

TACTICS, DIALECTICS, REPRESENTATION THEORY

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This article is devoted to the tactical game theoretical interpretation of dialectics. Dialectical games are considered as abstractly as well as models of the internal dialogue and reflection. The models related to the representation theory (representative dynamics) are specially investigated in detail, they correlate with the hypothesis on the dialectical features of human thinking in general and mathematical thought (the constructing of a solution of mathematical problem) in particular.

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INTRODUCTION

The mathematical formalism of interactive games, which extends one of ordinary games (see e.g.[1]) and is based on the concept of an interactive control, was recently proposed by the author [2] to take into account the complex composition of controls of a real human person, which are often complicated couplings of his/her cognitive and known controls with the unknown subconscious behavioral reactions. In the article [3] the dialogues as psycholinguistic phenomena were described in the interactive game theoretical terms. It allowed to transfer from the pure interactivity to the essentially new and much complex phenomenon of tactics [4]. The tactical games were introduced together with some useful methods of their constructing, examples were considered and applications specified. Generally, tactics may be regarded as "an art to manipulate the unknown, which is manifested by the interactivity, without making it known" [4], in particular, tactical actions are not completely understood even by the acting person.

The present article is devoted to the abstract foundations of tactics, precisely, to the interactive game theoretical interpretation of dialectics. The least is defined as a logical self-describing tactical game. A proper subclass of tactical games is extracted, they are the dialectical games, the tactical games, which are tactical extensions of dialectics. And if tactics is thought as an art to manipulate the unknown without making it known, dialectics may be thought as an art to comprehend such manipulations.

An interesting type of dialectical games is related to the representation theory, with a strong accent on its inverse problems and representative dynamics. The special attention is paid to such games as well as to the various sides of interaction of tactics, dialectics and representation theory that is the main theme of this article.

The present article finishes two-year researches of the author on the mathematical formulation of the game theoretical background for interactivity and tactics. It seems that the picture is complete now and all main concepts are introduced. However, many technical details should be clarified but the practice is essential here. Certainly, only the practice may determine the directions of future theoretical studies. Mathematical proof of any nontrivial statement claims a lot of time and efforts, and when the initial direction was misleading the whole activity is simply senseless. So let's "learn to labor and to wait".

I. TACTICS

1.1. Interactive games [2].

Definition 1 [2]. An *interactive system* (with n *interactive controls*) is a control system with n independent controls coupled with unknown or incompletely known feedbacks (the feedbacks as well as their couplings with controls are of a so complicated nature that their can not be described completely). An *interactive game* is a game with interactive controls of each player.

Below we shall consider only deterministic and differential interactive systems. In this case the general interactive system may be written in the form:

$$(1) \quad \dot{\varphi} = \Phi(\varphi, u_1, u_2, \dots, u_n),$$

where φ characterizes the state of the system and u_i are the interactive controls:

$$u_i(t) = u_i(u_i^\circ(t), [\varphi(\tau)]|_{\tau \leq t}),$$

i.e. the independent controls $u_i^\circ(t)$ coupled with the feedbacks on $[\varphi(\tau)]|_{\tau \leq t}$. One may suppose that the feedbacks are integrodifferential on t .

However, it is reasonable to consider the *differential interactive games*, whose feedbacks are purely differential. It means that

$$u_i(t) = u_i(u_i^\circ(t), \varphi(t), \dots, \varphi^{(k)}(t)).$$

A reduction of general interactive games to the differential ones via the introducing of the so-called *intention fields* was described in [2]. Below we shall consider the differential interactive games only if the opposite is not specified explicitly.

The interactive games introduced above may be generalized in the following ways.

The first way, which leads to the *indeterminate interactive games*, is based on the idea that the pure controls $u_i^\circ(t)$ and the interactive controls $u_i(t)$ should not be obligatory related in the considered way. More generally one should only postulate that there are some time-independent quantities $F_\alpha(u_i(t), u_i^\circ(t), \varphi(t), \dots, \varphi^{(k)}(t))$ for the independent magnitudes $u_i(t)$ and $u_i^\circ(t)$. Such claim is evidently weaker than one of Def.1. For instance, one may consider the inverse dependence of the pure and interactive controls: $u_i^\circ(t) = u_i^\circ(u_i(t), \varphi(t), \dots, \varphi^{(k)}(t))$.

The second way, which leads to the *coalition interactive games*, is based on the idea to consider the games with coalitions of actions and to claim that the interactive controls belong to such coalitions. In this case the evolution equations have the form

$$(2) \quad \dot{\varphi} = \Phi(\varphi, v_1, \dots, v_m),$$

where v_i is the interactive control of the i -th coalition. If the i -th coalition is defined by the subset I_i of all players then

$$v_i = v_i(\varphi(t), \dots, \varphi^{(k)}(t), u_j^\circ | j \in I_i).$$

Certainly, the intersections of different sets I_i may be non-empty so that any player may belong to several coalitions of actions. Def.1 gives the particular case when $I_i = \{i\}$.

Remark 1. One is able to consider interactive games of discrete time in the similar manner.

Remark 2. If one suspect that the explicit dependence of the feedbacks on the derivatives of φ is not correct because they are determined via the evolution equations governed by the interactive controls, it is reasonable to use the inverse dependence of pure and interactive controls.

Interactive games are games with incomplete information by their nature. However, this incompleteness is in the unknown feedbacks, not in the unknown states. The least situation is quite familiar to specialists in game theory and there is a lot of methods to have deal with it. For instance, the unknown states are interpreted as independent controls of the virtual players and some muppositions on their strategies are done. To transform interactive games into the games with an incomplete information on the states one can use the following trick, which is called the ε -representation of the interactive game.

Definition 2. The ε -representation of the differential interactive game is a representation of the interactive controls $u_i(t)$ in the form

$$u_i(t) = u_i(u_i^\circ(t), \varphi(t), \dots, \varphi^{(k)}(t); \varepsilon_i(t))$$

with the *known* function u_i of its arguments $u_i^\circ, \varphi, \dots, \varphi^{(k)}$ and ε_i , whereas

$$\varepsilon_i(t) = \varepsilon_i(u_i^\circ(t), \varphi(t), \dots, \varphi^{(k)}(t))$$

is the *unknown* function of u_i° and $\varphi, \dots, \varphi^{(k)}$.

ε_i are interpreted as parameters of feedbacks and, thus, characterize the internal *existential* states of players. It motivates the notation ε . Certainly, ε -parameters are not really states being the unknown functions of the states and pure controls, however, one may sometimes to apply the standard procedures of the theory of games with incomplete information on the states. For instance, it is possible to regard ε_i as controls of the virtual players. The naïvely introduced virtual players only double the ensemble of the real ones in the interactive games but in the coalition interactive games the collective virtual players are observed. More sophisticated procedures generate ensembles of virtual players of diverse structure.

Precisely, if the derivatives of φ are excluded from the feedbacks (at least, from the interactive controls u_i as functions of the pure controls, states and the ε -parameters) the evolution equation will have the form

$$(3) \quad \dot{\varphi}(t) = \Phi(\varphi, u_1(u_1^\circ(t), \varphi(t); \varepsilon_1(t)), \dots, u_n(u_n^\circ(t), \varphi(t); \varepsilon_n(t))),$$

so it is consistent to regard the equations as ones of the controlled system with the ordinary controls $u_1, \dots, u_n, \varepsilon_1, \dots, \varepsilon_n$. One may consider a new game postulating that these controls are independent. Such game will be called the ordinary differential game associated with the ε -representation of the interactive game.

Let us consider now an arbitrary ordinary differential game with the evolution equations

$$\dot{\varphi} = \Phi(\varphi, u_1, u_2, \dots, u_n),$$

where φ characterizes the state of the system and u_i are the ordinary controls. Let us fix a player. For simplicity of notations we shall suppose that it is the first one. As a rule the players have their algorithms of predictions of the behaviour of other players. For a fixed moment t_0 of time let us consider the prediction of the first player for the game. It consists of the predicted controls $u_{[t_0];i}^\circ(t)$ ($t > t_0$; $i \geq 2$) of all players and the predicted evolution of the system $\varphi_{[t_0]}^\circ(t)$. Let us fix Δt and consider the real and predicted controls for the moment $t_0 + \Delta t$. Of course, they may be different because other players use another algorithms for the game prediction. One may interpret the real controls $u_i(t)$ ($t = t_0 + \Delta t$; $i \geq 2$) of other players as interactive ones whereas the predicted controls $u_{[t_0];i}^\circ(t)$ as pure ones, i.e. to postulate their relation in the form:

$$u_i(t) = u_i(u_{[t-\Delta t];i}^\circ(t); \varphi_{[t_0]}^\circ(\tau) | \tau \leq t).$$

In particular, the feedbacks may be either reduced to differential form via the introducing of the intention fields or simply postulated to be differential. Thus, we

constructed an interactive game from the initial ordinary game. One may use $\varphi(\tau)$ as well as $\varphi_{[t_0]}^\circ(\tau)$ in the feedbacks.

Note that the controls of the first player may be also considered as interactive if the corrections to the predictions are taken into account when controls are chosen.

The obtained construction may be used in practice to make more adequate predictions. Namely, *a posteriori* analysis of the differential interactive games allows to make the short-term predictions in such games. One should use such predictions instead of the initial ones. Note that at the moment t_0 the first player knows the pure controls of other players at the interval $[t_0, t_0 + \Delta t]$ whereas their real freedom is interpreted as an interactivity of their controls $u_i(t)$. So it is reasonable to choose Δt not greater than the admissible time depth of the short-term predictions.

Naïvely, the proposed idea to improve the predictions is to consider deviations of the real behaviour of players from the predicted ones as a result of the interactivity, then to make the short-term predictions taking the interactivity into account and, thus, to receive the corrections to the initial predictions. Such corrections may be regarded as “psychological” though really they are a result of different methods of predictions used by players.

Remark 3. The interpretation of the ordinary differential game as an interactive game also allows to perform the strategical analysis of interactive games. Indeed, let us consider an arbitrary differential interactive game A . Specifying its ε -representation one is able to construct the associated ordinary differential game B with the doubled number of players. Making some predictions in the game B one transform it back into an interactive game C . Combination of the strategical long-term predictions in the game B with the short-term predictions in C is often sufficient to obtain the adequate strategical prognosis for A .

Remark 4. The interpretation of the ordinary differential game as an interactive game is especially useful in situations, when the goals of players are not known precisely to each other and some more or less rough suppositions are made.

Let’s expose another way to obtain an interactive game from the ordinary differential game. If one has an ordinary differential game with the states $\varphi(t)$ and the controls $u_i(t)$ it is possible to introduce some filtering procedure for controls, for instance, the exclusion of the high frequency components or the separation of frequencies from some special set. The result will be denote by $u_i^\circ(t)$. One may suspect that $u_i^\circ(t) = u_i^\circ(u_i(t); \varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), \dots, \varphi^{(k)}(t))$ and use *a posteriori* analysis for estimation of the feedback. In general, the filtering procedure may depend on $\varphi(t)$ and its derivatives but in any case the interactive game is constructed. Also one may consider the integrodifferential form of feedbacks: $u_i^\circ(t) = u_i^\circ(u_i(t), [\varphi(\tau)]|_{\tau \leq t})$.

Note that both versions to unravel an interactivity of the ordinary differential game may be combined.

Remark 5. The unraveling of the interactivity of ordinary differential games is an essential step for the understanding of the complex psychophysiological processes. The next step is to describe if necessary the interactivity in terms of intention fields and to unravel their structure (to solve the inverse problem of representation theory). It may be useful to interpret the intention fields as fields of interactive forces. This scheme may be applied to various interesting psychophysiological phenomena usually exposed in terms of Eastern (presumably, Chinese and Indian) traditions. Note that their use in medicine is certainly a tactical problem (see below).

1.2. Dialogues and verbalizable games [3]. Let us now expose the interactive game formalism for a description of dialogues as psycholinguistic phenomena [3]. First of all, note that one is able to consider interactive games of discrete time as well as interactive games of continuous time above.

Defintion 3A (the naïve definition of dialogues) [3]. The *dialogue* is a 2-person interactive game of discrete time with intention fields of continuous time.

The states and the controls of a dialogue correspond to the speech whereas the intention fields describe the understanding.

Let us give the formal mathematical definition of dialogues now.

Definition 3B (the formal definition of dialogues) [3]. The *dialogue* is a 2-person interactive game of discrete time of the form

$$(4) \quad \varphi_n = \Phi(\varphi_{n-1}, \vec{v}_n, \xi(\tau) | t_{n-1} \leq \tau \leq t_n).$$

Here $\varphi_n = \varphi(t_n)$ are the states of the system at the moments t_n ($t_0 < t_1 < t_2 < \dots < t_n < \dots$), $\vec{v}_n = \vec{v}(t_n) = (v_1(t_n), v_2(t_n))$ are the interactive controls at the same moments; $\xi(\tau)$ are the intention fields of continuous time with evolution equations

$$(5) \quad \dot{\xi}(t) = \Xi(\xi(t), \vec{u}(t)),$$

where $\vec{u}(t) = (u_1(t), u_2(t))$ are continuous interactive controls with ε -represented couplings of feedbacks:

$$u_i(t) = u_i(u_i^\circ(t), \xi(t); \varepsilon_i(t)).$$

The states φ_n and the interactive controls \vec{v}_n are certain *known* functions of the form

$$(6) \quad \begin{aligned} \varphi_n &= \varphi_n(\vec{\varepsilon}(\tau), \xi(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_n &= \vec{v}_n(\vec{u}^\circ(\tau), \xi(\tau) | t_{n-1} \leq \tau \leq t_n). \end{aligned}$$

Note that the most nontrivial part of mathematical formalization of dialogues is the claim that the states of the dialogue (which describe a speech) are certain “mean values” of the ε -parameters of the intention fields (which describe the understanding).

Important. The definition of dialogue may be generalized on arbitrary number of players and below we shall consider any number n of them, e.g. $n = 1$ or $n = 3$, though it slightly contradicts to the common meaning of the word “dialogue”.

An embedding of dialogues into the interactive game theoretical picture generates the reciprocal problem: how to interpret an arbitrary differential interactive game as a dialogue. Such interpretation will be called the *verbalization*.

Definition 4 [3]. A differential interactive game of the form

$$\dot{\varphi}(t) = \Phi(\varphi(t), \vec{u}(t))$$

with ε -represented couplings of feedbacks

$$u_i(t) = u_i(u_i^\circ(t), \varphi(t), \dot{\varphi}(t), \ddot{\varphi}(t), \dots, \varphi^{(k)}(t); \varepsilon_i(t))$$

is called *verbalizable* if there exist a *posteriori* partition $t_0 < t_1 < t_2 < \dots < t_n < \dots$ and the integrodifferential functionals

$$(7) \quad \begin{aligned} \omega_n(\vec{\varepsilon}(\tau), \varphi(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_n(\vec{u}^\circ(\tau), \varphi(\tau) | t_{n-1} \leq \tau \leq t_n) \end{aligned}$$

such that

$$(8) \quad \omega_n = \Omega(\omega_{n-1}, v_n; \varphi(\tau) | t_{n-1} \leq \tau \leq t_n).$$

The verbalizable differential interactive games realize a dialogue in sense of Def.3.

Remark 6. One may include ω_n explicitly into the evolution equations for φ

$$\dot{\varphi}(\tau) = \Phi(\varphi(\tau), \vec{u}(\tau); \omega_n), \quad \tau \in [t_n, t_{n+1}].$$

as well as into the feedbacks and their couplings.

The main heuristic hypothesis is that all differential interactive games “which appear in practice” are verbalizable. The verbalization means that the states of a differential interactive game are interpreted as intention fields of a hidden dialogue and the problem is to describe such dialogue completely. If a differential interactive game is verbalizable one is able to consider many linguistic (e.g. the formal grammar of a related hidden dialogue) or psycholinguistic (e.g. the dynamical correlation of various implications) aspects of it.

During the verbalization it is a problem to determine the moments t_i . A way to the solution lies in the structure of ε -representation. Let the space E of all admissible values of ε -parameters be a CW-complex. Then t_i are just the moments of transition of the ε -parameters to a new cell.

1.3. Tactical games [4]. Tactics as it will be defined below is derived from two independent concepts: the parametric interactive games and external controls on one hand and the comments to dialogues on another hand.

First of all, note that an interactive game may depend on the additional parameters. Such dependence is of two forms. First, parameters may appear in the evolution equations:

$$(9A) \quad \dot{\varphi} = \Phi(\varphi, u_1, u_2, \dots, u_n; \lambda).$$

Here, λ is a collective notation for parameters. Second, parameters may appear in feedbacks:

$$(9B) \quad u_i(t) = u_i(u_i^\circ(t), \varphi(t), \dots, \varphi^{(k)}(t); \lambda).$$

The dependence of u_i on λ is either unknown (incompletely known) or known. The least means that $\partial u_i / \partial \lambda$ may be expressed via u_i as a function of other variables (such expression are integrodifferential on these variables). Both variants of parametric dependence of interactive game may be combined together.

The additional parameters may realize the external controls. In this situation they depend on time:

$$\lambda = \lambda(t).$$

In practice, such situation appear in the teaching systems. The parameters are interpreted as controls of a teacher. This example is rather typical. It shows that the controls $\lambda(t)$ may be considered as “slow” whereas the interactive controls $u_i(t)$ as “quick”.

Of course, one is able to introduce the slow controls $\lambda(t)$, which belong to the same players as the interactive controls $u_i(t)$ or to their coalitions. And, certainly, the slow controls of discrete time may be considered. One may suspect that the discrete time controls λ_n realize a convenient approximation for the slow controls $\lambda(t)$, which is timer in practice.

The slow controls may be interactive.

If dependence of u_i on λ is known and one consider the ε -representation of feedbacks it is either postulated that ε -parameters do not depend on λ or claimed that $\partial\varepsilon/\partial\lambda$ is expressed via ε as a function of other arguments.

Now let us define comments to the dialogue. Let

$$\omega_n = \Omega(\omega_{n-1}, \vec{v}_n, \xi(\tau) | t_{n-1} \leq \tau \leq t_n)$$

be the n -person dialogue with the discrete time interactive controls \vec{v}_n and the intention fields governed by the evolution equations

$$\dot{\xi}(t) = \Xi(\xi(t), \vec{u}(t)),$$

where $\vec{u}(t)$ are the continuous interactive controls with ε -represented couplings of feedbacks:

$$u_i(t) = u_i(u_i^\circ(t), \xi(t); \varepsilon_i(t)).$$

The states φ_n and the interactive controls \vec{v}_n are expressed as

$$\begin{aligned} \omega_n &= \omega_n(\vec{\varepsilon}(\tau), \xi(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_n &= \vec{v}_n(\vec{u}^\circ(\tau), \xi(\tau) | t_{n-1} \leq \tau \leq t_n). \end{aligned}$$

The discrete time *comments* ϑ_n to the dialogue are defined recurrently as

$$(10) \quad \vartheta_n = \Theta(\vartheta_{n-1}, \omega_n, v_n).$$

Comments to the dialogue at the fixed moment t_n contain various information on the dialogue. For instance, one may to raise a problem to restore some features of a dialogue from certain comments or alternatively what features of a dialogue may be restored from the fixed comment.

The main difference of the comments ϑ_n from the states ω_n is the absence of expressions of the first via $\vec{\varepsilon}(\tau)$ and $\xi(\tau)$ ($t_{n-1} \leq \tau \leq t_n$).

Comments are applied to the verbalizable games in the same way.

Tactical games combine mechanisms of parametric interactive games and comments to the dialogue (verbalizable game).

Definition 5. The *tactical game* is a parametric verbalizable game with comments, in which the parameters are of discrete time and coincide with the comments.

It is really wonderful that such simple definition is applicable to a very huge class of phenomena. However, it is so! As it was marked above virtually all known forms of human activity such as scientific researches and economics, sport or military

actions, medicine, fine or martial arts, literature and music, theatre and dance, psychotherapy and even magic may be regarded as certain tactical games. Trying to improve the model I have no found any wider concept, whose using is necessary and effective, whereas the notion of tactical game may describe these phenomena very correctly.

Remark 7. The pairs (v_n, ϑ_n) of discrete time interactive controls and the comments will be called the *tactical actions*, whereas the continuous time interactive controls $u_i(t)$ will be called the *instant actions*. The tactical actions may be involved as in the evolution equations as in the interactivity.

Now we shall describe some operations over the tactical games (the tactical interaction, the tactical synthesis and the tactical extension).

Let us consider two tactical games defined by the evolution equations

$$\dot{\varphi}_1 = \Phi_1(\varphi_1, \vec{u}_1; \vartheta_1)$$

and

$$\dot{\varphi}_2 = \Phi_2(\varphi_2, \vec{u}_2; \vartheta_2)$$

with ε -represented couplings of feedbacks

$$u_{1,i} = u_{1,i}(u_{1,i}^\circ, \varphi_1, \dot{\varphi}_1, \ddot{\varphi}_1, \dots, \varphi_1^{(k)}; \varepsilon_{1,i}, \vartheta_1)$$

and

$$u_{2,i} = u_{2,i}(u_{2,i}^\circ, \varphi_2, \dot{\varphi}_2, \ddot{\varphi}_2, \dots, \varphi_2^{(k)}; \varepsilon_{2,i}, \vartheta_2).$$

The integrodifferential functionals (7) have the form

$$\begin{aligned} \omega_{j,n}(\vec{\varepsilon}_j(\tau), \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_{j,n}(\vec{u}_j^\circ(\tau), \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n) \end{aligned}$$

and the relations (8)

$$\omega_{j,n} = \Omega_j(\omega_{j,n-1}, v_{j,n}; \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n)$$

hold ($j = 1, 2$). The comments ϑ_1 and ϑ_2 are defined recurrently as

$$\vartheta_{1,n} = \Theta_1(\vartheta_{1,n-1}, \omega_{1,n}, v_{1,n})$$

and

$$\vartheta_{2,n} = \Theta_2(\vartheta_{2,n-1}, \omega_{2,n}, v_{2,n}).$$

The *tactical interaction* is realized by the addition of the interaction terms into the recurrent formulas for ϑ_j to produce the interdetermination of comments:

$$(11A) \quad \vartheta_{1,n} = \Theta_1(\vartheta_{1,n-1}, \omega_{1,n}, v_{1,n}) + \tilde{\Theta}_{1,2}^{\text{int}}(\vartheta_{1,n-1}, \vartheta_{2,n-1}, \omega_{1,n}, v_{1,n})$$

and

$$(11B) \quad \vartheta_{2,n} = \Theta_2(\vartheta_{2,n-1}, \omega_{2,n}, v_{2,n}) + \tilde{\Theta}_{2,1}^{\text{int}}(\vartheta_{2,n-1}, \vartheta_{1,n-1}, \omega_{2,n}, v_{2,n}).$$

Let us consider N control systems represented as tactical games defined by the evolution equations

$$\dot{\varphi}_j = \Phi_j(\varphi_j, \vec{u}_j; \vartheta_j)$$

($j = 1, 2, \dots, N$) with ε -represented couplings of feedbacks

$$u_{j,i} = u_{j,i}(u_{j,i}^\circ, \varphi_j, \dot{\varphi}_j, \ddot{\varphi}_j, \dots, \varphi_j^{(k)}; \varepsilon_{j,i}, \vartheta_j).$$

The integrodifferential functionals (7) have the form

$$\begin{aligned} \omega_{j,n}(\vec{\varepsilon}_j(\tau), \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_{j,n}(\vec{u}_j^\circ(\tau), \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n) \end{aligned}$$

and the relations (8)

$$\omega_{j,n} = \Omega_j(\omega_{j,n-1}, v_{j,n}; \varphi_j(\tau) | t_{n-1} \leq \tau \leq t_n)$$

hold. The comments ϑ_j are defined recurrently as

$$(12) \quad \vartheta_{j,n} = \Theta_j(\vartheta_{j,n-1}, \omega_{j,n}, v_{j,n}).$$

The *tactical synthesis* is realized by the redefinition of the recurrent formulas for ϑ_j to produce the unification of comments:

$$\vartheta_{j,n} = \tilde{\Theta}_j(\vartheta_{1,n-1}, \dots, \vartheta_{N,n-1}, \omega_{1,n}, \dots, \omega_{N,n}, v_{1,n}, \dots, v_{N,n}).$$

The functions $\tilde{\Theta} = (\tilde{\Theta}_1, \dots, \tilde{\Theta}_N)$ determines the synthesis. It may have various internal structure, which is characterized by the set of real arguments of functions $\tilde{\Theta}_j$ and their hierarchical structure. It presupposes that $\tilde{\Theta}_j$ depend not on all triples $(\vartheta_j, \omega_j, v_j)$ and various triples may appear in $\tilde{\Theta}_j$ in coalitions of different form and nature. One may think that the functions $\tilde{\Theta} = (\tilde{\Theta}_1, \dots, \tilde{\Theta}_N)$ are constructed from the functions $\Theta = (\Theta_1, \dots, \Theta_N)$ using some operations, which realize the synthesis. Tactical interaction is a form of tactical synthesis of two games.

If under the tactical synthesis of two games (in particular, it may be a tactical interaction) $\tilde{\Theta}_1 = \Theta_1$ we shall say that the *tactical extension* of the first game is realized.

II. DIALECTICS AND DIALECTICAL GAMES

To unravel the abstract foundations of tactics is much more than to define it. Such foundations should provide the description of tactical phenomena that presupposes a possibility of their repeating and reproducing, whereas definitions only established the rules of games. The following paragraph is an attempt to initiate the investigation of such foundations. The crucial role is played by the game theoretical definition of dialectics.

Definition 6. *Dialectics* is a logical self-describing tactical game.

”Logical” means that the game is purely cognitive and does not need obligatory any material substrate or, otherwise, that its objects are notions. ”Self-describing” means that all notions as well as procedures of their transformation, which constitute the game evolution, are derived from themselves and that self-description as a definition of all appearing objects from themselves is simultaneously the goal of the game.

The main question is whether dialectics exists. Its existence should be considered as a postulate.

Postulate. *Dialectics exists.*

Apparently this postulate can not be derived in any way from the standard mathematical foundations such as set theory, mathematical logic and arithmetics. Moreover, there are no reasons to believe that it is possible to prove formally (i.e. by means of mathematical logic) that it does not contradict to them. However, if the postulate is adopted its compatibility with mathematical logic may be received dialectically.

Another question is one of the equivalence of various forms of dialectics. It is not a fact that dialectics is unique.

Now let us define dialectical games.

Definition 7. A *dialectical game* is a tactical extension of dialectics.

It means that dialectical games use some dialectical procedures combined with others, otherwords, have elements of self-description. Dialectical games are not obligatory logical. For instance, one may consider dialectical games of perception, dialectical computer games, etc.

Note that dialectics as a tactical game itself is not given *before* but is self-constructing *during* the dialectical game. It is its difference from the formal logic.

Remark 8. Dialectics allows to give self-consistent descriptions of any tactical games and such descriptions are dialectical games. So one may think dialectics as a formal cause of descriptibility of tactical phenomena.

Thus, if tactics is thinking as "an art to manipulate the unknown, which is manifested by the interactivity, without making it known" [4], dialectics should be regarded as "an art to comprehend these manipulations" and, moreover, as "a kinaesthetic art to comprehend the unknown itself via manipulations over it", dialectics is an abstract and kinaesthetic contemplation of the unknown. Hence, the dialectical games may be an effective tools for analysis and controlling of the internal dialogue and the reflection because dialectics may be regarded as an *interiorization of tactics*.

In fact, tactical games tends to be dialectical because any purposeful activity presuppose some form of its understanding based on its description. Because description of a tactical phenomena is also a tactical game, to make such description self-consistent one should use dialectics (or already constructed dialectical game). Hence, tactics is effectiveness as much as it uses dialectics.

Let us describe a general form of the dialectical game. Such game is defined by the evolution equations

$$\dot{\varphi} = \Phi(\varphi, \vec{u}; \vartheta)$$

with ε -represented couplings of feedbacks

$$u_i = u_i(u_i^{\circ}, \varphi, \dot{\varphi}, \ddot{\varphi}, \dots, \varphi^{(k)}; \varepsilon_i, \vartheta)$$

The integrodifferential functionals (7) have the form

$$\begin{aligned} \omega_n(\vec{\varepsilon}(\tau), \varphi(\tau) | t_{n-1} \leq \tau \leq t_n), \\ \vec{v}_n(\vec{u}^{\circ}(\tau), \varphi(\tau) | t_{n-1} \leq \tau \leq t_n) \end{aligned}$$

and the relations (8)

$$\omega_n = \Omega(\omega_{n-1}, v_n; \varphi(\tau) | t_{n-1} \leq \tau \leq t_n).$$

The comments ϑ_n are defined recurrently as

$$(13) \quad \vartheta_n = \Theta(\vartheta_{n-1}, \delta_n, \omega_n, v_n),$$

where $\{\delta_n\}$ are self-describing objects of dialectics.

Remark 9. In the dialectical game the tactical actions (the pairs (v_n, ϑ_n)) are constructed using the dialectical objects δ_n besides v_n and ω_n so dialectics appears as a logic of tactics.

Remark 10. Often a dialectical game is a tactical extension of both dialectics and a concrete tactical game of any nature.

Remark 11 (for a discussion): Dialectics may be regarded not only as a formal cause of descriptibility of tactical phenomena but also as a formal cause of these phenomena themselves. For an active cause of tactical (interactive) phenomena the notion "māyā" was adopted. One may think that both causes form the same *linguodynamical or logodynamical cause* of tactics, and for such linguodynamical unity of dialectics as a logical self-describing tactical game and māyā as a universal dynamical principle of all subject-object relations the related term "*lilā*" is very convenient. Because māyā is not neither object nor subject, and it is not active, there are no any energetical or dialogical aspects of lilā itself, besides ones between real subjects and objects. Lilā is thought as purely dynamical, kinaesthetical and contemplative dialectics of māyā. Note that māyā is objectivized in various forms: as intention fields or as fields of interactive forces. All these objects may be visualized. The visualizations of lilā are the dialectical perception games, combining visual, dynamical, kinaesthetical and dialectically contemplative aspects.

III. DIALECTICAL GAMES AND REPRESENTATION THEORY

Let us describe a type of dialectical games, which may be considered simultaneously as illustrations of general concepts and as models for the internal dialogue and the reflection.

First of all, some additional constructions, which realize a link between control and representation theory, are necessary.

3.1. Representative dynamics [5].

Definition 8. Let $\mathbf{X} = \mathbf{X}(t) = (X_1(t), \dots, X_m(t))$ ($X_i(t) \in \text{Mat}_n(\mathbb{C})$) be the time-dependent vector of m complex $n \times n$ matrices. The *representative dynamics* is a controlled system (with constraints) of the form

$$(14) \quad \dot{\mathbf{X}}(t) = F(\mathbf{X}(t), a(t))$$

with the *fixed* initial data $\mathbf{X}(t_0)$, where the control parameter $a(t) = (\mathfrak{A}(t), \mathbf{e}(t))$ is the pair of *any* associative algebra $\mathfrak{A}(t)$ from the fixed class of such algebras \mathbb{A} , $\mathbf{e}(t) = (e_1(t), \dots, e_m(t))$ is *any* set of algebraic generators of the algebra $\mathfrak{A}(t)$ (one may claim $\mathbf{e}(t)$ to be an algebraic basis in $\mathfrak{A}(t)$) such that the mapping $e_i(t) \mapsto X_i(t)$ may be extended to the representation $T(t) : \mathfrak{A}(t) \mapsto \text{Mat}_n(\mathbb{C})$ of the algebra $\mathfrak{A}(t)$ in the matrix algebra $\text{Mat}_n(\mathbb{C})$ (this is a constraint on the control $a(t)$).

Certainly, the claim that (1) is a representative dynamics restricts the choice of the function F and initial data $\mathbf{X}(t_0)$ because for each moment t *any* admissible choice of the pair $(\mathfrak{A}(t), \mathbf{e}(t))$ should provide that the set of admissible pairs will not be empty in future.

Remark 12. It is possible to consider the infinite dimensional algebras of operators instead of $\text{Mat}_n(\mathbb{C})$ or their subalgebras (e.g. commutative algebras of functions if all algebras $\mathfrak{A}(t)$ are commutative).

Remark 13. Let us consider the following equivalence on the set \mathcal{A} of all admissible $a = (\mathfrak{A}, \mathbf{e})$. The pairs $a_1 = (\mathfrak{A}_1, \mathbf{e}_1)$ and $a_2 = (\mathfrak{A}_2, \mathbf{e}_2)$ will be equivalent iff the algebras A_1 and A_2 are isomorphic under an isomorphism which maps the linear space V_1 spanned by the elements of \mathbf{e}_1 onto the linear space V_2 spanned by the elements of \mathbf{e}_2 . Then the equivalence divides the time interval $[t_0, t_1]$, on which the representative dynamics is considered, onto the subsets, on which the pairs $a(t)$ are equivalent.

Remark 14. Representative dynamics combines structural and functional features. The first are accumulated in the class \mathbb{A} and the least are expressed by the function F . Both are interrelated. The situation is similar to one in the group theory of special functions. However, the difference is essential: in the representative dynamics the functional aspects are not derived from the structural ones and hve an independent origin.

Let us now describe the dynamical inverse problem of representation theory for controlled systems following the general ideology of the inverse problems of representation theory [6].

Definition 9. Let

$$(15) \quad \dot{x} = \varphi(x, u),$$

be the controlled system, where x is the time-dependent m -dimensional complex vector and u is the control parameter. *Dynamical inverse problem of representation theory for the controlled system* (15) is to construct a representative dynamics

$$\dot{\mathbf{X}} = F(\mathbf{X}, a)$$

and the function

$$a = a(u, x) \quad \text{such that} \quad \varphi(x, u) = f(x, a(u, x)),$$

where the operator function F is defined by the Weyl (symmetric) symbol f as a function of m non-commuting variables X_1, \dots, X_m .

Remark 15. If the controls are absent and the pair $a(t) = (\mathfrak{A}(t), \mathbf{e}(t))$ is time-independent, Def.9 is reduced to the definition of the dynamical inverse problem of representation theory [6].

Remark 16. One may consider dynamical inverse problem of representation theory for games, the interactively controlled systems and interactive games.

Remark 17. If the function φ contained some constants $c_\alpha \in \mathbb{C}$ then one may interpret them as time-independent variables and include the matrices $C_\alpha \in \text{Mat}_n(\mathbb{C})$ instead of them in the operator function F (compare with the quantization of constants [6]).

3.2. Tactical representative dynamics as dialectical game. Note that a representative dynamics may be considered as a parametric one. The parameter is a class \mathbb{A} of associative algebras. Such view allows to define the tactical representative dynamics.

Definition 10. The *tactical representative dynamics* is a tactical game, which is a representative dynamics as a verbalizable interactive game, with the variable classes \mathbb{A}_n of associative algebras as the comments.

The change of the class \mathbb{A}_n may be caused by the fact that the equations of representative dynamics become insolvable in the class \mathbb{A}_{n-1} .

One may describe solutions of many mathematical problems as tactical representative dynamics. The keypoint is an algorithm of the constructing of comments, i.e. how to derive a class \mathbb{A}_n from the preceding one \mathbb{A}_{n-1} . Though in the simplest case the recurrent formulas for comments are represented as

$$\vartheta_n = \Theta(\vartheta_{n-1}),$$

where ϑ_n is the pair (\mathbb{A}_n, η_n) of the class \mathbb{A}_n and additional parameters η_n , the general form is

$$\vartheta_n = \Theta(\vartheta_{n-1}, \omega_n, v_n)$$

as it is in the definition of a tactical game or explicitly

$$\begin{cases} \mathbb{A}_n = \mathbf{A}(\mathbb{A}_{n-1}, \eta_{n-1}, \omega_n, v_n) \\ \eta_n = \mathbf{H}(\eta_{n-1}, \mathbb{A}_{n-1}, \omega_n, v_n) \end{cases}$$

However, it is intuitively clear that the dialectical objects should be involved into this procedure because there are no any self-consistent ways to obtain any non-trivially and principally new class \mathbb{A}_n from the preceding one \mathbb{A}_{n-1} so \mathbb{A}_n should be self-derived from \mathbb{A}_{n-1} and the transition $\mathbb{A}_{n-1} \Rightarrow \mathbb{A}_n$ should be described in the internal terms of the class \mathbb{A}_n . So the tactical representative dynamics

corresponding to the solution of a mathematical problem should be a dialectical game and the algorithm of the comments' constructing should have the form

$$\vartheta_n = \Theta(\vartheta_{n-1}, \delta_n, \omega_n, v_n),$$

where δ_n are dialectical objects. Explicitely,

$$\begin{cases} \mathbb{A}_n = \mathbf{A}(\mathbb{A}_{n-1}, \eta_{n-1}, \delta_n, \omega_n, v_n) \\ \eta_n = \mathbf{H}(\eta_{n-1}, \mathbb{A}_{n-1}, \delta_n, \omega_n, v_n) \end{cases}$$

Such tactical representative dynamics will be called *dialectical*. The expressions for \mathbb{A}_n describe its synthesis, which is the dialectical synthesis of a class \mathbb{A}_n of associative algebras.

Remark 18. The dialectical representative dynamics may be considered as a model for the internal dialogue and reflection of a human. Here, the representations in mathematical and psychological sense coincide. The associative algebras $\mathfrak{A}(t)$ from the classes \mathbb{A}_n describe the cognitive aspect of the process. The recurrent formulas for comments correspond to the deduction.

Remark 19. The tactical (dialectical) representative dynamics solves the inverse problem of representation theory for the tactical (dialectical) games.

Remark 20. One can describe a dialectical object δ by the concrete transitions (syntheses) $\mathbb{A}_n \Rightarrow \mathbb{A}_{n+1}$, which it governs.

CONCLUSIONS

Thus, the tactical game theoretical interpretation of dialectics is given. Dialectical games are considered as abstractly as well as models of the internal dialogue and reflection. The models related to the representation theory (representative dynamics) are specially investigated in detail, they correlate with the hypothesis on the dialectical features of human thinking in general and mathematical thought (the constructing of a solution of mathematical problem) in particular.

Also the discussed subjects are significant for the understanding of tactical features of various psychophysiological processes and their couplings with cognitive ones during the functioning of the videocognitive and analogous integrated interactive systems and, thus in fact, for the clarification of the psychophysical nature of cognitive activity.

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