QUANTUM CO-ADJOINT ORBITS OF THE REAL DIAMOND GROUP

NGUYEN VIET HAI

Abstract. We present explicit formulas for deformation quantization on the co-adjoint orbits of the real diamond Lie group. From this we obtain quantum half-plans, quantum hyperbolic cylinders, quantum hyperbolic paraboloids via Fedosov deformation quantization and finally, the corresponding unitary representations of this group.

1. Introduction

Let us first recall that it was Hermann Weyl(see [W]), who introduced a mapping from classical observables (i.e. functions on the phase space $\mathbb{R}^{2n}$) to quantum observables (i.e. normal operators in the Hilbert space $L^2(\mathbb{R}^n)$). The idea was to express functions on $\mathbb{R}^{2n}$ as Fourier transforms and then, by using the inverse Fourier transforms to correspond this correspondence to functions on characters, i.e. one-dimensional representations of the Heisenberg group, parameterized by the Planck constant $\hbar$ - and finally to present them as elements in the corresponding infinite-dimensional representations of the Heisenberg group. This profound idea was later retrieved by Moyal, who have seen that the symbols of the commutators or of the products of operators are of the the form of *sine* (or *exponential*) functions of the bidifferential operators (of the Poisson brackets) of the corresponding symbols.

In the early 70's Berezin has treated the general mathematical definition of quantization as a kind of a functor from the category of classical mechanics to a certain category of associative algebras. About the same time as F. A. Berezin, M. Flato, M. G. Fronsdal, F. Bayen, A. Lichnerowicz and D. Sternheimer considered quantization as a deformation of the commutative products of classical observables into a noncommutative $\star$-products which are parameterized by the Planck constant $\hbar$ and satisfy the *correspondence principle*. They systematically developed the notion of deformation quantization as a theory of $\star$-products and gave an independent formulation of quantum mechanics based on this notion (see[RT]).

It was proved by Gerstenhaber that a formal deformation quantization exists on an arbitrary symplectic manifold, see for example [F] for a detailed explaination. It

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is however formal and quite complecate in general. We would like to simplify it in some particular cases.

From the orbit method, it is well-known that coadjoint orbits are homogeneous symplectic manifolds with respect to the natural Kirillov structure form on coadjoint orbits. A natural question is to associate in a reasonable way to these orbits some quantum objects, what could be called *quantum co-adjoint orbits*. In particular, in [DH1] and [DH2] we obtained "quantum half-planes" and "quantum punctured complex planes", associated with the affine transformation groups of the real or complex straightlines. In this paper we will therefore continue to realize the problem for the real diamond Lie group. This group has a lot of nontrivial 2-dimensional coadjoint orbits, which are the half-planes, the hyperbolic cylinders and the hyperbolic paraboloids. We should find out explicit formulas for each of these orbits. Our main result therefore is the fact that by using $\star$-product we can construct the corresponding quantum half-plans, quantum hyperbolic cylinders, quantum hyperbolic paraboloids and by an *exact computation* we can find out explicit $\star$-product formulas and then, the complete list of irreducible unitary representations of this group. It is useful to do here a remark that there is a general theory for exponetial and compact groups. But our consideration concerning with non-exponential and noncompact Lie group and associated G-homogeneous symplectic manifolds.

Let us in few words describe the structure od the paper. We introduce some preliminary results in §2. Then, the adapted chart and in particular, Hamiltonian functions in canonical coordinates of the co-adjoint orbit Ω_F are exposed in §3. The operators $\hat{\ell}_A$ which define the representations of the real diamond Lie algebra are constructed in §4 and finally, by exponentiating them, we obtain the corresponding unitary representations of the real diamond Lie group $\mathbb{R} \ltimes \mathbb{H}_3$.

2. Preliminary results

The so called real diamond Lie algebra is the 4-dimensional solvable Lie algebra $\mathfrak{g}$ with basis X, Y, Z, T satisfying the following commutation relations:

$$[X, Y] = Z, [T, X] = -X, [T, Y] = Y,$$
$$[Z, X] = [Z, Y] = [T, Z] = 0.$$

These relations show that this real diamond Lie algebra $\mathbb{R} \ltimes \mathfrak{h}_3$ is an extension of the one-dimensional Lie algebra $\mathbb{R}T$ by the Heisenberg algebra $\mathfrak{h}_3$ with basis X, Y, Z, where the action of T on Heisenberg algebra $\mathfrak{h}_3$ is defined by the matrix

$$ad_T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We introduce the following notations. The real diamond Lie algebra is isomorphic to $\mathbb{R}^4$ as vector spaces. The coordinates in this standard basis is denote by (a, b, c, d). We identify its dual vector space $\mathfrak{g}^*$ with $\mathbb{R}^4$ with the help of the dual basis X^*, Y^*, Z^*, T^* and with the local coordinates as $(\alpha, \beta, \gamma, \delta)$. Thus, the general form of an element of $\mathfrak{g}$ is $U = aX + bY + cZ + dT$ and the general form of an element of $\mathfrak{g}^*$ is $F = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$. The co-adjoint action of $G = \mathbb{R} \ltimes \mathbb{H}_3$ on $\mathfrak{g}^*$ is given (see e.g. [Ki1]) by

$$\langle K(g)F, U\rangle = \langle F, \mathrm{Ad}(g^{-1})U\rangle, \quad \forall F \in \mathfrak{g}^*, g \in G \text{ and } U \in \mathfrak{g} = Lie(\mathbb{R} \ltimes \mathbb{H}_3).$$

Denote the co-adjoint orbit of G in $\mathfrak{g}$, passing through F by

$$\Omega_F = K(G)F := \{K(g)F | g \in G\}.$$

By a direct computation one obtains (see [D]):

- Each point of the line $\alpha = \beta = \gamma = 0$ is a 0-dimensional co-adjoint orbit

(1) $$\Omega^1 = \Omega_{(0,0,0,\delta)}.$$

- The set $\alpha \neq 0, \beta = \gamma = 0$ is union of 2-dimensional co-adjoint orbits ,which are just the **half-planes**

(2) $$\Omega^2 = \{(x, 0, 0, t) \quad | \quad x, t \in \mathbb{R}, \alpha x > 0\}.$$

- The set $\alpha = \gamma = 0, \beta \neq 0$ is a union of 2-dimensional co-adjoint orbits, which are **half-planes**

(3) $$\Omega^3 = \{(0, y, 0, t) \quad | \quad y, t \in \mathbb{R}, \beta y > 0\}.$$

- The set $\alpha\beta \neq 0, \gamma = 0$ is decomposed into a family of 2-dimensional co-adjoint orbits, which are **hyperbolic cylinders**

(4) $$\Omega^4 = \{(x, y, 0, t) \quad |x, y, t \in \mathbb{R} \quad \& \quad \alpha x > 0, \beta y > 0, xy = \alpha\beta\}.$$

- The open set$\gamma \neq 0$ is decomposed into a family of 2-dimensional co-adjoint orbits ,which are just the **hyperbolic paraboloids**

(5) $$\Omega^5 = \{(x, y, \gamma, t) \quad |x, y, t \in \mathbb{R} \quad \& \quad xy - \alpha\beta = \gamma(t - \delta)\}.$$

Thus, the real diamond Lie algebra belongs to the class of MD_4-algebras , i.e. every K-orbit of the corresponding Lie group has dimension 0 or maximal (see [D]).

Let us consider now the problem of deformation quantization on half-planes, hyperbolic cylinders, hyperbolic paraboloids. In order to do this, we shall construct on each of these orbits a canonical Darboux coordinate system (p, q) and a class of Hamiltonian functions in these coordinates.

3. Hamiltonian functions in canonical coordinates of the orbits Ω_F

Each element $A \in \mathfrak{g}$ can be considered as the restriction of the corresponding linear functional $\tilde{A}$ onto co-adjoint orbits, considered as a subset of g^* ,$\tilde{A}(F) = \langle F, A\rangle$. It is well-known that this function is just the Hamiltonnian function, associated with the Hamiltonian vector field ξ_A, defined by the formula

$$(\xi_A f)(x) := \frac{d}{dt} f(x \exp(tA))|_{t=0}, \forall f \in C^\infty(\Omega_F).$$

It is well-known the relation $\xi_A(f) = \{\tilde{A}, f\}, \forall f \in C^\infty(\Omega_F)$. Denote by ψ the symplectomorphism from $\mathbb{R}^2$ onto Ω_F

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) \in \Omega_F,$$

we have:

Proposition 3.1. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p, q) of the orbit Ω_F is of the form*

$$\tilde{A} \circ \psi(p, q) = \begin{cases} dp + a\alpha e^{-q}, & \text{if} \quad \Omega_F = \Omega^2 \\ dp + b\beta e^{q}, & \text{if} \quad \Omega_F = \Omega^3 \\ dp + a\alpha e^{-q} + b\beta e^{q}, & \text{if} \quad \Omega_F = \Omega^4 \\ (d \pm b\gamma e^{q})p \pm a e^{-q} \pm b(\alpha\beta - \gamma\delta)e^{q} + c\gamma, & \text{if} \quad \Omega_F = \Omega^5 \end{cases}$$

2. In the canonical coordinates (p, q) of the orbit Ω_F, the Kirillov form ω is coincided with the standard form $dp \wedge dq$.

Proof. 1. We adapt the diffeomorphism ψ to each of the following cases (for 2-dimensional co-adjoint orbits, only)

- With $\alpha \neq 0, \beta = \gamma = 0$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (\alpha e^{-q}, 0, 0, p) \in \Omega^2$$

Element $F \in \mathfrak{g}^*$ is of the form $F = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*$, hence the value of the function $f_A = \tilde{A}$ on the element $A = aX + bY + cZ + dT$ is $\tilde{A}(F) = \langle F, A \rangle =$

$$\langle \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*, aX + bY + cZ + dT \rangle = \alpha a + \beta b + \gamma c + \delta d.$$

It follows that

$$\tilde{A} \circ \psi(p, q) = a\alpha e^{-q} + dp, \tag{6}$$

- With $\alpha = \gamma = 0, \beta \neq 0$,

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (0, \beta e^{q}, 0, p) \in \Omega^3.$$

$\tilde{A}(F) = \langle F, A \rangle = \alpha a + \beta b + \gamma c + \delta d$. From this,

$$\tilde{A} \circ \psi(p, q) = b\beta e^{q} + dp \tag{7}$$

- With $\alpha\beta \neq 0, \gamma = 0$,

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (\alpha e^{-q}, \beta e^{q}, 0, p) \in \Omega^4.$$

$$\tilde{A} \circ \psi(p, q) = a\alpha e^{-q} + b\beta e^{q} + dp \tag{8}$$

- At last, if $\gamma \neq 0$, we consider the orbit with the first coordinate $x > 0$

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (e^{-q}, (\alpha\beta + \gamma p - \gamma\delta)e^{q}, \gamma, p) \in \Omega^5.$$

We have

$$\tilde{A} \circ \psi(p, q) = a e^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^{q} + c\gamma + dp = \tag{9}$$

$$= (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma.$$

The case $x < 0$ is similarly treated:

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (-e^{-q}, -(\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \Omega^5.$$

$$\tilde{A} \circ \psi(p, q) = -ae^{-q} - b(\alpha\beta + \gamma p - \gamma\delta)e^q + c\gamma + dp = \tag{10}$$

$$= (d - b\gamma e^q)p - ae^{-q} - b(\alpha\beta - \gamma\delta)e^q + c\gamma.$$

2. We consider only the following case (the rest are similar):

$$(p, q) \in \mathbb{R}^2 \mapsto \psi(p, q) = (e^{-q}, (\alpha\beta + \gamma p - \gamma\delta)e^q, \gamma, p) \in \Omega^5.$$

$$\tilde{A} \circ \psi(p, q) = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma.$$

In canonical Darboux coordinates (p, q) ,

$$F' = e^{-q}X^* + (\alpha\beta + \gamma p - \gamma\delta)e^q Y^* + \gamma Z^* + pT^* \quad \in \Omega^5,$$

and for $A = aX + bY + cZ + dT, \quad B = a'X + b'Y + c'Z + d'T$, we have $\langle F', [A, B]\rangle =$ $= \langle e^{-q}X^* + (\alpha\beta + \gamma p - \gamma\delta)e^q Y^* + \gamma Z^* + pT^*, (ad' - da')X + (db' - bd')Y + (ab' - ba')Z\rangle$.

It follows therefore that

$$\langle F', [A, B]\rangle = (ad' - da')e^{-q} + (db' - bd')(\alpha\beta + \gamma p - \gamma\delta)e^q + \gamma(ab' - ba'). \tag{11}$$

On the other hand,

$$\xi_A(f) = \{\tilde{A}, f\} = (d + b\gamma e^q)\frac{\partial f}{\partial q} - [-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial f}{\partial p}$$

$$\xi_B(f) = \{\tilde{B}, f\} = (d' + b'\gamma e^q)\frac{\partial f}{\partial q} - [-a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial f}{\partial p}.$$

From this, consider two vector fields

$$\xi_A = (d + b\gamma e^q)\frac{\partial}{\partial q} - [-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial}{\partial p},$$

$$\xi_B = (d' + b'\gamma e^q)\frac{\partial}{\partial q} - [-a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial}{\partial p}.$$

We have

$$\xi_A \otimes \xi_B = (d + b\gamma e^q)(d' + b'\gamma e^q)\frac{\partial}{\partial q} \otimes \frac{\partial}{\partial q} + \tag{12}$$

$$+[(ad' - da')e^{-q} + (db' - d'b)(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q} +$$

$$+[-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q][-a'e^{-q} + b'(\alpha\beta + \gamma p - \gamma\delta)e^q]\frac{\partial}{\partial p} \otimes \frac{\partial}{\partial p}$$

From (11) and (12) we conclude that in the canonical coordinates the Kirillov form is just the standard symplectic form $\omega = dp \wedge dq$. The proposition is therefore proved. □

DEFINITION 3.2. *Each chart ψ^{-1} on Ω_F which satisfy 1. and 2. of proposition 3.1 is called an adapted chart on Ω_F.*

In the next section we shall see that each adapted chart carries the Moyal $\star$-product from $\mathbb{R}^2$ onto Ω_F.

4. Moyal $\star$-product and representations of $G = \mathbb{R} \ltimes \mathbb{H}_3$.

Let us denote by Λ the 2-tensor associated with the Kirillov standard form $\omega = dp \wedge dq$ in canonical Darboux coordinates. Let us consider the well-known Moyal $\star$-product of two smooth functions $u, v \in C^\infty(\mathbb{R}^{2n})$ (see e.g [AC1],[DH1]), defined by

$$u \star v = u.v + \sum_{r \geq 1} \frac{1}{r!}(\frac{1}{2i})^r P^r(u, v),$$

where

$$P^1(u, v) = \{u, v\}$$

$$P^r(u, v) := \Lambda^{i_1 j_1}\Lambda^{i_2 j_2} \dots \Lambda^{i_r j_r} \partial^r_{i_1 i_2 \dots i_r} u \partial^r_{j_1 j_2 \dots j_r} v,$$

with

$$\partial^r_{i_1 i_2 \dots i_r} := \frac{\partial^r}{\partial x^{i_1} \dots \partial x^{i_r}}; \quad x := (p, q) = (p_1, \dots, p_n, q^1, \dots, q^n)$$

using multi-index notation. It is well-known that this series converges in the Schwartz distribution spaces $\mathcal{S}(\mathbb{R}^{2n})$. Furthermore, it was obtained the results (see e.g [AC1]): If $u, v \in \mathcal{S}(\mathbb{R}^{2n})$, then

- $\overline{u} \star \overline{v} = \overline{v \star u}$
- $\int (u \star v)(\xi) d\xi = \int uv d\xi$
- $\ell_u : \quad \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$, defined by $\ell_u(v) = u \star v$ is continuous in $L^2(\mathbb{R}^{2n}, d\xi)$ and then can be extended to a bounded linear operator (still denoted by ℓ_u) on $L^2(\mathbb{R}^{2n}, d\xi)$.

We apply this to the special case $n = 1$, $x = (x^1, x^2) = (p, q)$

Proposition 4.1. *In the above mentioned canonical Darboux coordinates (p, q) on the orbit Ω_F, the Moyal $\star$-product satisfies the relation*

$$i\tilde{A} \star i\tilde{B} - i\tilde{B} \star i\tilde{A} = i\widetilde{[A, B]}, \forall A, B \in \mathfrak{g} = Lie(\mathbb{R} \ltimes \mathbb{H}_3).$$

Proof. We prove the proposition for the orbit Ω^5, $\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma$ (the other cases are proved similar). Consider the elements $A = aX + bY + cZ + dT, \quad B = a'X + b'Y + c'Z + d'T,$. Then as said above, the corresponding Hamiltonian functions are

$$\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma$$

$$\tilde{B} = (d' + b'\gamma e^q)p + a'e^{-q} + b'(\alpha\beta - \gamma\delta)e^q + c'\gamma$$

It is easy then to see that

$$\begin{aligned}
&P^0(\tilde{A},\tilde{B}) = \tilde{A}.\tilde{B}\\
&P^1(\tilde{A},\tilde{B}) = \{\tilde{A},\tilde{B}\} = \partial_p\tilde{A}\partial_q\tilde{B} - \partial_q\tilde{A}\partial_p\tilde{B} =\\
&= (d+b\gamma e^q)[-a'e^{-q} + b'(\alpha\beta+\gamma p-\gamma\delta)e^q] -\\
&-(d'+b'\gamma e^q)[-ae^{-q} + b(\alpha\beta+\gamma p-\gamma\delta)e^q] =\\
&= [(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-ba')\gamma]\\
&P^2(\tilde{A},\tilde{B}) = \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\tilde{B} + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\tilde{B} + \Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\tilde{B} +\\
&+\Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\tilde{B} = -2bb'\gamma^2e^{2q}\\
&P^3(\tilde{A},\tilde{B}) = \Lambda^{12}\Lambda^{12}\Lambda^{12}\partial^3_{ppp}\tilde{A}\partial^3_{qqq}\tilde{B} + \Lambda^{12}\Lambda^{12}\Lambda^{21}\partial^3_{ppq}\tilde{A}\partial^3_{qqp}\tilde{B} +\\
&+\Lambda^{12}\Lambda^{21}\Lambda^{12}\partial^3_{pqp}\tilde{A}\partial^3_{qpq}\tilde{B} + \Lambda^{21}\Lambda^{12}\Lambda^{12}\partial^3_{qpp}\tilde{A}\partial^3_{pqq}\tilde{B} +\\
&+\Lambda^{21}\Lambda^{21}\Lambda^{12}\partial^3_{qqp}\tilde{A}\partial^3_{ppq}\tilde{B} + \Lambda^{21}\Lambda^{12}\Lambda^{21}\partial^3_{qpq}\tilde{A}\partial^3_{pqp}\tilde{B} +\\
&+\Lambda^{12}\Lambda^{21}\Lambda^{21}\partial^3_{pqq}\tilde{A}\partial^3_{qpp}\tilde{B} + \Lambda^{21}\Lambda^{21}\Lambda^{21}\partial^3_{qqq}\tilde{A}\partial^3_{ppp}\tilde{B} = 0.
\end{aligned}$$

By analogy, we have

$$P^k(\tilde{A},\tilde{B}) = 0, \forall k \geq 4.$$

Thus,

$$\begin{aligned}
i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A} &= \frac{1}{2i}[P^1(i\tilde{A},i\tilde{B}) - P^1(i\tilde{B},i\tilde{A})]\\
&= i[(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-a'b)\gamma].
\end{aligned}$$

On the other hand, as

$$\begin{aligned}
[A,B] &= [aX+bY+cZ+dT, a'X+b'Y+c'Z+d'T]\\
&= (ad'-da')X + (db'-d'b)Y + (ab'-a'b)Z
\end{aligned}$$

we obtain

$$\begin{aligned}
&i[(ad'-da')e^{-q} + (db'-d'b)(\alpha\beta+\gamma p-\gamma\delta)e^q + (ab'-a'b)\gamma]\\
&= i\widetilde{[A,B]} = i\tilde{A}\star i\tilde{B} - i\tilde{B}\star i\tilde{A}.
\end{aligned}$$

The proposition is hence proved. □

Consequently, to each adapted chart, we associate a G-covariant $\star$-product.Then there exists a representation τ of G in Aut $N[[\nu]]$,(see [G]) such that (here $\nu = \frac{i}{2}$):

$$\tau(g)(u\star v) = \tau(g)u\star\tau(g)v.$$

For each $A \in Lie(\mathbb{R}\ltimes\mathbb{H}_3)$, the corresponding Hamiltonian function is $\tilde{A}$ and we can put $\ell_A(u) = i\tilde{A}\star u$,$u \in L^2(\mathbb{R}^2, \frac{dpdq}{2\pi})^\infty$. It is then continuated to the whole space $L^2(\mathbb{R}^2, \frac{dpdq}{2\pi})$. Because of the relation in Proposition (4.1), we have

Corollary 4.2.

(13) $$\ell_{[A,B]} = \ell_A\star\ell_B - \ell_B\star\ell_A := [\ell_A,\ell_B]^\star$$

This implies that the correspondence $A \in Lie(\mathbb{R} \ltimes \mathbb{H}_3) \mapsto \ell_A = i\tilde{A} \star .$ is a representation of the Lie algebra $Lie(\mathbb{R} \ltimes \mathbb{H}_3)$ on the space $N[[\frac{i}{2}]]$ of formal power series in the parameter $\nu = \frac{i}{2}$(i.e $\hbar = 1$) with coefficients in $N = C^\infty(M, \mathbb{R})$ [G].

Let us denote by $\mathcal{F}_p(f)$ the partial Fourier transform of the function f from the variable p to the variable x(see e.g[MV]), i.e.

$$\mathcal{F}_p(f)(x,q) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipx} f(p,q) dp.$$

Let us denote by $\mathcal{F}_p^{-1}(f)(p,q)$ the inverse Fourier transform.

Lemma 4.3. *1.*$\partial_p \mathcal{F}_p^{-1}(f) = i\mathcal{F}_p^{-1}(x.f)$
2.$\mathcal{F}_p(p.v) = i\partial_x \mathcal{F}_p(v)$
3.$\forall k \geq 2$ *,then* $P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) =$

$$= \begin{cases} a\alpha e^{-q} \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by (6)} \\ (-1)^k b\beta e^q \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by (7)} \\ [a\alpha e^{-q} + (-1)^k b\beta e^q] \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by (8)} \\ (-1)^{k-1} k.b\gamma e^q \partial^k_{qp\dots p} \mathcal{F}_p^{-1}(f) + & \\ +[ae^{-q} + (-1)^k b(\alpha\beta + \gamma p - \gamma\delta)e^q] \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f) & \text{if } \tilde{A} \text{ is defined by (9)} \end{cases}$$

Proof. The first two formulas are well-known from theory of Fourier transforms.

Let us prove 3. Remark that $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the standard symplectic Darboux coordinates (p,q) on the orbit Ω_F, then

- If $\tilde{A} = a\alpha e^{-q} + dp$

$$P^2(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \Lambda^{12}\Lambda^{12} \partial^2_{pp}\tilde{A} \partial^2_{qq} \mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21} \partial^2_{pq}\tilde{A} \partial^2_{qp} \mathcal{F}_p^{-1}(f)) +$$
$$\Lambda^{21}\Lambda^{12} \partial^2_{qp}\tilde{A} \partial^2_{pq} \mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21} \partial^2_{qq}\tilde{A} \partial^2_{pp} \mathcal{F}_p^{-1}(f)) = a\alpha e^{-q} \partial^2_{pp} \mathcal{F}_p^{-1}(f) =$$
$$P^3(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^6 a\alpha e^{-q} \partial^3_{ppp} \mathcal{F}_p^{-1}(f) = a\alpha e^{-q} \partial^3_{ppp} \mathcal{F}_p^{-1}(f)$$

  and $P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = a\alpha e^{-q} \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f) \quad \forall k \geq 4,$
- If $\tilde{A} = b\beta e^q + dp.$

$$P^k(\tilde{A}, \mathcal{F}_p^{-1}(f)) = (-1)^k b\beta e^q \partial^k_{p\dots p} \mathcal{F}_p^{-1}(f)$$

  with $\quad \forall k \geq 2$
- If $\tilde{A} = a\alpha e^{-q} + b\beta e^q + dp,$

$$P^2(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \Lambda^{12}\Lambda^{12} \partial^2_{pp}\tilde{A} \partial^2_{qq} \mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21} \partial^2_{pq}\tilde{A} \partial^2_{qp} \mathcal{F}_p^{-1}(f)) +$$
$$\Lambda^{21}\Lambda^{12} \partial^2_{qp}\tilde{A} \partial^2_{pq} \mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21} \partial^2_{qq}\tilde{A} \partial^2_{pp} \mathcal{F}_p^{-1}(f)) =$$
$$= [a\alpha e^{-q} + (-1)^2 b\beta e^q] \partial^2_{pp} \mathcal{F}_p^{-1}(f)$$
$$P^3(\tilde{A}, \mathcal{F}_p^{-1}(f)) = [a\alpha e^{-q} + (-1)^3 b\beta e^q] \partial^3_{ppp} \mathcal{F}_p^{-1}(f)$$

By analogy we have

$$P^k(\tilde{A},\mathcal{F}_p^{-1}(f)) = [a\alpha e^{-q} + (-1)^k b\beta e^q]\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f)), \quad \forall \quad k\geq 3.$$

- If $\tilde{A} = (d + b\gamma e^q)p + ae^{-q} + b(\alpha\beta - \gamma\delta)e^q + c\gamma$,

$$P^2(\tilde{A},\mathcal{F}_p^{-1}(f)) = \Lambda^{12}\Lambda^{12}\partial^2_{pp}\tilde{A}\partial^2_{qq}\mathcal{F}_p^{-1}(f)) + \Lambda^{12}\Lambda^{21}\partial^2_{pq}\tilde{A}\partial^2_{qp}\mathcal{F}_p^{-1}(f)) +$$
$$\Lambda^{21}\Lambda^{12}\partial^2_{qp}\tilde{A}\partial^2_{pq}\mathcal{F}_p^{-1}(f)) + \Lambda^{21}\Lambda^{21}\partial^2_{qq}\tilde{A}\partial^2_{pp}\mathcal{F}_p^{-1}(f)) =$$
$$= (-1)2.b\gamma e^q\partial_{qp}\mathcal{F}_p^{-1}(f) + [ae^{-q} + (-1)^2 b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial^2_{pp}\mathcal{F}_p^{-1}(f).$$
$$P^3(\tilde{A},\mathcal{F}_p^{-1}(f)) = (-1)^2.3b\gamma e^q\partial_{qpp}\mathcal{F}_p^{-1}(f) +$$
$$+[ae^{-q} + (-1)^3 b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial^3_{ppp}\mathcal{F}_p^{-1}(f).$$

From this we also obtain :

$$P^k(\tilde{A},\mathcal{F}_p^{-1}(f)) =$$
$$(-1)^{k-1}.k.b\gamma e^q\partial^k_{qp\ldots p}\mathcal{F}_p^{-1}(f) +$$
$$+[ae^{-q} + (-1)^k b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial^k_{p\ldots p}\mathcal{F}_p^{-1}(f). \quad \forall \quad k\geq 3$$

The lemma is therefore proved. □

We study now the convergence of the formal power series. In order to do this, we look at the $\star$-product of $i\tilde{A}$ as the $\star$-product of symbols and define the differential operators corresponding to $i\tilde{A}$.

Theorem 4.4. *For each $A \in Lie(\mathbb{R}\ltimes\mathbb{H}_3)$ and for each compactly supported C^∞ function $f \in C_0^\infty(\mathbb{R}^2)$, putting $\hat{\ell}_A(f) := \mathcal{F}_p\circ\ell_A\circ\mathcal{F}_p^{-1}(f)$,we have*

$$\hat{\ell}_A(f) = \begin{cases} [d(\frac{1}{2}\partial_q - \partial_x) + ia\alpha e^{-(q-\frac{x}{2})}]f & \text{if } \tilde{A} \text{ is defined by (6)} \\ [d(\frac{1}{2}\partial_q - \partial_x) + ib\beta e^{(q-\frac{x}{2})}]f & \text{if } \tilde{A} \text{ is defined by (7)} \\ [d(\frac{1}{2}\partial_q - \partial_x) + i(a\alpha e^{-(q-\frac{x}{2})} + b\beta e^{(q-\frac{x}{2})})]f & \text{if } \tilde{A} \text{ is defined by (8)} \\ [(d + b\gamma e^{q-\frac{x}{2}})(\frac{1}{2}\partial_q - \partial_x)]f + & \\ +i[ae^{-(q-\frac{x}{2})} + b(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + c\gamma]f & \text{if } \tilde{A} \text{ is defined by (9)} \\ [(d - b\gamma e^{q-\frac{x}{2}})(\frac{1}{2}\partial_q - \partial_x)]f + & \\ +i[-ae^{-(q-\frac{x}{2})} - b(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + c\gamma]f & \text{if } \tilde{A} \text{ is defined by (10)} \end{cases}$$

Proof. Applying Lemma (4.3),we have :

1. If $\tilde{A} = a\alpha e^{-q} + dp$ then

$$\hat{\ell}_A(f) := \mathcal{F}_p\circ\ell_A\circ\mathcal{F}_p^{-1}(f) = \mathcal{F}_p(i\tilde{A}\star\mathcal{F}_p^{-1}(f)) = i\mathcal{F}_p\left(\sum_{r\geq 0}\left(\frac{1}{2i}\right)^r\frac{1}{r!}P^r(\tilde{A},\mathcal{F}_p^{-1}(f))\right) =$$

$$= i\mathcal{F}_p\Big\{(a\alpha e^{-q} + dp)\mathcal{F}_p^{-1}(f) + \tfrac{1}{1!}\tfrac{1}{2i}[d\partial_q\mathcal{F}_p^{-1}(f) + a\alpha e^{-q}\partial_p\mathcal{F}_p^{-1}(f)] +$$
$$+\tfrac{1}{2!}(\tfrac{1}{2i})^2.a\alpha e^{-q}\partial_p p^2\mathcal{F}_p^{-1}(f) + \cdots + \tfrac{1}{r!}(\tfrac{1}{2i})^r a\alpha e^{-q}\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f) + \ldots\Big\} =$$
$$= i\Big\{a\alpha e^{-q}f + d\mathcal{F}_p(p.\mathcal{F}_p^{-1}(f)) + \tfrac{1}{1!}\tfrac{1}{2i}[d\partial_q f + a\alpha e^{-q}\mathcal{F}_p(\partial_p\mathcal{F}_p^{-1}(f))] +$$
$$+\tfrac{1}{2!}(\tfrac{1}{2i})^2.a\alpha e^{-q}\mathcal{F}_p(\partial^2_{pp}\mathcal{F}_p^{-1}(f)) + +\tfrac{1}{3!}(\tfrac{1}{2i})^3.a\alpha e^{-q}\mathcal{F}_p(\partial^3_{ppp}\mathcal{F}_p^{-1}(f)) + \ldots$$
$$+\tfrac{1}{r!}(\tfrac{1}{2i})^r.a\alpha e^{-q}\mathcal{F}_p(\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f)) + \ldots\Big\} =$$
$$= d(\tfrac{1}{2}\partial_q - \partial_x)f + ia\alpha e^{-q}[1 + \tfrac{x}{2} + \tfrac{1}{2!}(\tfrac{x}{2})^2 + \cdots + \tfrac{1}{r!}(\tfrac{1}{x})^r + \ldots]f =$$
$$= d(\tfrac{1}{2}\partial_q - \partial_x)f + ia\alpha e^{-q}e^{\frac{x}{2}}f = d(\tfrac{1}{2}\partial_q - \partial_x)f + ia\alpha e^{-(q-\frac{x}{2})}f$$

2. If $\tilde{A} = b\beta e^q + dp$ then

$$\hat{\ell}_A(f) = d(\frac{1}{2}\partial_q - \partial_x)f + ib\beta e^{q-\frac{x}{2}}f$$

3. For each $\tilde{A} = a\alpha e^{-q} + b\beta e^q + dp$, we have:

$$\hat{\ell}_A = i\mathcal{F}_p\Big\{(a\alpha e^{-q} + b\beta e^q + dp)\mathcal{F}_p^{-1}(f) + \tfrac{1}{2i}[d\partial_q\mathcal{F}_p^{-1}(f) - (-a\alpha e^{-q} +$$
$$+b\beta e^q)\partial_p\mathcal{F}_p^{-1}(f)] + \tfrac{1}{2!}(\tfrac{1}{2i})^2[a\alpha e^{-q} + (-1)^2 b\beta e^q]\partial^2_{pp}\mathcal{F}_p^{-1}(f) + \cdots +$$
$$+\tfrac{1}{r!}(\tfrac{1}{2i})^r[a\alpha e^{-q} + (-1)^r b\beta e^q]\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f) + \ldots\Big\}$$
$$= ia\alpha e^{-q}.f + id\mathcal{F}_p\big(p.\mathcal{F}_p^{-1}(f)\big) + ib\beta e^q f + \tfrac{1}{2}d\partial_q f + \tfrac{1}{2}a\alpha e^{-q}\mathcal{F}_p\big(\partial_p\mathcal{F}_p^{-1}(f)\big) -$$
$$-\tfrac{1}{2}b\beta e^q\mathcal{F}_p\big(\partial_p\mathcal{F}_p^{-1}(f)\big) + \ldots i\tfrac{1}{r!}(\tfrac{1}{2i})^r a\alpha e^{-q}\mathcal{F}_p\big(\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f)\big) +$$
$$+i\tfrac{1}{r!}(\tfrac{-1}{2i})^r b\beta e^q\mathcal{F}_p\big(\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f)\big) + \cdots = d(\tfrac{1}{2}\partial_q - \partial_x) + ia\alpha e^{-q}[1 + \tfrac{x}{2} + \ldots$$
$$+\tfrac{1}{r!}(\tfrac{x}{2})^r + \ldots] + ib\beta e^q[1 + (\tfrac{-x}{2}) + \cdots + \tfrac{1}{r!}(\tfrac{-x}{2})^r + \ldots]$$
$$= d(\tfrac{1}{2}\partial_q - \partial_x) + i[a\alpha e^{-(q-\frac{x}{2})} + b\beta e^{q-\frac{x}{2}}]f.$$

4. For each $\tilde{A}$ is as in (9) , remark that

$$P^0(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \tilde{A}.\mathcal{F}_p^{-1}(f);$$

$$P^1(\tilde{A}, \mathcal{F}_p^{-1}(f)) = \{\tilde{A}, \mathcal{F}_p^{-1}(f)\} =$$

$$(d + b\gamma e^q)\partial_q\mathcal{F}_p^{-1}(f) - [-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial_p\mathcal{F}_p^{-1}(f)$$

and applying Lemma (4.3), we obtain:

$$\hat{\ell}_A(f) = i\Big\{\mathcal{F}_p\Big([dp + ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q + c\gamma]\mathcal{F}_p^{-1}(f)\Big) +$$
$$+\tfrac{1}{2i}\tfrac{1}{1!}\mathcal{F}_p\Big([d + b\gamma e^q]\partial_q\mathcal{F}_p^{-1}(f) - [-ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial_p\mathcal{F}_p^{-1}(f)\Big) +$$
$$+(\tfrac{1}{2i})^2\tfrac{1}{2!}\mathcal{F}_p\Big(-2b\gamma e^q\partial^2_{pq}\mathcal{F}_p^{-1}(f) + [ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\partial^2_{pp}(\mathcal{F}_p^{-1}(f)\Big) + \ldots$$
$$+(\tfrac{1}{2i})^r\tfrac{1}{r!}\mathcal{F}_p\Big((-1)^{r-1}rb\gamma e^q\partial^r_{p\ldots pq}\mathcal{F}_p^{-1}(f) + (-1)^r[(-1)^r ae^{-q} + b(\alpha\beta + \gamma p - \gamma\delta)e^q]\times$$
$$\times\partial^r_{p\ldots p}\mathcal{F}_p^{-1}(f)\Big) + \ldots\Big\} =$$

$$= i\Big\{ae^{-q}f + b(\alpha\beta - \gamma\delta)e^q f + d\mathcal{F}_p\big(p\mathcal{F}_p^{-1}(f)\big) + b\gamma e^q\mathcal{F}_p\big(p\mathcal{F}_p^{-1}(f)\big)$$
$$+\tfrac{1}{2i}\tfrac{1}{1!}(d + b\gamma e^q)\partial_q f - \tfrac{1}{2i}[-ae^{-q}ixf + b(\alpha\beta - \gamma\delta)e^q ixf + b\gamma e^q\mathcal{F}_p\big(p\mathcal{F}_p^{-1}(xf)\big)] +$$
$$+(\tfrac{1}{2i})^2\tfrac{1}{2!}(-2b\gamma e^q)\mathcal{F}_p\left(\partial^2_{pq}\mathcal{F}_p^{-1}(f)\right) + (\tfrac{1}{2i})^2\tfrac{1}{2!}[ae^{-q}(ix)^2 f + b(\alpha\beta - \gamma\delta)e^q(ix)^2 f +$$
$$+b\gamma e^q\mathcal{F}_p\left(pi^2\mathcal{F}_p^{-1}(x^2 f)\right)] + \cdots + (\tfrac{1}{2i})^r\tfrac{1}{r!}(-1)^{r-1}rb\gamma e^q\partial^r_{p\ldots pq}\mathcal{F}_p^{-1}(f)$$
$$+(\tfrac{1}{2i})^r\tfrac{1}{r!}[ae^{-q}(ix)^r f + (-1)^r b(\alpha\beta - \gamma\delta)e^q(ix)^r f + b\gamma e^q\mathcal{F}_p\left(p(ix)^r\mathcal{F}_p^{-1}(f)\right)] + \ldots\Big\}$$
$$= i[ae^{-q}(1 + \tfrac{1}{2!}\tfrac{x}{2} + \cdots + \tfrac{1}{r!}(\tfrac{x}{2})^r\ldots)f] + i[b(\alpha\beta - \gamma\delta)e^q(1 - \tfrac{1}{2!}\tfrac{x}{2} + \cdots +$$
$$+(-1)^r\tfrac{1}{r!}(\tfrac{x}{2})^r\ldots)f] + ic\gamma f + i^2 d\partial_x f + \tfrac{1}{2}d\partial_q f + ib\gamma e^q[i\partial_x f - \tfrac{1}{2i}\mathcal{F}_p\left(pi\mathcal{F}_p^{-1}(xf)\right) +$$
$$+\cdots + (\tfrac{1}{2i})^r\tfrac{1}{r!}(-1)^r\mathcal{F}_p\left(pi^r\mathcal{F}_p^{-1}(x^r f)\right) + \ldots] = d(\tfrac{1}{2}\partial_q - \partial_x)f +$$
$$+[iae^{-(q-\frac{x}{2})} + ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}}]f + ic\gamma f + \tfrac{1}{2}e^{-\frac{x}{2}}b\gamma e^q\partial_q f - b\gamma e^q e^{-\frac{x}{2}}\partial_x f$$
$$= (d + b\gamma e^{q-\frac{x}{2}})(\tfrac{1}{2}\partial_q - \partial_x)f + [iae^{-(q-\frac{x}{2})} + ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + ic\gamma]f$$

5. At last, if $\tilde{A}$ is defined by (10) then :

$$\hat{\ell}_A(f) = (d - b\gamma e^{q-\frac{x}{2}})(\frac{1}{2}\partial_q - \partial_x)f + [-iae^{-(q-\frac{x}{2})} - ib(\alpha\beta - \gamma\delta)e^{q-\frac{x}{2}} + ic\gamma]f$$

The theorem is therefore proved. □

Remark 4.5. Setting new variables $s = q - \frac{x}{2}$, $t = q + \frac{x}{2}$, we have

$$\hat{\ell}_A(f) = \begin{cases} \big(d\partial_s + ia\alpha e^{-s}\big)f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (6)} \\ \big(d\partial_s + ib\beta e^{s}\big)f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (7)} \\ \big(d\partial_s + i[a\alpha e^{-s} + b\beta e^s]\big)f|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (8)} \\ \Big((d + b\gamma e^s)\partial_s + & \\ i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)f|_{(s,t)}. & \text{if } \tilde{A} \text{ is defined by (9)} \\ \Big((d - b\gamma e^s)\partial_s + & \\ i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)f|_{(s,t)}. & \text{if } \tilde{A} \text{ is defined by (10)} \end{cases}$$

Theorem 4.6. *With above notations we obtain the operators :*

$$\hat{\ell}_A = \begin{cases} \hat{\ell}_A^{(2)} = \big(d\partial_s + ia\alpha e^{-s}\big)|_{(s,t)} \\ \hat{\ell}_A^{(3)} = \big(d\partial_s + ib\beta e^{s}\big)|_{(s,t)} \\ \hat{\ell}_A^{(4)} = \big(d\partial_s + i[a\alpha e^{-s} + b\beta e^s]\big)|_{(s,t)} \\ \hat{\ell}_A^{(5)} = \Big((d + b\gamma e^s)\partial_s + i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)|_{(s,t)}. \\ \hat{\ell}_A^{(5')} = \Big((d - b\gamma e^s)\partial_s + i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^s + c\gamma]\Big)|_{(s,t)} \end{cases}$$

which provides the representations of the Lie algebra $\mathfrak{g}=Lie(\mathbb{R} \ltimes \mathbb{H}_3)$.

Furthermore, $\forall A, B \in \mathfrak{g}$,

$$\hat{\ell}_A \circ \hat{\ell}_B - \hat{\ell}_B \circ \hat{\ell}_A = \hat{\ell}_{[A,B]}$$

Proof For each compactly supported C^∞ function $f \in C_0^\infty(\mathbb{R}^2)$ and for $A, B \in Lie(\mathbb{R} \ltimes \mathbb{H}_3)$,we have

$$\hat{\ell}_{(\mu_1 A+\mu_2 B)}(f) = \mathcal{F}_p \circ \ell_{(\mu_1 A+\mu_2 B)} \circ \mathcal{F}_p^{-1}(f) = \mathcal{F}_p\Big(i(\mu_1 \widetilde{A+\mu_2} B) \star \mathcal{F}_p^{-1}\Big) =$$

$$= \mu_1 \mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}(f) + \mu_2 \mathcal{F}_p \circ \ell_B \circ \mathcal{F}_p^{-1}(f) = \mu_1 \hat{\ell}_A(f) + \mu_2 \hat{\ell}_B(f) \quad \forall \mu_1, \mu_2 \in \mathbb{R}.$$

Moreover,

$$\hat{\ell}_A \circ \hat{\ell}_B(f) - \hat{\ell}_B \circ \hat{\ell}_A(f) = \hat{\ell}_A\Big(\mathcal{F}_p \circ \ell_B \circ \mathcal{F}_p^{-1}(f)\Big) - \hat{\ell}_B\Big(\mathcal{F}_p \circ \ell_A \circ \mathcal{F}_p^{-1}(f)\Big) =$$

$$= \mathcal{F}_p\Big(i\tilde{A} \star (i\tilde{B} \star \mathcal{F}_p^{-1}(f)\Big) - \mathcal{F}_p\Big(i\tilde{B} \star (i\tilde{A} \star \mathcal{F}_p^{-1}(f)\Big) = \mathcal{F}_p\Big(i\widetilde{[A,B]} \star \mathcal{F}_p^{-1}(f)\Big) = \hat{\ell}_{[A,B]}(f)$$

□

DEFINITION 4.7. Let Ω_F^λ be K-orbits of the real diamond Lie group G. With A runs over the Lie algebra $\mathfrak{g} = Lie(G)$,

- $(\Omega^2, \hat{\ell}_A^{(2)}); (\Omega^3, \hat{\ell}_A^{(3)})$ are called the quantum half-planes,
- $(\Omega^4, \hat{\ell}_A^{(4)})$ - quantum hyperbolic cylinder,
- $(\Omega^5, \hat{\ell}_A^{(5)}, \hat{\ell}_A^{(5')})$ - quantum hyperbolic paraboloid,

with respect to the co-adjoint action of Lie group G. In the other words, $(\Omega_F, \hat{\ell}_A)$, with A running over the Lie algebra $\mathfrak{g}$ is called a *quantum co-adjoint orbit* of Lie group G.

As G=$\mathbb{R} \ltimes \mathbb{H}_3$ is connected and simply connected, we obtain a unitary representations T of G defined by the following formula

$$T(\exp A) := \exp(\hat{\ell}_A); \quad A \in \mathfrak{g}$$

More detail,

$$\exp(\hat{\ell}_A) = \begin{cases} \exp(d\partial_s + ia\alpha e^{-s})|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (6)} \\ \exp(d\partial_s + ib\beta e^{s})|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (7)} \\ \exp(d\partial_s + i[a\alpha e^{-s} + b\beta e^{s}])|_{(s,t)} & \text{if } \tilde{A} \text{ is defined by (8)} \\ \exp((d + b\gamma e^{s})\partial_s + & \\ i[ae^{-s} + b(\alpha\beta - \gamma\delta)e^{s} + c\gamma])|_{(s,t)}. & \text{if } \tilde{A} \text{ is defined by (9)} \\ \exp((d - b\gamma e^{s})\partial_s + & \\ i[-ae^{-s} - b(\alpha\beta - \gamma\delta)e^{s} + c\gamma])|_{(s,t)}. & \text{if } \tilde{A} \text{ is defined by (10)} \end{cases}$$

This means that we refind all the representations $T(\exp A)$ of the real diamond Lie group $\mathbb{R} \ltimes \mathbb{H}_3$, those could implicitly obtained from (induction) orbit method induction. What we did here gives us more precise analytic formulas in this case for orbit method induction.

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Haiphong Teacher's Training College ,Haiphong city,Vietnam
E-mail address: nguyen_viet_hai@yahoo.com