Chow groups of weighted hypersurfaces.

Marco Leoni

Abstract — We extend a result of to Esnault-Levine-Viehweg concerning the Chow groups of hypersurfaces in projective space to those in weighted projective spaces.

1 Introduction

The purpose of this paper is to generalize to the case of weighted projective spaces over an algebraically closed field $\mathbf{K}$ the following result from [ELV]:

Theorem 1.1 [ELV], Th. 4.6. *Let $X \subset \mathbf{P}^n$ be a hypersurface of degree $d \geq 3$ and let $s \leq n-1$ be an integer such that:*

$$\left(\begin{array}{c} s+d \\ s+1 \end{array} \right) \leq n.$$

Then $\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q}$.

Let $Q = (q_0, \ldots, q_n) \in \mathbf{N}^{n+1}$. Let $\mu_a := \{z \in \mathbf{K} | z^a = 1\}$ and set $\mu := \prod_{i=1}^n \mu_{q_i}$. The weighted projective space $\mathbf{P}(Q)$ can be realized either as the quotient $\mathbf{P}^n/\mu$ (with the action defined by componentwise multiplication) or as the quotient $\mathbf{K}^{n+1}/\mathbf{K}^*$, the action being defined by $t\cdot(x_0, \ldots x_n) := (t^{q_0}x_0, \ldots, t^{q_n}x_n)$. The map $\varphi_Q : [t_0 : \ldots : t_n]_\mu \to [t_0^{q_0}, \ldots, t_n^{q_n}]_{\mathbf{K}^*}$ gives the isomorphism between the two representations. One deduces from this that there is a one to one correspondence between hypersurfaces $X := \{f = 0\}$ of $\mathbf{K}^{n+1}/\mathbf{K}^*$ and those of $\mathbf{P}^n/\mu$ defined by the zeroes of the polynomial $f'([t_0 : \ldots ; t_n]_\mu) := f([t_0^{q_0}, \ldots, t_n^{q_n}]_{\mathbf{K}^*})$. If f' is smooth, the hypersurface $\{f = 0\}$ is seen to be *quasismooth*: the cone $\mathcal{C}_X := \{x \in \mathbf{K}^{n+1} | f(x) = 0\}$ has one singularity in the origin.

2 The Main Result.

Theorem 2.1 *For a smooth irreducible weighted hypersurface X' of degree $d \geq 3$ in $\mathbf{P}^n$ and $\forall l \in \mathbf{N}$ such that:*

$$\left(\begin{array}{c} d+l \\ 1+l \end{array} \right) \leq \sum_{j=0}^n q_j - 1$$

one has:

$$\mathrm{CH}_l(X) \otimes \mathbf{Q} = \mathbf{Q}$$

where $X = X'/\mu$.

Proof:
Let:

$$N := (\sum_{j=0}^{n} q_j) - 1$$

$$N_r := \left\{ \begin{array}{cc} 0 & r = -1 \\ \sum_{j=0}^{r} q_j & \forall r = 0, \ldots, n \end{array} \right.$$

Remark in particular that $N_n = N + 1$, and that $N_r - N_{r-1} = q_r \ \forall r = 0, \ldots, n$. Define a rational map:

$$\sigma_Q : \mathbf{P}^N \to \mathbf{P}(Q)$$

by:

$$(\sigma_Q([t_0 : \ldots : t_N]))_r := \prod_{j=N_{r-1}}^{N_r - 1} t_j \ \forall r = 0, \ldots, n.$$

Set:

$$\mathcal{J}_Q := \{(j_0, \ldots, j_n) \in \mathbf{N}^{n+1} : N_{r-1} \leq j_r \leq N_r - 1 \, \forall r = 0, \ldots, n\}$$

and consider $\forall J \in \mathcal{J}_Q$, the subvarieties:

$$Z_J := \left\{ t \in \mathbf{P}^N : t_{j_0} = \ldots = t_{j_n} = 0 \right\}$$

$$Z_Q := \cup_{J \in \mathcal{J}_Q} Z_J$$

It is clear that σ_Q is only defined on $\mathbf{P}^N - Z_Q$. This map is well-defined on $\mathbf{P}^N - Z_Q$: indeed, if one considers lt_j instead of t_j for a nonzero l, one has:

$$\prod_{j=N_{r-1}}^{N_r - 1} lt_j = l^{q_r} \prod_{j=N_{r-1}}^{N_r - 1} t_j$$

so that modulo the weighted action of $\mathbf{K}^*$ these two quantities coincide.

Also, σ_Q is onto, since if $x \in \mathbf{P}(Q)$ and $(x_0, \ldots, x_n)$ is a representative in ${\mathbf{K}^{n+1}}^*$, one may choose, $\forall r = 0, \ldots, n$, some $q_r - 1$ variables freely and the last one such that $x_r = \prod_{j=N_{r-1}}^{N_r - 1} t_j$. So:

$$\forall x \in \mathbf{P}(Q), \dim(\sigma_Q^{-1}(x)) = \sum_{r=0}^{n} (q_r - 1) = N - n$$

Let $X \subset \mathbf{P}(Q)$ be a weighted homogeneous hypersurface of Q-degree $d \geq 3$. If X is defined by the weighted homogeneous polynomial $f = f(x_0, \ldots, x_n)$, we define $\tilde{X}$ in $\mathbf{P}^N$ by the polynomial $\tilde{f} = \tilde{f}(t_0 : \ldots : t_N)$, of the same degree, obtained by replacing x_k by $\prod_{j=N_{k-1}}^{N_k - 1} t_j$. The map σ_Q induces a rational map:

$$\sigma_Q : \tilde{X} \to X.$$

Let R be the plane in $\mathbf{P}^N$ defined by the equations:

$$t_{N_{r-1}} = \ldots = t_{N_r - 1} \quad \forall r = 0, \ldots, n$$

The number of equations which define it is:

$$\sum_{r=0}^{n}(N_r - 1 - N_{r-1}) = \sum_{r=0}^{n} q_r - (n+1) = N - n$$

Let $S := R \cap \tilde{X}$. Then this linear space has dimension n and has, by construction, the fundamental property that $S \cap Z_Q = \emptyset$:

$$\forall t \in Z_Q, \forall r | 0 \leq r \leq n, \exists i \text{ such that } N_{r-1} \leq i \leq N_r - 1 \text{ for which } t_i = 0$$

But then in S, $t_{N_{r-1}} = 0$ also and all the other t_j with j in the rth string are also zero. This for every r.

Let

$$u : \mathrm{Bl}_{Z_Q}(\tilde{X}) \to \tilde{X}$$

be the blow-up along Z_Q turning σ_Q into a morphism:

$$\begin{array}{ccc} \mathrm{Bl}_{Z_Q}(\tilde{X}) & \stackrel{\hat{\sigma}_Q}{\to} & X \\ \downarrow u & & || \\ \tilde{X} & \stackrel{\sigma_Q}{\to} & X \end{array}$$

Let:

$$l_0 := \max_{l \in \mathbf{N}} \left\{ \left(\begin{array}{c} l+d \\ l+1 \end{array} \right) \leq N \right\} \cap \{ l \in \mathbf{N} | l \leq n \}$$

We know from [ELV], Theorem 4.6., that if $s \leq l_0$, then:

$$\mathrm{CH}_s(\tilde{X}) \otimes \mathbf{Q} = \mathbf{Q}$$

So let's take $\gamma \in \mathrm{CH}_s(X) \otimes \mathbf{Q}$ where $s \leq l_0$. Set $\tilde{\gamma} := \tau^{-1}(\gamma)$, being $\tau := \sigma_Q|_S$.

Certainly $\tilde{\gamma}$ is an s-cycle on $\tilde{X}$ which is supported on S. Therefore there is some $a \in \mathbf{Q}$ and a $\Gamma \in \mathrm{Gr}(s+1)$ such that:

$$\tilde{\gamma} \sim_{\tilde{X}} [\Gamma \cap \tilde{X}] = a\Gamma \cdot \tilde{X}$$

Since $\mathrm{CH}_s(\mathbf{P}^N) \otimes \mathbf{Q} = \mathbf{Q}$, one can eventually replace Γ by another $(s+1)$-plane which is transversal to Z_Q. Therefore we may assume that the proper transform of Γ under the blow-up along Z_Q, which I denote by $\hat{\Gamma}$, is isomorphic to Γ itself. Certainly $\hat{\tilde{\gamma}} \simeq \tilde{\gamma}$ because $\tilde{\gamma} \cap Z_Q = \emptyset$. Therefore we deduce:

$$\hat{\tilde{\gamma}} \sim_{\mathrm{Bl}_{Z_Q}(\tilde{X})} b\hat{\Gamma} \cdot \mathrm{Bl}_{Z_Q}(\tilde{X})$$

Since X' is smooth, and since μ is a finite group, by [FU], Ex 11.4.7., we have a "moving lemma" on $X = X'/\mu$. Therefore we can move γ inside X in such a way that that it is not in the ramification locus of $\hat{\sigma}_Q$. Hence $\hat{\sigma}_Q$ is finite of a certain nonzero degree, say e.

So we deduce:

$$\hat{\sigma}_{Q*}\hat{\tilde{\gamma}} = e\gamma$$

while:

$$\hat{\sigma}_{Q*}(\hat{\Gamma}' \cdot \mathrm{Bl}_{Z_Q}(\tilde{X})) = eH_{s+1} \cdot eX$$

being H_{s+1} the generator of $\mathrm{CH}_{s+1}(\mathbf{P}(Q)) \otimes \mathbf{Q} = \mathbf{Q}$.

Therefore $\gamma \sim_X be^2 H_{s+1} \cdot X = tH_s$, with H_s generator of $\mathbf{P}(Q) \otimes \mathbf{Q} = \mathbf{Q}$. This shows $\mathrm{CH}_s(X) \otimes \mathbf{Q} = \mathbf{Q}$ $\forall s \leq l_0$.

QED

Remark 2.1 In the preceding proof a moving Lemma is used; for this reason X' should have at most quotient singularities. One can probably avoid this as to arrive at the true generalization of the result in [ELV] valid irrespective of the singularities.

Essentially the same method also works for complete intersections so that appropriate analogues of [ELV] Prop. 3.5 and Thm. 4.6. hold. In view of technical complications we preferred to state and give the proof for hypersurfaces onl

References:

[ELV] H.Esnault- M.Levine- E.Viehweg "*Chow groups of projective varieties of very small degree*" Duke Math. Journal 87 n.1 (1997) 29-58.

[FU] W.Fulton "*Intersection Theory*", Springer-Verlag, Berlin, 1984.