

**NONUNIFORMIZABLE SKEW CYLINDERS.
A COUNTEREXAMPLE TO THE
SIMULTANEOUS UNIFORMIZATION PROBLEM**

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ABSTRACT. At the end of 1960^{ths} Yu.S.Ilyashenko stated the problem: is it true that for any one-dimensional holomorphic foliation with singularities on a Stein manifold leaves intersecting a transversal disc can be uniformized by family of simply connected domains in $\overline{\mathbb{C}}$ so that the uniformization function would depend holomorphically on the transversal parameter? In the present paper we give a two-dimensional counterexample. This together with the previous result of Ilyashenko (Lemma 1) implies existence of a counterexample given by a foliation on affine (projective) algebraic surface by level curves of a rational function with singularities deleted. This implies that *Bers' simultaneous uniformization theorem* [9] for topologically trivial holomorphic fibrations by compact Riemann surfaces does not extend to general fibrations by compact Riemann surfaces with singularities.

1. Skew cylinders and simultaneous uniformization.

By T_g , $g \geq 2$, denote the Teichmüller space of closed Riemann surfaces of genus g . The tautological fibration over T_g is the union of all the Riemann surfaces: the fiber over each point of T_g is the corresponding Riemann surface. The tautological fibration admits a natural complex structure, and so does the union of the universal coverings of its fibers. The latter is fibered over T_g with fibers conformally equivalent to disc.

In 1960 L.Bers [9] proved the classical simultaneous uniformization theorem saying that fibers of the tautological fibration admit a uniformization that depends holomorphically on the Teichmüller space parameter. Namely, he proved that the union of the universal coverings of the fibers is biholomorphically fiberwise equivalent to a subdomain in the direct product $\overline{\mathbb{C}} \times T_g$. This theorem implies the analogous statement for any topologically trivial rational fibration by algebraic curves. In particular, for any two-dimensional smooth projective surface S fibered by algebraic curves with singularities and any transversal disc $D \subset S$ intersecting only nonsingular fibers the union of the universal coverings of the fibers intersecting D is biholomorphically fiberwise equivalent to a subdomain in the direct product $\overline{\mathbb{C}} \times D$.

It appears that the generalization of the last statement to topologically nontrivial fibrations by algebraic curves *with singularities* is wrong in general.

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Theorem 1. *There exists a smooth algebraic surface S (of dimension 2) in \mathbb{C}^n (\mathbb{P}^n) fibered by algebraic curves with singularities such that the family of Riemann surfaces thus obtained does not admit simultaneous uniformization by family of simply connected domains in $\overline{\mathbb{C}}$ such that the uniformizing function depends holomorphically on the parameter. (The number n may be chosen to be equal to 5.)*

More precisely, there exists a polynomial P in \mathbb{C}^n , $P|_S \neq \text{const}$, and an inverse mapping $i : D \rightarrow S$ of the unit disc D in the image of P , $P \circ i = \text{Id}$, with the following properties:

- 1) *The image $i(D)$ is transversal to the foliation $P = \text{const}$ by level curves of P .*
- 2) *Let M be the union of the universal coverings of the leaves intersecting $i(D)$ of the foliation $P = \text{const}$ equipped with the natural complex structure. There is no biholomorphic mapping of M onto a domain in $\overline{\mathbb{C}} \times D$ that forms a commutative diagram with P and the standard direct product projection.*

Theorem 1 is proved at the end of the Section.

Definition 1. Let D be a simply-connected domain in complex line, M be a two-dimensional complex manifold, $p : M \rightarrow D$ be a proper holomorphic surjection having nonzero derivative. We say that the triple (M, p, D) is a *skew cylinder* with the base D and the total space M , if

- 1) all the level sets of the map p are simply connected holomorphic curves;
- 2) the "fibration" M admits a section, i.e., a holomorphic mapping $i : D \rightarrow M$ such that $p \circ i = \text{Id}$.

Example 1. Let S be a smooth affine (projective) two-dimensional complex surface, P be a nonconstant rational function, D be a disc in the image of P , $i : D \rightarrow S$ be an inverse mapping transversal to the foliation $P = \text{const}$, $P \circ i = \text{Id}$. Let M be the union of the universal coverings of the leaves intersecting D equipped with the natural complex structure. Then the manifold M equipped with the standard projection $P : M \rightarrow D$ is a skew cylinder. A skew cylinder thus constructed is called *affine (projective) algebraic*.

Definition 2. A skew cylinder (M, p, D) is said to be *uniformizable*, if there exists a biholomorphic mapping $u : M \rightarrow \overline{\mathbb{C}} \times D$ (not necessarily "onto") that forms a commutative diagram with the projections to D .

Definition 3. A skew cylinder is said to be *Stein*, if its total space is Stein.

The main result of the paper is the following

Theorem 2. *There exists a nonuniformizable Stein skew cylinder.*

Theorem 2 is proved in Subsection 2.4.

Definition 4. Let (M, p, D) be a skew cylinder, $B \subset M$ be its subdomain. Then B is called a *subcylinder*, if the triple $(B, p, p(B))$ is a skew cylinder.

Definition 5. Two skew cylinders are said to be *equivalent*, if there exists a biholomorphism of the total space of the one onto that of the other that forms a commutative diagram with the projections.

Lemma 1 (Yu.S.Ilyashenko). ¹ *Any compact subcylinder of a Stein skew cylinder is equivalent to a subcylinder of an affine (projective) algebraic skew cylinder.*

¹From unpublished paper by Yu.S.Ilyashenko (late 1960^{ths})

The dimension of the ambient affine (projective) space of the corresponding surface (see Example 1) may be chosen to be equal to 5.

Lemma 1 is proved in Subsection 2.5.

Proposition 1. *Let a Stein skew cylinder be exhausted by increasing sequence of uniformizable subcylinders. Then it is uniformizable.*

A version of Proposition 1 was proved by Yu.S.Ilyashenko in [6]. The proof of its present version is a modification of Ilyashenko's proof (his argument that uses normality of the space of normalized \mathbb{C} -valued univalent functions in unit disc [6] is replaced by analogous one using normality of the space of normalized $\overline{\mathbb{C}}$ -valued univalent functions $\phi: \phi(0) = \phi''(0) = 0, \phi'(0) = 1$).

Proof of Theorem 1. By Theorem 2, there exists a nonuniformizable Stein skew cylinder. By Proposition 1, it has a nonuniformizable compact subcylinder. By Lemma 1, this subcylinder is equivalent to a subcylinder of an affine (projective) algebraic skew cylinder. The last subcylinder is nonuniformizable as well. The same holds for the ambient algebraic cylinder. This proves Theorem 1.

2. NONUNIFORMIZABILITY

In this Section we prove Theorem 2 (Subsection 2.4) and Lemma 1 (Subsection 2.5).

In Subsections (2.1-2.3) we give a survey of previous results and state an open question (Subsection 2.3).

2.1. Previous results for algebraic skew cylinders.

In 1973 Yu.S.Ilyashenko [5] extended Bers' simultaneous uniformization theorem to the tautological fibration with a unique singular fiber over disc embedded in the Deligne-Mumford compactification of the moduli space of Riemann surfaces. This implies the positive solution of the simultaneous uniformization problem for foliation with singularities by level curves of rational function and leaves over a disc in its image under the assumption that *only one of these leaves is singular* and has at most simple double point singularities.

2.2. Previous results for general skew cylinders.

It is easy to construct nonstein nonuniformizable skew cylinders.

Example 2. Let D be unit disc, $f: D \rightarrow \mathbb{C}$ be arbitrary nonholomorphic function (say, $f(z) = \bar{z}$). Let M be the universal covering over the complement to the graph of f in the direct product $\mathbb{C} \times D$. The manifold M admits a natural structure of skew cylinder, and the latter is nonuniformizable.

Other examples of nonstein nonuniformizable skew cylinders with fibers conformally equivalent to \mathbb{C} may be found in [6].

Yu.S.Ilyashenko [6] proposed a conjecture that any Stein skew cylinder is uniformizable. In 1969 T.Nishino [10] independently proved the positive answer for skew cylinders with fibers \mathbb{C} .

A.A.Shcherbakov [14] proved that a Stein skew cylinder can be exhausted by a growing sequence of compact subcylinders with smooth strictly pseudoconvex boundaries.

Proposition 1 together with Theorem 2 imply the following

Corollary 1. *There exists a compact skew cylinder with a smooth strictly pseudoconvex boundary that is nonuniformizable.*

2.3. Foliations on \mathbb{P}^2 and uniformization problem for them.

It is well known that in the generic case the arrangement of phase curves of a polynomial vector field in \mathbb{C}^n (or more generally, leaves of foliation with singularities by analytic curves) is quite complicated. For example, there is an open domain U in the space of polynomial vector fields in \mathbb{C}^2 such that each phase curve of a generic polynomial vector field from U is everywhere dense in the phase space [1,2,3]. A statement of this type follows from a theorem of I.Nakai [16]. Recently F.Loray and J.Rebelo proved in their joint paper [13] that there is an open subset V in the space of one-dimensional holomorphic foliations with singularities on \mathbb{P}^n of fixed degree greater than 1 such that for any foliation from V each leaf is dense.

Definition 6. Let S be a two-dimensional complex manifold, F be a one-dimensional holomorphic foliation with isolated singularities on S . Let D be an embedded disc in S transversal to F . The *universal covering manifold* associated to the triple (S, F, D) is the union of the universal coverings over the leaves intersecting D with marked points in D . More precisely, it is the set of all the triples consisting of a point $z \in D$, a point z' of the leaf containing z , and a homotopy class of a path connecting z to z' in the leaf.

Remark 1. In the condition of the previous Definition let S be Stein (or projective). Then for any transversal disc D the corresponding universal covering manifold admits a natural structure of complex manifold; it is Stein if S is Stein. These statements hold in any dimension for one-dimensional holomorphic foliations with singular sets of complex codimension at least 2. For Stein S they were proved by Ilyashenko ([4], [15]). For projective S the first statement was proved by myself and a little later by E.M.Chirka (in unpublished papers). In both cases the universal covering manifold has a natural skew cylinder structure with the base D .

Question 1. *Does Theorem 1 hold with $S = \mathbb{C}^2$ or \mathbb{P}^2 ? Does there exist a polynomial (holomorphic) vector field in \mathbb{C}^2 (or a one-dimensional holomorphic foliation with isolated singularities on \mathbb{P}^2) and a transversal disc such that the corresponding universal covering manifold is not uniformizable?*

2.4. Proof of Theorem 2.

Let D be unit disc in complex line. For the proof of Theorem 2 we show that there exists a closed subset $K \subset \mathbb{C} \times D$ fibered over D by discs (an infinite number of them degenerates into single points) such that the universal covering over its complement $\mathbb{C} \times D \setminus K$ (equipped with the canonic projection to D) is a nonuniformizable Stein skew cylinder. We prove this statement for the following set K constructed by Bo Berndtsson and T.J.Ransford in their joint paper [7].

Theorem 3 [7]. *Let D be unit disc in complex line with the coordinate z , p be the standard projection to D of the direct product $\mathbb{C} \times D$. Let $E_+ = \{\frac{1}{2}, \frac{n}{2n+1}\}_{n \in \mathbb{N}}$, $E_- = -E_+ \subset D$. There exists a closed subset $K \subset \mathbb{C} \times \overline{D}$ such that*

- 1) *the complement $M' = \mathbb{C} \times D \setminus K$ is pseudoconvex;*
- 2) *for any $z \notin E_+ \cup E_-$ the fiber $K \cap p^{-1}(z)$ is a disc;*
- 3) *for any $z \in E_+$ $K \cap p^{-1}(z) = 0 \times z$;*
- 4) *for any $z \in E_-$ $K \cap p^{-1}(z) = 1 \times z$.*

For the completeness of presentation, we recall the construction of the set K from [7]. Let w be the coordinate in the fiber \mathbb{C} in the direct product $\mathbb{C} \times D$. Let $u(z) = \ln |z - \frac{1}{2}| + \ln |z + \frac{1}{2}| + \sum_{n=1}^{+\infty} 2^{-n} (\ln |z - \frac{n}{2n+1}| + \ln |z + \frac{n}{2n+1}|)$, $A \in \mathbb{R}^+$. The function u is harmonic and is equal to $-\infty$ at E_{\pm} . Let $\psi : D \rightarrow \mathbb{C}$ be a C^{∞} function with bounded derivatives (up to the second order) that is constant in a neighborhood of each one of the sets E_{\pm} so that $\psi|_{E_+} = 0$, $\psi|_{E_-} = 1$. Define

$$K = \{|w - \psi(z)| \leq e^{u(z)+|z|^2+A}\}.$$

The fibers of K over E_+ (E_-) are single points where the coordinate w is equal to 0 and 1 respectively. If A is large enough, then $\mathbb{C} \times D \setminus K$ is pseudoconvex. The proof of this statement (presented in [7]) is a straightforward calculation of the Levi form together with the argument on approximation of the harmonic function u by decreasing sequence of smooth subharmonic functions.

The nonuniformizable skew cylinder we are looking for is the universal covering over the complement $M' = \mathbb{C} \times D \setminus K$ (denote this covering by M). Equivalently, it is the universal covering manifold for the foliation on M' by the level curves of the projection to D (all these level curves intersect one and the same transversal disc $N \times D$ corresponding to N large enough). Indeed, M is Stein. This follows from pseudoconvexity of the complement $\mathbb{C} \times D \setminus K$ and the classical theorem due to Stein [12] saying that a covering over a Stein manifold is Stein. Let us prove that M is nonuniformizable by contradiction. Suppose the contrary, i.e., there exists a uniformization $u : M \rightarrow \overline{\mathbb{C}} \times D$. Let $\pi : \overline{\mathbb{C}} \times D \rightarrow \overline{\mathbb{C}}$ be the standard projection, $f = \pi \circ u$ be the corresponding coordinate component of u . The fibers of the cylinder M over E_{\pm} are conformally equivalent to complex plane. Let w be the coordinate in \mathbb{C} (we consider it as a coordinate on $M' \subset \mathbb{C} \times D$). Consider the chart $\ln w$ on the fibers of M . It is a well-defined 1-to-1 chart on the fibers over E_+ , and this is not the case for the fibers over E_- , where this chart is multivalued and has branch points. Therefore, for any $z \in E_+$ the restriction to the fiber over z of the function f (which is univalent) is Möbius in the chart $\ln w$, and this is not the case for the fibers over E_- .

Let Sf be the Schwartzian derivative of f in the coordinate $\ln w$. It is a holomorphic function on M . It vanishes identically on all the fibers of M over E_+ and on no fiber over no point in E_- by the previous statement. The first one of the two last statements implies that $Sf \equiv 0$ (the set E_+ contains the limit point $\frac{1}{2}$), which contradicts the second statement. This proves Theorem 2.

2.5. Proof of Lemma 1. The proof of Lemma 1 presented below essentially repeats that due to Yu.S.Ilyashenko².

Let (M, p, D) be a Stein skew cylinder. The manifold M is Stein. We consider that it is embedded as a submanifold in \mathbb{C}^n . (Without loss of generality one can consider that $n = 5$ by Bishop-Narasimhan embedding theorem.) Let $B \subset M$ be a subcylinder with compact closure. Without loss of generality we consider that its base $p(B)$ is unit disc. We will construct a smooth affine algebraic surface $S \subset \mathbb{C}^n$ and a polynomial P such that there exists a biholomorphism $h : B \rightarrow S$ (not onto) that forms a commutative diagram with p and P . Consider the foliation $P = const$ (with singularities) by level curves of P , or its extension up to a foliation with isolated singularities on the projective closure of S (the surface S constructed

²From unpublished paper by Yu.S.Ilyashenko (late 1960^{ths})

below may be chosen to have a smooth projective closure). Let M_B be the union of the universal coverings of the leaves intersecting the image $h(B)$. Then M_B is an algebraic skew cylinder from Lemma 1 we are looking for.

The surface M is a complete intersection: there exist $n-2$ holomorphic functions f_1, \dots, f_{n-2} in \mathbb{C}^n such that M is the transversal intersection of their zero level hypersurfaces (corollary 1.5 from the paper [8]). Let $i : D \rightarrow M$ be the section of M . The projection $p : M \rightarrow D$ extends up to a holomorphic function (still denoted by p) on \mathbb{C}^n by classical theorem on extension of analytic function on submanifold of Stein manifold [11].

To construct the surface S and the polynomial P , we use the fact that the functions $f_i, i = 1, \dots, n-2$, and p can be approximated by polynomials uniformly on each ball in \mathbb{C}^n . Let us fix a ball \tilde{B} containing \bar{B} and denote the corresponding approximating polynomials by F_i and P respectively. Let us denote by S the irreducible component of the conjoint zero set of the polynomials F_i that approaches B , as $F_i \rightarrow f_i, P \rightarrow p$. Let us prove that there exists a biholomorphic mapping $h : \bar{B} \rightarrow S$ that forms a commutative diagram with the projections p and P . To construct this biholomorphism h , we use the following

Proposition 2. *Let (M, p, D) be a Stein skew cylinder, $i : D \rightarrow M$ be its section (see Definition 1). There exists a holomorphic function f on M with nonvanishing derivative along each fiber $p^{-1}(z), z \in D$, such that $f|_{i(D)} = 0$.*

Proof. Let $(M, p, D), i$ be as in Proposition 2. Let us construct a function f on M that satisfies the statements Proposition 2.

A generic holomorphic function on M has nonidentically-vanishing derivative along each fiber (since M is Stein). Let us fix such a function $g(x)$ that in addition vanishes at the image $i(D)$ of the section (one can achieve this by changing g to the function $g - g \circ i \circ p$). By Σ denote the subset of points in M where the restriction of the differential dg vanishes on the line tangent to the fiber. In the case, when Σ is empty, g is a function we are looking for. Suppose $\Sigma \neq \emptyset$. Then Σ is a hypersurface such that no its irreducible component is contained in a fiber. By definition, the *multiplicity* of its irreducible component Σ' is said to be the order of zero of the restriction to the fiber of the differential dg at a generic point of Σ' . There exists a holomorphic function on M that vanishes at Σ with the prescribed multiplicities at its irreducible components (let us fix this function and denote it by F). This follows from the theorem saying that any holomorphic line bundle over a contractible Stein manifold is trivial. (This is a corollary of the classical theorem on triviality of the first $\bar{\partial}$ -cohomology group for Stein manifolds [11].) By construction, the ratio $\frac{dg}{F}$ is a nowhere vanishing 1-form holomorphic on each fiber. The function

$$f(x) = \int_{i \circ p(x)}^x \frac{dg}{F}$$

on M is a one we are looking for (the integration is made along a path in the fiber containing x that starts at $i \circ p(x)$ and ends at x). Proposition 2 is proved.

Let us construct the fiberwise biholomorphism $h : \bar{B} \rightarrow S$. To do this, consider a function f from Proposition 2 and its extension to \mathbb{C}^n (denoted by the same symbol f). The function f is locally (but not globally) univalent on each fiber. By construction, the level curves of f in \mathbb{C}^n are transversal to the fibers of M . The biholomorphism h we are looking for is defined by the following pair of equations:

$p(x) = P(h(x))$, $f(x) = f(h(x))$. It is well defined provided that the above approximations of the surface $M \cap \tilde{B}$ and the function $p|_{\tilde{B}}$ by $S \cap \tilde{B}$ and P respectively are accurate enough and the more accurate they are, the more close h is to identity. It forms a commutative diagram with p and P by construction. Lemma 1 is proved.

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