mathfrak{A}$. The other factors do not contribute since $\det(F)$ does not contain λ or μ and α_i are independent. ■

7 Symmetric form on Stab_F

Definition 10 *Let $\mathfrak{A}$ be an associative algebra. For an element $F \in \mathfrak{A}^*$ we define*

$$\mathrm{Stab}_F = \{a \in \mathfrak{A} : \forall x \in \mathfrak{A} \Rightarrow F(ax) = F(xa)\}$$

Proposition 13 *1. Stab_F is a subalgebra in $\mathfrak{A}$.*

2. If $\mathfrak{A}$ possesses a unity then Stab_F possesses a unity.

Definition 11 *We define the form Q_F on Stab_F by the following formula:*

$$Q_F(a, b) = F(ab)$$

Theorem 14 [**Properties of the form** Q_F] *The form Q_F possesses the following properties:*

1. *Q_F is symmetric.*
2. *Q_F is ad invariant.*

Proof The proof is not difficult:

1.

$$Q_F(a, b) = F(ab) = F(ba) = Q_F(b, a)$$

2.

$$\begin{aligned} Q_F(\mathrm{ad}_c\, a, b) &= F((\mathrm{ad}_c\, a)b) = F((ca - ac)b) = \\ &= F(cab - acb) = F(abc - acb) = \\ &= -F(a(\mathrm{ad}_c\, b)) = -Q_F(a, \mathrm{ad}_c\, b) \end{aligned}$$

■

Theorem 15 *The index of associative algebra $\mathfrak{A}$ is equal to the maximal power N when $(\lambda+\mu)^N$ divides $\chi_{\mathfrak{A}}$ if and only if the quadratic form Q_F is non-degenerate for a generic $F \in \mathfrak{A}^*$.*

Corollary An easy source of examples of associative algebras $\mathfrak{A}$ that satisfy this condition are algebras with unity with index equal to 1. Indeed, the dimension of stabilizer of such algebra is 1 (which is thus generated by unity) and it is not difficult to find generic F that does not vanish on the stabilizer (which is the same for all generic F).

In particular for these algebras $\operatorname{ind}(\mathrm{Mat}_n \otimes \mathfrak{A}) = n$.

Proof We will show that non-degeneracy of Q_F for a particular F implies that the maximal power of $\lambda+\mu$ that divides $\chi(\cdot,\cdot,F)$ (evaluated on F) is equal to $\dim \mathrm{Stab}_F$. This will imply the statement of the theorem in case F is generic.

Let A be the multiplication table of $\mathfrak{A}$ evaluated in F. Let us choose a basis $\{e_i\}$ in Stab_F and complement it with vectors $\{w_j\}$ to the full basis in $\mathfrak{A}$ in such a way that $(A-A^T)$ restricted to $\{w_j\}$ is a non-degenerate skew-symmetric form.

Then

$$\chi(\lambda,\mu,F) = \det(\lambda A + \mu A^T) = \det((\lambda+\mu)A - \mu(A-A^T))$$

The matrix $A-A^T$ is a block matrix with respect to partition of basis in $\{e_i\}$ and $\{w_j\}$. The only non-zero block is the block $\{w_j\}\times\{w_j\}$. Thus the only way for the term

$$(\lambda+\mu)^{\dim \mathrm{Stab}_F}\,\mu^{\dim \mathfrak{A} - \dim \mathrm{Stab}_F}$$

to appear in $\det((\lambda+\mu)A - \mu(A-A^T))$ is when A restricted to $\{e_i\}$ is non-degenerate. But $A|_{\{e_i\}} = Q_F$. ∎

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