ANY 3-MANIFOLD 1-DOMINATES AT MOST FINITELY MANY GEOMETRIC 3-MANIFOLDS

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§1 Introduction.

Maps between 3-manifolds has been studied by many people long times ago, and become an active subject again after Thurston's revolution on 3-manifold theory. We refer to [BW], [LWZ1] for various results and references on the subject.

This paper addresses the following natural question which was raised around 1990, see also Kirby's Problem List, [K, 3.100].

Question 1. *Let M be a closed orientable 3-manifold. Are there at most finitely many closed, irreducible and orientable 3-manifolds N such that there exists a degree one map $f : M \to N$?*

Remarks on the conditions in Question 1.

(i) If Poincare Conjecture fails, i.e., there is a homotopy 3-sphere N which is not S^3, then one can get infinitely many reducible homotopy 3-spheres by doing connected sums on N. Since there always exists degree one map from a 3-manifold M to a homotopy 3-sphere, the condition "irreducible" on the target N is posed to avoid this unclear case.

(ii) The condition "closed" is posed on M and N just for simplicity. Indeed we can replace "closed" by "compact", and meanwhile replace "degree one map" by "degree one proper map". A map $f : M \to N$ between compact manifolds is proper if $f^{-1}(\partial N) = \partial M$.

For simplicity, we adapt the following definition from [BW]. Let M and N be two compact orientable 3-manifolds. Say M *(1-)dominates* N if there is a proper map $f : M \to N$ of non-zero degree (degree 1).

A closed orientable 3-manifold is called *geometric* if it admits one of the following geometries: H^3 (hyperbolic), $\widetilde{PSL_2}(\mathbb{R})$, $H^2 \times E^1$, Sol, Nil, E^3, $S^2 \times E^1$, S^3 (spherical). Thurston's geometrization conjecture claims that any closed, irreducible, and orientable 3-manifolds is either geometric or can be decomposed by

both authors are supported by Outstanding Youth Grants of NNSFC and Grants of QSSTF.

the Jaco-Shalen-Johannson torus decomposition so that each piece is geometric. (For details see [Th2], [Th3] or [Sc].) All geometric 3-manifolds are precisely the Seifert manifolds except those carry hyperbolic geometry or Sol geometry, and all geometric manifolds have infinite fundamental group except those carry spherical geometry.

It is natural to study Question 1 when the targets are geometric first. There are many partial results of Question 1:

(i) The answer is affirmative if both the domain and the target are Seifert manifolds with infinite fundamental group [Ro], which is based on Waldhausen's 3-manifold topology argument.

(ii) The answer is affirmative if the domain is non-Haken and the target is geometric [RW], which is based on Culler-Shalen's character variety theory of 3-manifold groups. Since the domain is non-Haken, then the geometry of the target must be either hyperbolic or spherical.

Note also that there are additional conditions posed on the domains in (i) and (ii). Two substantial result to the Question 1 are obtained recently, where no additional conditions are posed on the domains.

(iii) The answer is affirmative if the targets are hyperbolic [So], which is based on the argument of Thurston's original approach on the deformation of acylindrical manifolds.

(iv) The answer is affirmative if the target are spherical [LWZ2], which is based on the old knowledges of linking pair of 3-manifolds and of combinatorial groups.

In this paper we will prove the affirmative answer to Question 1 when the targets are all the remaining geometric 3-manifolds. The main result of this paper is the following.

Theorem 1. *Any orientable closed* 3*-manifold* M 1*-dominates at most finitely many closed orientable 3-manifolds which are either Seifert manifolds with infinite fundamental group or Sol manifolds.*

Combining Theorem 1 with the results of [So] and [LWZ2] we obtain the following assertion.

Corollary 1. *Any closed orientable* 3*-manifold* 1*-dominates at most finitely many geometric* 3*-manifolds.*

If an irreducible 3-manifold has non-trivial JSJ-decomposition, then each decomposition piece is either a hyperbolic 3-manifold or a Seifert manifolds with torus boundary. From the proof of Theorem 1, we have the following corollary, which should be useful in the discussion of non-trivial JSJ-torus decomposition case.

Corollary 2. *Any compact orientable* 3*-manifold* M *dominates at most finitely many Seifert manifolds with non-empty boundary or zero Euler number.*

In Section 2, we first explain that in the proof of the main result, one need only to deal with Seifert manifolds with orientable orbifold base and the torus bundle over the circle with Anosov monodromy. Then we present various known results about degree one map, Seifert manifolds, Thurston norm and volume of representations, including the brief descriptions of Thurston norm and of volume of representations. Those results will be used in the proof of the main results. The proof of the main result is given Section 3.

§2. Reductions and preliminary results.

Each Seifert manifold has an orbifold base which is either orientable or non-orientable.

Lemma 1. *If there is a closed orientable 3-manifold 1-dominates infinitely many closed orientable Seifert manifolds with non-orientable orbifold base, then there is a closed orientable 3-manifold 1-dominates infinitely many closed orientable Seifert manifolds with orientable orbifold bases.*

Proof. Suppose $f_j : M \to N_j$ is degree one map for all $j \in \mathbb{N}$, $p_j : N_j \to O_j$ is the projection from the closed orientable Seifert manifolds onto the non-orientable orbifold base O_j, and N_i and N_j are not homeomorphic if $i \neq j$. Let $\tilde{q}_j : \tilde{O}_j \to O_j$ be the unique orientable double cover of O_j and $q_j : \tilde{N}_j \to N_j$ be the double covering which covers $\tilde{q}_j$. Then $\tilde{N}_j$ is a closed orientable Seifert manifold with orientable orbifold base $\tilde{O}_j$. Since $f_j : M \to N_j$ is of degree one, $f_{j*} : \pi_1(M) \to \pi_1(N_j)$ is onto. This implies that the index of $f_*^{-1}(\pi_1(\tilde{N}_j))$ in $\pi_1(M)$ is two. Let $\tilde{M}_j$ be the double cover of M corresponding to the subgroup $f_*^{-1}(\pi_1(\tilde{N}_j))$. Then $f_j : M \to N_j$ can be covered by a degree one map $\tilde{f}_j : \tilde{M}_j \to \tilde{N}_j$. Since any finitely presented group has only finitely many subgroup of given index, $\pi_1(M)$ has only finitely many subgroup of index 2. It follows that there are only finitely many homeomorphic types among $\{\tilde{M}_j; j \in \mathbb{N}\}$. By passing to a subsequence, we may assume all $\tilde{M}_j = \tilde{M}$ and we have degree one map $f_j : \tilde{M} \to \tilde{N}_j, j \in \mathbb{N}$. Since any double covering is a regular covering, each orientable Seifert manifold double covers at most finitely many Seifert manifolds by [MS]. It follows the homeomorphic types of $\{\tilde{N}_j\}$ are infinite. □

Each Sol manifold is either a tours bundle over the circle or a union of two twisted I-bundle over Klein bottle.

Lemma 2. *If there is a closed orientable 3-manifold 1-dominates infinitely many Sol manifold which are unions of two twisted* I*-bundle over Klein bottle, then there* *is a closed orientable 3-manifolds 1-dominates infinitely many Sol manifolds which are torus bundle over S^1.*

Proof. Since each union of twisted I-bundle over Klein bottle is double covered by a torus bundle over the circle, the rest of the proof is the exactly same as that we did in the proof of Lemma 1. □

Let N be an orientable Seifert fibered space with orientable orbifold base F_g with n exceptional fibers. Then N has the standard form $(g, b; a_1, b_1; a_2, b_2; .....; a_n, b_n)$, $a_i > b_i > 0$. There are two invariants associated with N: the Euler characteristic of the orbifold base

$$\chi_N = 2 - 2g - \sum_{i=1}^{n}(1 - \frac{1}{a_i}),$$

and the Euler number of N

$$e(N) = -b - \sum_{i=1}^{n} \frac{b_i}{a_i}.$$

We now give a brief description for the volume of representation (see [Re], [Gr], [Th3] for more details). Let G be a semisimple Lie group and $X = G/K$, where K is the maximum compact subgroup of G. For any orientable closed manifold M and any representation $\phi : \pi_1(M) \to G$, there is a flat X-bundle over M, $M \times_\phi X = \frac{\tilde{M} \times X}{\pi_1(M)}$, with structure group G, where $\tilde{M}$ is the universal cover of M, $\pi_1(M)$ acting on the first factor $\tilde{M}$ by covering transformations, and by ϕ on the second factor X. For simplicity, we assume that $dimX = dimM = 3$ and X is contractible. Let ω' be the G-invariant volume form on X, which is a closed 3-form. Let $q : \tilde{M} \times X$ be the projection to the second factor. Then $q^*(\omega')$ is a $\pi_1(M)$-invariant closed 3-form on $\tilde{M} \times X$, and which induces a 3-form ω on $M \times_\phi X$. Let $s : M \to M \times_\phi X$ be a section. (Since X is contractible, such a section exists and all such sections are homotopic.) We call $\int_M s^*(\omega) \in \mathbb{R}$ the volume of the representation ϕ, denoted by $Vol(\phi)$, clearly it is independent of the choice of the section s. Define

$$Vol_G(M) = \max\{|Vol(\phi)|; \phi : \pi_1(M) \to G\}.$$

Note if some $\phi : \pi_1(M) \to G$ is discrete and faithful, then M support the geometry of (G, X) and $Vol_G(M) = Vol(\phi)$. We get the famous Gromov norm in the case $(G, X) = (PSL_2(\mathbb{C}), H^3)$, and we are interested the case $(G, X) = (\widetilde{PSL_2(\mathbb{R})} \ltimes \mathbb{Z}, \widetilde{PSL_2(\mathbb{R})})$ in this paper. For short we use $SV(M)$ to denote $Vol_{\widetilde{PSL_2(\mathbb{R})} \ltimes \mathbb{Z}}(M)$.

Lemma 3. *Let M and N be closed orientable 3-manifolds. If $f : M \to N$ is a degree one map, then*

(1) $TorH_1(M,\mathbb{Z}) = A \oplus TorH_1(N,\mathbb{Z})$, where $TorH_1$ is the torsion part of H_1.

If $f : M \to N$ is a map of degree $d \neq 0$, then

(2) $SV(M) \geqslant dSV(N)$.

(3) $[\pi_1(N) : f_*(\pi_1(M))]|d$.

Proof. For (1) see [Br, 1.2.5 Theorem]. For (2) see [BG] or [Re]. (3) is well-known and can be obtained directly by applying covering space argument. □

Let N be a Seifert manifold with the standard form

$$(g, b; a_1, b_1; a_2, b_2; ......; a_n, b_n), \qquad a_i > b_i > 0.$$

Lemma 4.

(1) N supporting the geometry of either $\widetilde{PSL_2}\mathbb{R}$, or Nil, *or $H^2 \times E^1$ is characterized by either $e(N) \neq 0$ and $\chi_N < 0$, or $e(N) \neq 0$ and $\chi_N = 0$, or $e(N) = 0$ and $\chi_N < 0$ respectively.*

(2) If $e(N) \neq 0$, then the order of the torsion part of $H_1(M,\mathbb{Z})$,

$$|TorH_1(N,\mathbb{Z})| = \left| \left(\prod_{i=1}^{n} a_i \right) \left(b + \sum_{i=1}^{n} \frac{b_i}{a_i} \right) \right| = \left| e(N) \prod_{i=1}^{n} a_i \right|.$$

(3) If N supports the geometry of $\widetilde{PSL_2}(\mathbb{R})$, Then

$$SV(N) = \left| \frac{\chi_N^2}{e(N)} \right|.$$

(4) The equation $\chi_N = 2 - 2g - \sum_{i=1}^{n}(1 - \frac{1}{a_i}) = 0$ has only finitely many solutions $(g, a_1, ..., a_n)$.

(5) If $\chi_N < 0$, then $\chi_N \leqslant -\frac{1}{42}$.

Proof. For (1) see [Sc]. For (2) see [LWZ1, 3.1]. For (3) see [BG]. (4) and (5) are well-known and can be obtained by elementary algebra. □

Now we give a brief description on Thurston norm. In a closed oriented 3-manifold N, each element $y \in H_2(N,\mathbb{Z})$ can be represented by an embedded oriented surface F. Let $\chi_-(F) = \max\{0, -\chi(F)\}$ if F is connected, otherwise $\chi_-(F) = \sum \chi_- F_i$, where F_i are components of F. Then let

$$X(y) = min\{\chi_-(F); F \text{ is an embedded surface representing } y\}.$$

Similarly we can define $X_s(y)$ if we replace "embedded surfaces" by "singular surfaces" in the definition of X (see [Th1] for details). X and X_s can be extended to the second homology H_2 with real coefficent and are often called Thurston norm and Thurston singular norm respectively.

Lemma 5.
(1) X is a pseudonorm on $H_2(M,\mathbb{R})$, in particular $mX(y)-nX(z)\leqslant X(my+nz)\leqslant mX(y)+nX(z)$.
(2) $X=X_s$.

Proof. For (1) see [Th1], and for (2) see [Ga]. □

Recall that there are only finitely many 3-manifolds support the geometries of $S^2\times E^1$ and E^3, and a torus bundle over the circle is a Sol manifold if and only if the gluing map is Anosov. With Lemma 1 and Lemma 2, (1) of Lemma 4 to prove Theorem 1, we need only to prove the following

Proposition 1. *Any orientable closed 3-manifold M 1-dominates at most finitely many closed orientable 3-manifolds N_j, where N_j belongs to one of the following classes:*
(a) Seifert manifolds with orientable orbifold bases with Euler number $e=0$ and Euler characteristic $\chi<0$.
(b) Seifert manifolds with orientable orbifold bases with Euler number $e\neq 0$ and Euler characteristic $\chi\leqslant 0$.
(c) torus bundle over the circle with Anosov monodromy.

§3. Proof of the Theorems.

In this section we will prove Proposition 1. Suppose contrarily that there is an orientable closed 3-manifold M 1-dominates infinitely many 3-manifolds N_j, where N_j is subject to the conditions in Theorem 1. By passing to a subsequence we may assume that all N_i's belong to one of the following classes:

(a) Seifert manifolds with orientable orbifold bases with Euler number $e=0$ and Euler characteristic $\chi<0$.

(b) Seifert manifolds with orientable orbifold bases with Euler number $e\neq 0$ and Euler characteristic $\chi\leqslant 0$.

(c) torus bundle over the circle with Anosov monodromy.

We will show that none of those three cases can happen.

Let $f_j: M\to N_j$ be a degree one map defining 1-domination. By (1) of Lemma 3, we have

$$|TorH_1(M,\mathbb{Z})|\geqslant|TorH_1(N_j,\mathbb{Z})|. \tag{1}$$

In the first two case, we have the Seifert manifold

$$N_j=(g_j,b_j;a_{j1},b_{j1};......a_{jn_j},b_{jn_j}),\qquad a_{ji}>b_{ji}>0.$$

Since $f_{i*} : \pi_1(M) \to \pi_1(N_j)$ is surjective by (3) of Lemma 3, the rank of $\pi_1(N_j)$ is at most the rank of $\pi_1(M)$. The rank $\pi_1(N_j)$ is at least $2g_j + n_j - 2$ [BZ], so g_j's and n_j's are bounded. Passing to a subsequence we may assume that $g_j = g$ and $n_j = n$ and we have

$$N_j = (g, b_j; a_{j1}, b_{j1}; ......a_{jn}, b_{jn}), \qquad a_{ji} > b_{ji} > 0 \tag{2}$$

Below we use e_j to denote $e(N_j)$ and χ_j to denote χ_{N_j}.

Proof of Case (a).

Each homology class y of $H_2(N_j, \mathbb{Z})$ can be presented by an incompressible surface. Since N_j is irreducible Seifert manifold, each incompressible surface is either a vertical torus (foliated by Seifert circles), or a horizontal surfaces (transversal to all Seifert circles) [p. 109, J]. Since $e_j = 0$, N_j admits horizontal surfaces.

Let O_j be the orbifold base of N_j and C_j be a regular fiber of N_j. Suppose also that O_j, C_j and N_j are compatible oriented.

Let F_j be the horizontal surface of N_j, and $p_j : F_j \to O_j$ is the branched covering. Then we have $\chi(F_j) = |d| \times \chi_j < 0$, where $d = deg(p_j)$ equals to $F_j \cap C_j$, the algebraic intersection number of F_j and a regular Seifert fiber of N_j. Note that $|F_j \cap C_j|$, the absolute value of algebraic intersection number, is precisely the geometric intersection number.

Suppose further F_j is a minimal genus horizontal surface of N_j, thus F_j is characterized by that $|F_j \cap C_j| > 0$ is minimal.

Let X_j be the Thurston norm on $H_2(N_j, \mathbb{R})$. Let $V_j = \{y \in H_2(N_j, \mathbb{Z}); X_j(y) = 0\}$, which is generated by vertical tori. Then V_j is a subgroup of $H_2(N_j, \mathbb{Z})$.

Lemma 6. $H_2(N_j, \mathbb{Z}) = \langle [F_j] \rangle + V_j$.

Proof. Pick any homology class $y \in H_2(N_j, \mathbb{Z})$. If $X_j(y) = 0$, then $y \in V_j$. Suppose $X_j(y) \neq 0$. Let F be a oriented horizontal surface representing y with $-\chi(F) = X_j(y)$. We may assume that the degree of $p_j : F \to O_j$ is positive (otherwise replace y by $-y$). Then $(l+1)X_j([F_j]) > X_j(y) \geqslant lX_j([F_j])$ for some positive integer l, that is

$$F_j \cap C_j > (F - lF_j) \cap C_j \geqslant 0.$$

Since $F_j \cap C_j$ is minimal among all positive intersections, we have $(F - lF_j) \cap C_j$ is zero, and therefore the minimal genus incompressible surface which represents $[F - lF_j]$ must be a union of tori. That is $[F - lF_j] \in V_j$ and $y = [lF_j] + [F - lF_j]$. □

Lemma 7. *If $f : M \to N$ is a map of degree $d \neq 0$, then $f_* : H_*(M, \mathbb{R}) \to H_*(N, \mathbb{R})$ is surjective.*

Proof. Recall that Poincare duality $P : H^{n-q}(N,\mathbb{R}) \to H_q(N,\mathbb{R})$ is given by $z^{n-q} \to z^{n-q} \cap [N]$, where $[N]$ ($[M]$) is the fundamental class class of $H_n(N,\mathbb{R})$ ($H_n(M,\mathbb{R})$) and we also have the formula

$$f_*(f^* z^p \cap [M]) = z^p \cap f_*[M] \tag{3}$$

for any map $f : M \to N$.

Let $z_q \in H_q(N,\mathbb{R})$. Let $y_q = \frac{1}{d} f^* \circ P^{-1}(z_g) \cap [M]$. Then by (3) we have

$$f_*(y_q) = f_*(\frac{1}{d} f^* \circ P_N^{-1}(z_q) \cap [M])$$

$$= \frac{1}{d} P^{-1}(z_q) \cap f_*[M] = \frac{1}{d} P^{-1}(z_q) \cap d[N] = z_q. \quad \square$$

Let X_M be the Thurston norm on $H_2(M,\mathbb{R})$, and X_{sj} be the Thurston singular norm on $H_2(N_j,\mathbb{R})$. Let $z_1, ..., z_m$ be a basis of $H_1(M,\mathbb{Z})$ and S_i be a surface representing z_i with $\chi_-(S_i) = X_M(z_i)$, for $i = 1, ..., n$.

Let $y_i = [f_j(S_i)] = l_{ji}[F_j] + v_{ij}$, where $v_{ij} \in V_j$. By (1) of Lemma 5, we have

$$X_j(y_i) = X_j(l_{ji}[F_j] + v_{ji}) \leqslant l_{ij} X_j([F_j]) - X_j(v_{ji}) = l_{ij} X_j([F_j]) \tag{4}$$

Then by (2) of Lemma 5 and the definition of Thurston singular norm, we have

$$X_j(y_i) = X_{sj}(y_i) \leqslant \chi_-(S_i) = X_M(z_i) \tag{5}$$

Combine (4) and (5), we have

$$X_M(z_i) \geqslant X_j(y_i) \geqslant l_{ji} X_j([F_j]). \tag{6}$$

Let $L = \max\{X_M(z_i); i = 1, ..., m\}$. If $X_j([F_j]) > L$, then (6) implies that $l_{ji} = 0$, and therefore $y_i = [f_j(S_i)] = v_{ij}$. It follows that $f_{j*}(H_2(M,\mathbb{Z})) \subset V_j$. It contradicts Lemma 7 that $f_{j*} : H_2(M,\mathbb{R}) \to H_2(N_j,\mathbb{R})$ is surjective.

So $L \geqslant X_j(F_j) > 0$. By passing to a subsequence we may assume that all $X_j(F_j)$ are the same, therefore all F_j's have the same homeomorphic types, denoted by S.

Cutting N_j along the horizontal surface S, we obtained an I-bundle over S, and therefore N_j can be presented as a surface bundle over S^1 with fiber S and monodromy $g_j : S \to S$. Since N_j is a Seifert manifold, g_j must be a periodic map [VI. 26., J]. However it is well-known that there are only finitely many periodic maps on the given surface S up to conjugacy. Since any two conjugated gluing map provide the homeomrphic 3-manifolds, there are only finitely many homeomorphic types among all N_j's. We reach a contradiction. We have proved that Case (a) cannot happen. □

Proof of Corollary 2. First note that Lemma 7 is stated for any degree $d \neq 0$, and is still true for proper maps between manifolds with non-empty boundaries. Then note that for manifolds with boundary, Thurston norm was established and Lemma 5 is still valid ([Th1] and [Ga]). Finally note that if N_j is a Seifert manifold with boundary, then N_j always contains a horizontal embedded surface. With those three facts. The proof of Corollary 2 is the same as the proof of Case (a). □

Proof for Case (b).

By (2) of Lemma 4, we have

$$|TorH_1(N_j, \mathbb{Z})| = \left|\left(\prod a_{ji}\right)\left(b_j + \sum \frac{b_{ji}}{a_{ji}}\right)\right| = \left|e_j \prod_{i=1}^{n} a_{ji}\right| \tag{7}$$

If all a_{ji}'s are uniformly bounded, then all b_{ji}'s are uniformly bounded. Since we assume that all N_j's are in different homeomorphic type, we must have that b_j is unbounded.

Since $\prod a_{ji} > 1$ and $|\sum \frac{b_{ji}}{a_{ji}}| \leqslant n$, we have

$$\left|\left(\prod a_{ji}\right)\left(b_j + \sum \frac{b_{ji}}{a_{ji}}\right)\right| \geqslant |b_j - n| \tag{8}$$

By (1) we have $|TorH_1(N_j, \mathbb{Z})|$ is bounded for all j, and by (7) and (8) we have $|TorH_1(N_j, \mathbb{Z})|$ is unbounded. We reach a contradiction.

By (4) of Lemma 4, we have ruled out the situation that $\chi = 0$. Below we assume that $\chi_j < 0$, i.e., all N_j support the $\widetilde{PSL_2\mathbb{R}}$ geometry.

Now we assume that some a_{ji} tends to infinite as j tends to infinite up to a subsequence. Then $|\prod a_{ji}|$ tends to infinite as j tends to infinite. To be not contradicted with (1) and (7), We must have

$$|e_j| = \left|b_j + \sum \frac{b_{ji}}{a_{ji}}\right| \to 0 \qquad \text{as } j \to \infty \tag{9}$$

Since $\chi_j < 0$, we have $\chi_j \leqslant -\frac{1}{42}$ by (5) of Lemma 4, and then $|\chi_j| \geqslant \frac{1}{42}$. Then by (5) of Lemma 4, we have

$$SV(N_j) = \left|\frac{\chi_j^2}{e_j}\right| \geqslant \left|\frac{1}{(42)^2 e_j}\right|,$$

which is tends to infinite as j tends to infinite. But by (2) of Lemma 4 we have

$$SV(M) \geqslant SV(N_j) \tag{11}$$

i.e. $SV(N_j)$ is uniformly bounded for all j. We reach a contradiction again. We have proved that Case (b) cannot happen. $\square$

Proof of Case (c).

Now N_j is a torus T bundle over the circle with Anosov map g_j, denoted as (T, g_j) sometimes. Fix a basis of $H_1(T, \mathbb{Z})$. Let $SL_2(\mathbb{Z})$ be the group of 2 by 2 invertible integer matrices. Let $A_j \in SL_2(\mathbb{Z})$ presents g_j under the chosen basis of $H_1(T, \mathbb{Z})$. Then A_j has two real eigenvalues of λ_j and λ_j^{-1}, with $|\lambda_j| > 1$.

Using HHN extension one can calculate directly that

$$TorH_1(N_j, \mathbb{Z}) = \frac{H_1(T, \mathbb{Z})}{\langle I - A_j \rangle},$$

where I is the unit of $SL_2(\mathbb{Z})$, and the first Betti number of N_j is 1.

Then by linear algebra we have

$$|TorH_1(N_j, \mathbb{Z})| = |I - A_j| = |(1 - \lambda_j)(1 - \lambda_j^{-1})| = |(2 - (\lambda_j + \lambda_j^{-1})| \tag{12}$$

By (1), $|TorH_1(N_j, \mathbb{Z})|$ is uniformly bounded, then by (12), the absolute value of the trace A_i, $|\lambda_j + \lambda_j^{-1}|$, is uniformly bounded, say by some constant $k > 0$.

Let (T, g) and (T, g') be two torus bundles over the circle with Anosov maps g and g'. Let A and A' are matrices associated with g and g' under the given basis of $H_1(T, \mathbb{Z})$. If $A = BA'B^{-1}$, for some $B \in SL_2(\mathbb{Z})$, then $g = hg'h^{-1}$, is induced by $h : T \to T$ is a homeomorphism realizing B. It follows easily that (T, g) and (T, g') are homeomorphic.

Now a contradiction in this case will follows by the following lemma.

Lemma 8. *There are only finitely many conjugacy classes in $SL_2(\mathbb{Z})$ representing Anosov maps with the absolute values of the traces are bounded by $k > 0$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since A represents an Anosov map, $bc \neq 0$. Suppose $|a| > k$. Since $|a + d| \leqslant k$, we have $|d| < 2|a|$, and then $|ad| < 2a^2$. The fact $ad - bc = 1$ implies that $|bc| < 2a^2 + 1$. In particular, either $|b|$ or $|c|$ is at most $\sqrt{2}|a|$. If $|b| \leqslant \sqrt{2}a$, let $C = \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$, where we chose 1 if $ab > 0$ and -1 if $ab < 0$. Then $CAC^{-1} = \begin{pmatrix} a \mp b & * \\ * & * \end{pmatrix}$ and we can make $|a \pm b| \leqslant |a|$. If $|c| \leqslant \sqrt{2}a$, let $C = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $CAC^{-1} = \begin{pmatrix} a \mp c & * \\ * & * \end{pmatrix}$ and we can make $|a \pm c| \leqslant |a|$. This concludes that if $|a| > k$, we always can get $A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ which is conjugate to A in $SL_2(\mathbb{Z})$ and $|a_1| < |a|$.

Therefore to prove the Lemma, we may assume that $|a| \leqslant k$. Then similarly since $|a + d| \leqslant k$ we have $|d| \leqslant 2k$, and then $|ad| \leqslant 2k^2$. Since $ad - bc = 1$, $|bc| \leqslant 2k^2 + 1$. In particular all entries are bounded by $2k^2 + 1$. Clearly there are only finitely many such elements in $SL_2(\mathbb{Z})$.

We have proved that Case (c) cannot happen. □

We have completed the proof of Proposition 1, and therefore the proof of Theorem 1.

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