Foliations with Transversal Quaternionic Structures

by
Paolo Piccinni and Izu Vaisman

ABSTRACT. We consider manifolds equipped with a foliation $\mathcal{F}$ of codimension $4q$, and an almost quaternionic structure Q on the transversal bundle of $\mathcal{F}$. After discussing conditions of projectability and integrability of Q, we study the transversal twistor space $Z\mathcal{F}$ which, by definition, consists of the Q-compatible almost complex structures. We show that $Z\mathcal{F}$ can be endowed with a lifted foliation $\widehat{\mathcal{F}}$ and two natural almost complex structures J_1, J_2 on the transversal bundle of $\widehat{\mathcal{F}}$. We establish the conditions which ensure the projectability of J_1 and J_2, and the integrability of J_1 (J_2 is never integrable).

1 Preliminaries

We recall the basic definitions of quaternionic geometry e.g., [3, 20]. The general framework is the C^∞ category.

An *almost hypercomplex structure* on a $4q$-dimensional differentiable manifold N^{4q} is an ordered triple $H = (I_1, I_2, I_3)$ of almost complex structures satisfying the quaternionic identities $I_\alpha \circ I_\beta = I_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and cyclic permutations. If the structures I_1, I_2, I_3 are integrable, H is said to be a *hypercomplex structure*.

* *1991 Mathematics Subject Classification* 53 C 12, 53 C 25.

Key words and phrases: Foliation,Quaternionic Structure, Twistor Space.

Acknowledgements. This paper was started while the second author visited the Istituto di Matematica, Università di Roma 1, under the auspices of the CNR, Italy. He is expressing here his thanks to the hosts. The first author acknowledges support by MURST of Italy and support and hospitality by the Erwin Schrödinger Institute for Mathematical Physics in Vienna.

If $H = (I_1, I_2, I_3)$ is an almost hypercomplex structure, any triple (J_1, J_2, J_3) obtained from (I_1, I_2, I_3) by multiplying by a matrix of $SO(3)$ is again an almost hypercomplex structure. Moreover, there exists a set of *compatible almost complex structures* associated with a given almost hypercomplex structure namely, the set of all $J = a_1I_1 + a_2I_2 + a_3I_3$ where a_1, a_2, a_3 are functions satisfying $a_1^2 + a_2^2 + a_3^2 = 1$.

An *almost quaternionic structure* on the manifold N^{4q} is a rank 3 vector subbundle Q of the endomorphism bundle $End(TN)$ locally spanned by almost hypercomplex structures $H = (I_1, I_2, I_3)$ which are related by $SO(3)$-matrices on the intersections of trivializing open sets. A *quaternionic structure* on the manifold N^{4q} is an almost quaternionic structure such that there exists a torsionless connection ∇ of TN which, when extended to the vector bundle $End(TN)$, preserves the subbundle Q i.e., $\nabla Q \subseteq Q$. The existence of the Q-preserving torsionless connection ∇ is not equivalent with the integrability of Q as a G-structure. The existence of a *flat* torsionless connection which preserves Q implies that Q can be obtained from local quaternionic coordinates on N. If an almost quaternionic structure Q is fixed on N^{4q}, the *local bases* (I_1, I_2, I_3) which span the vector bundle Q are also called *local compatible almost hypercomplex structures*, and any local $J = a_1I_1 + a_2I_2 + a_3I_3$ with $a_1^2 + a_2^2 + a_3^2 = 1$ is called a *local Q-compatible almost complex structure*.

A Riemannian metric g on a (almost) hypercomplex manifold (N, H) is *(almost) hyperhermitian*, respectively *(almost) hyperkähler*, if it is (almost) Hermitian, respectively, (almost) Kählerian, with respect to all the structures I_α, $\alpha = 1, 2, 3$, of H. (Then, g also is compatible with any H-compatible structure J.) Similarly, on an almost quaternionic manifold (N, Q), the metric g is *quaternion Hermitian* if it is Hermitian with respect to the local bases (I_α) of Q, and it is *quaternion Kähler* if it is quaternion Hermitian and Q is parallel (i.e., $\nabla Q \subseteq Q$) with respect to the Levi-Civita connection ∇ of g. (In both cases, the property says nothing about the integrability of the structures I_α.) The terms are also used for manifolds endowed with the respective structures. Of course, a hyperkähler manifold necessarily is hypercomplex, and a quaternion Kähler manifold necessarily is quaternionic.

The twistor space ZN of an almost quaternionic manifold (N, Q) is defined as the manifold of the Q-compatible almost complex structures of the tangent spaces of N.Thus, ZN is an S^2-bundle associated with the vector bundle Q, where Q has the metric which makes the local bases $H = (I_1, I_2, I_3)$ orthonormal bases [3, 20].

Now, let us consider a C^∞ manifold M^{p+4q}, equipped with a p-dimensional foliation $\mathcal{F}$. Denote by $L = T\mathcal{F}$ the tangent bundle of $\mathcal{F}$, and by $\nu\mathcal{F} = TM/L$ its transversal vector bundle of rank $4q$. We will often identify the transversal bundle $\nu\mathcal{F}$ with a complementary distribution E of L i.e., a splitting of the exact sequence

$$0 \to L = T\mathcal{F} \overset{\subseteq}{\to} TM \overset{\pi_\nu}{\to} \nu\mathcal{F} \to 0.$$

Almost hypercomplex and almost quaternionic structures can be defined similarly on vector bundles of rank $4q$. Accordingly, they will be reductions of the structure group of the bundle to G, where the group $G = GL(q, \mathbf{H})$ for the almost hypercomplex structures and

$$G = GL(q, \mathbf{H}) \cdot Sp(1) = \frac{GL(q, \mathbf{H}) \times Sp(1)}{\pm Id}$$

for the almost quaternionic structures ($\mathbf{H}$ is the algebra of the quaternions). Furthermore, almost hyperhermitian and quaternion Hermitian structures correspond to the structure groups $Sp(q)$ and $Sp(q) \cdot Sp(1)$, respectively.

We will consider structures of these types on the transversal bundle $\nu\mathcal{F}$ of a foliation $\mathcal{F}$, and refer to them as *transversal almost hypercomplex, transversal almost quaternionic, etc. structures* of the foliation $\mathcal{F}$. (Such structures sporadically appeared in the literature e.g., [12].)

In what follows, we use *Bott connections* [4] $D : \Gamma TM \times \Gamma\nu\mathcal{F} \to \Gamma\nu\mathcal{F}$. A Bott connection is a connection on the transversal bundle $\nu\mathcal{F}$ which extends the *partial connection* $\overset{\circ}{D}\colon \Gamma L \times \Gamma\nu\mathcal{F} \to \Gamma\nu\mathcal{F}$ given by

$$\overset{\circ}{D}_Y\, s = \pi_\nu[Y, X_s], \tag{1.1}$$

where Y is a tangent vector field of the leaves of the foliation $\mathcal{F}$, and X_s is any vector field on M such that $\pi_\nu X_s = s$, $s \in \Gamma\nu\mathcal{F}$. ($\Gamma$ always denotes spaces of global cross sections of vector bundles.) Notice that an identification $\nu\mathcal{F} \approx E$, where $TM = E \oplus L$, implies the replacement of (1.1) by

$$\overset{\circ}{D}_Y\, X = \pi[Y, X], \quad X \in \Gamma E, \tag{1.1$'$}$$

π being the projection $\pi : TM \to E$.

A Riemannian metric g splits $TM = T\mathcal{F} \oplus T^\perp\mathcal{F}$, and we will take $E = T^\perp\mathcal{F} \approx \nu\mathcal{F}$. Then, in particular, $\overset{\circ}{D}$ can be extended to a Bott connection D, by defining $D_X = \pi \circ \nabla_X$ ($X \in \Gamma E$), where ∇ is the Levi-Civita connection of g. For Riemannian foliations, this Bott connection D is the unique torsionless metric connection of the normal bundle $T^{\perp}\mathcal{F} \approx \nu\mathcal{F}$ [11].

1.1 Definition. (a) A transversal almost hypercomplex structure $H = (I_1, I_2, I_3)$ of $\mathcal{F}$ is *projectable* if the partial connection $\overset{\circ}{D}$ preserves the structures I_α, i. e. $\overset{\circ}{D} I_\alpha = 0$, $\alpha = 1, 2, 3$.
(b) A transversal almost quaternionic structure $Q \subseteq End(\nu\mathcal{F})$ *is projectable* if $\overset{\circ}{D}$ preserves Q: $\overset{\circ}{D} Q \subseteq Q$.

The projectability condition of Q can be formulated in terms of local bases $H = (I_1, I_2, I_3)$. Namely, Q is projectable iff, for any choice of a local basis $H = (I_1, I_2, I_3)$, there exist local 1-forms α, β, γ such that:

$$\overset{\circ}{D} I_1 = \alpha I_2 + \beta I_3, \quad \overset{\circ}{D} I_2 = -\alpha I_1 + \gamma I_3, \quad \overset{\circ}{D} I_3 = -\beta I_1 - \gamma I_2. \tag{1.2}$$

As a matter of fact, if equations of the type (1.2) hold for some choice of H similar equations hold for any choice of H.

If $J \in \Gamma End(\nu\mathcal{F})$ we may also see it as a cross section of $End\, E$, and for $Y \in \Gamma T\mathcal{F}$, $s \in \Gamma\nu\mathcal{F}$ we have

$$(\overset{\circ}{D}_Y J)s = \overset{\circ}{D}_Y Js - J \overset{\circ}{D}_Y s = \pi_\nu[Y, JX_s] - J\pi_\nu[Y, X_s] \tag{1.3}$$

for any $X_s \in \Gamma TM$ such that $s = \pi_\nu X_s$. A cross section s is *projectable* if s projects to a tangent vector field of any local *space of slices* of the foliation $\mathcal{F}$ [11]. From (1.3), it follows that J is projectable iff Js is projectable whenever s is projectable. Therefore, projectability in the sense of Definition 1.1 (a) means that we have a structure which is the lift of almost hypercomplex structures of the local slice spaces. The same is true in the case of Definition 1.1 (b) (see Proposition 3.1 later on).

Accordingly, we will give

1.2 Definition. A projectable, transversal, almost hypercomplex or almost quaternionic structure of a foliation $\mathcal{F}$ is *integrable* if the projected structures of the local slice spaces are hypercomplex or quaternionic, respectively.

If integrability holds, the word *almost* will be omitted.

2 Examples

We begin by an example of a foliation with a transversal almost hypercomplex structure. Let $\mathcal{F}$ be a transversally holomorphic foliation of real codimension $2q$ on a manifold N^{p+2q}. This means that, on N, there are local coordinates $(y^u, z^a, \bar{z}^a)$, where (y^u) are real coordinates, and (z^a) are complex coordinates with holomorphic transition functions, such that $\mathcal{F}$ is defined by $z^a = const.$, $\bar{z}^a = const.$ Furthermore, assume that g is a bundle-like Riemannian metric on N, which is $\mathcal{F}$-transversally Kähler i.e.,

$$g = g_{a\bar{b}}(z,\bar{z})dz^a \otimes d\bar{z}^b + g_{\bar{a}b}(z,\bar{z})d\bar{z}^a \otimes dz^b + \cdots, \tag{2.1}$$

where the first two terms define a Kähler metric in the coordinates (z), and the remaining (unexplicited) terms contain dy^u. (In this paper, we use the Einstein summation convention.)

Now, consider the manifold M defined by the total space of the conormal bundle of the foliation $\mathcal{F}$ i.e., the annihilator of the tangent bundle $T\mathcal{F}$. If E is the g-orthogonal bundle of $\mathcal{F}$, then $M = E^*$. On M, there exists a natural lift $\mathcal{F}^*$ of $\mathcal{F}$ such that the leaves of $\mathcal{F}^*$ are covering spaces of the leaves of $\mathcal{F}$. Moreover, the local slice spaces of $\mathcal{F}^*$ are cotangent bundles of Kähler manifolds. It is not difficult to construct an almost hypercomplex structure on such a cotangent bundle. If we do this on the local slice spaces, the obtained structures glue up to a projectable almost hypercomplex structure transversal to $\mathcal{F}^*$. The construction of the almost hypercomplex structure of the cotangent bundle of a Kähler manifold was described in [23]. For the reader's convenience, we present here the previously mentioned transversal structure of $\mathcal{F}^*$ directly.

As in [21, 22], let

$$\theta^u = dy^u + t^u_a dz^a + \bar{t}^u_a d\bar{z}^a \tag{2.2}$$

be a basis of the annihilator of E. Then, $\forall \zeta \in E^*$ we have

$$\zeta = \zeta_a dz^a + \bar{\zeta}_a d\bar{z}^a, \tag{2.3}$$

$(y^u, z^a, \bar{z}^a, \zeta_a, \bar{\zeta}_a)$ are local coordinates on M, and the system of equations

$$z^a = const., \ \bar{z}^a = const., \ \zeta_a = const., \ \bar{\zeta}_a = const.$$

defines the foliation $\mathcal{F}^*$. Obviously, $\mathcal{F}^*$ is again a transversally holomorphic foliation, and we will denote by I_1 the corresponding transversal complex structure. That is, I_1 is a complex structure on the transversal bundle $\nu\mathcal{F}^*$ which, in turn, may be identified with the complementary bundle S of $T\mathcal{F}^*$ given by the equations $\theta^u = 0$ on $M = E^*$. As usual, we may identify (S, I_1) with the holomorphic part $S_{1,0}$ of $S \otimes_{\mathbf{R}} \mathbf{C}$.

Then, S also has a canonical symplectic structure namely, if seen on M, (2.3) is a 1-form which may be viewed as the $\mathcal{F}^*$-*transversal Liouville form* ζ, and $-d\zeta$ is the mentioned symplectic structure. When transferred to $S_{1,0}$, these structures go to $\lambda = \zeta_a dz^a$ and $\omega = dz^a \wedge d\zeta_a$, respectively.

Furthermore, the Levi-Civita connection of the transversal Kählerian part of g yields a connection on E with a corresponding horizontal distribution $\mathcal{H}$ on E^* given by [23]

$$d\zeta_a - \Gamma^c_{ab}\zeta_c dz^b = 0, \;\; d\bar{\zeta}_a - \bar{\Gamma}^c_{ab}\bar{\zeta}_c d\bar{z}^b = 0, \tag{2.4}$$

where the coefficients Γ are the Christoffel symbols. The equations (2.4) completed by $\theta^u = 0$ define the *horizontal part* $\mathcal{H}_S$ of the bundle S. Of course, S is tangent to the fibers of E^* i.e., it contains the *vertical distribution*, say $\mathcal{V}$, of this bundle. As a matter of fact, we have $S = \mathcal{H}_S \oplus \mathcal{V}$.

Now, continuing with the identification $(S, I_1) \approx S_{1,0}$, we see that a new complex structure I_2 of S can be obtained by asking

$$I_2/_{\mathcal{H}_S} = \overline{\sharp_\omega \circ \flat_g}, \;\; I_2^2 = -Id, \tag{2.5}$$

where the *musical isomorphisms* are defined as in Riemannian geometry and the bar denotes complex conjugation (of course, only the transversal part of g is used).

Finally, the same computations as in [23] show that $(I_1, I_2, I_3 := I_1 \circ I_2)$ is an almost hypercomplex structure on the vector bundle S, and this is the announced example

Now, we will describe two classes of examples of foliations with projectable, transversal quaternionic structure, which come from 3-Sasakian and quaternion Hermitian-Weyl geometry, respectively (cf. [5, 6, 17, 18]).

A triple (ξ^1, ξ^2, ξ^3) of orthonormal Killing vector fields on a $(4q+3)$-dimensional Riemannian manifold $(\mathcal{S}, g)$ is said to define a 3-*Sasakian structure* if their brackets satisfy the identities

$$[\xi^\alpha, \xi^\beta] = 2\xi^\gamma$$

($(\alpha,\beta,\gamma) = (1,2,3)$ and cyclic permutations), and, furthermore, the dual 1-forms $\eta^\alpha = \flat_g \xi^\alpha$ satisfy the equations

$$(\nabla_Y \Phi^\alpha) Z = \eta^\alpha(Z) Y - g(Y,Z)\xi^\alpha \tag{2.6}$$

($\alpha = 1,2,3$) where ∇ is the Levi-Civita connection of g, and $\Phi^\alpha = \nabla \xi^\alpha \in End(T\mathcal{S})$.

A manifold $(\mathcal{S}, g)$ with a 3-Sasakian structure is a 3-*Sasakian manifold*, and the vector fields ξ^1, ξ^2, ξ^3 span a foliation $\mathcal{V}$ of $\mathcal{S}$. Furthermore, $\mathcal{V}$ is invariant by the endomorphisms Φ^α, and it has the orthogonal distribution E defined by $\eta^1 = 0$, $\eta^2 = 0$, $\eta^3 = 0$. Therefore, E also is Φ^α-invariant. Following Proposition 1.2.4 of [5], one has

$$(\Phi^\alpha)^2 = -I + \eta^\alpha \otimes \xi^\alpha,$$

and $(I_1 = -\Phi^1/_E,\ I_2 = -\Phi^2/_E,\ I_3 = -\Phi^3/_E)$ is an almost hypercomplex structure on the distribution $E = T^\perp \mathcal{F}$.

Now, we will check that, although not every I_1, I_2, I_3 is projectable, the vector bundle Q spanned by these structures is projectable (see also [5, 6]) hence, the foliation $\mathcal{V}$ has a projectable, transversal quaternionic structure.

Let X be a projectable cross section of E i.e., $[\xi^\alpha, X] \in \Gamma T\mathcal{V}$ for $\alpha = 1,2,3$. Then

$$\begin{aligned}(\overset{\circ}{D}_{\xi^\alpha} \Phi^\beta)X &= \pi[\xi^\alpha, \Phi^\beta X] = \pi(\nabla_{\xi^\alpha}(\Phi^\beta X) - \nabla_{\Phi^\beta X}\xi^\alpha) \\ &= \pi((\nabla_{\xi^\alpha}\Phi^\beta)X + \Phi^\beta(\nabla_{\xi^\alpha}X) - \Phi^\alpha \circ \Phi^\beta X) \\ &\overset{(2.6)}{=} \pi\{\Phi^\beta(\nabla_X \xi^\alpha + [\xi^\alpha, X]) - \Phi^\alpha \circ \Phi^\beta X\} \\ &= (\Phi^\beta \circ \Phi^\alpha - \Phi^\alpha \circ \Phi^\beta)X = 2(1 - \delta_{\alpha\beta})\Phi^\gamma X,\end{aligned}$$

where if $\alpha \neq \beta$ then (α,β,γ) is a cyclic permutations of $(1,2,3)$. The last equality holds because the structures I^α satisfy the quaternionic identities.

We recall that compact 3-Sasakian manifolds $\mathcal{S}^{4q+3}$ where the foliation $\mathcal{V}$ has all the leaves compact project onto a compact positive quaternion Kähler orbifold N^{4q}, and the leaves of $\mathcal{V}$ are homogeneous 3-dimensional spherical space forms. In the case of a regular foliation $\mathcal{V}$, the leaf space N^{4q} is a positive quaternion Kähler manifold. Thus, the simplest example of a foliation with projectable transversal quaternionic structure is the Hopf fibration $S^{4q+3} \to \mathbf{H}P^q$. An example of a 3-Sasakian manifold where $\mathcal{V}$ is not regular, but still all the leaves are compact, is the following. Consider the action of $\mathbf{Z}_3$ on the sphere $S^7 = \{(h_0, h_1) \in \mathbf{H}^2 \ / \ h_0\bar{h}_0 + h_1\bar{h}_1 = 1 \}$ generated by $(h_0, h_1) \mapsto (e^{\frac{2\pi i}{3}} h_0, e^{\frac{4\pi i}{3}} h_1)$. This action preserves the 3-Sasakian structure of S^7, therefore, the quotient $\mathbf{Z}_3 \backslash S^7$ is a 3-Sasakian manifold, and its foliation $\mathcal{V}$ admits a projectable transversal quaternion Kähler structure. In fact this structure projects to the orbifold $\mathbf{Z}_3 \backslash \mathbf{H}P^1$ defined by the induced action of Z_3. We refer the reader to [5, 6] for all these facts.

As a matter of fact, the projection on a quaternion Kähler manifold always holds locally (cf. [5], Theorem 2.3.4). This shows that the transversal almost quaternionic structure of the foliation $\mathcal{V}$ of an arbitrary 3-Sasakian manifold always is an integrable i.e., a quaternionic, structure.

A second class of examples of foliations with a projectable transversal quaternionic structure is that of the *locally conformal quaternion Kähler manifolds* M^{4q+4}. This means that M is endowed with an almost quaternionic structure Q and a metric g, which is Hermitian with respect to the local compatible almost complex structures of Q, and such that, over some open neighborhoods $\{U_i\}$ which cover M ($M = \cup_i U_i$), g is conformally related to local quaternion Kähler metrics:

$$g_{|U_i} = e^{f_i} g'_i,$$

where g'_i is quaternion Kähler on U_i and $f_i \in C^\infty(U_i)$.

Such a structure defines the so called Lee 1-form ω, where $\omega_{|U_i} = df_i$. ω appears as a factor in the exterior differential $d\Theta = 2\omega \wedge \Theta$ of the *Kähler 4-form* $\Theta = \sum_{\alpha=1}^{3} \Omega_\alpha \wedge \Omega_\alpha$, where Ω_α are the Kähler forms of the local bases (I_α) $(\alpha = 1, 2, 3)$ of Q. In the compact case and if g is not globally conformal quaternion Kähler, a result of P. Gauduchon yields a metric in the conformal class of g such that its Lee form ω is parallel with respect to the Levi-Civita connection of the new metric [7, 17]. With this choice and the normalization $|\omega| = 1$, the Lee vector field $\xi := \sharp_g \omega$, and the local vector fields $\xi^\alpha = I_\alpha \xi$ define a 4-dimensional foliation $\mathcal{V}$ (cf. [17], Proposition 1.7) whose orthogonal bundle E has a quaternionic structure Q_E induced by the structure Q of M (again, see [17]).

Moreover, the Lie derivative formulas of Proposition 1.7 of [17] allow for an easy verification of the fact that

$$\overset{\circ}{D}_\xi (Q_E) \subseteq Q_E, \overset{\circ}{D}_{\xi^\alpha} (Q_E) \subseteq Q_E,$$

for any Bott connection D of E.

Therefore, $\mathcal{V}$ is a foliation with a projectable transversal almost quaternionic structure. Moreover, it follows from the proof of Theorem 5.1 of [17] that this transversal structure is, in fact, integrable.

It is easy to give examples where the foliation $\mathcal{V}$ is not a fibration over an orbifold. The simplest examples of locally conformal hyperkähler (hence, implicitly, quaternion Kähler) manifolds are quotients $(\mathbf{H}^2 - \{0\})/\mathbf{Z}$, where $\mathbf{Z}$ is an infinite cyclic group which preserves the metric $g = (h_0\bar{h}_0 + h_1\bar{h}_1)^{-1}g_0$, conformal to the standard flat metric g_0. If we take $\mathbf{Z}$ generated by $(h_0, h_1) \mapsto (2h_0, 2e^{\sqrt{2}\pi i}h_1)$, we get a foliation $\mathcal{V}$ with non compact leaves. Thus, no orbifold structure is obtained on the leaf space. However, the transversal quaternionic structure of the foliation $\mathcal{V}$ defined above is still projectable. (See [17] for more explanations.)

3 Projectability

In this section we continue to use the notation of Section 1, and we discuss the notion of projectability of a transversal almost quaternionic structure $Q \subset End(\nu\mathcal{F})$ introduced by Definition 1.1.

3.1 Proposition. *The almost quaternionic structure $Q \subset End(\nu\mathcal{F})$ is projectable iff Q has local compatible, projectable, almost hypercomplex structures (J_α) $(\alpha = 1, 2, 3)$.*

Proof. Since local systems (I_1, I_2, I_3), (J_1, J_2, J_3) of Q-compatible almost hypercomplex structures are $SO(3)$-related, it follows that if $\overset{\circ}{D} J_\alpha = 0$ then $\overset{\circ}{D} I_\alpha$ are given by expressions of the type (1.2), i. e. Q is projectable. Conversely, since $\overset{\circ}{D}$ is a flat partial connection [4, 11], the condition $\overset{\circ}{D} Q \subset Q$ insures that $\overset{\circ}{D}$ induces a flat partial connection on the vector bundle Q. Accordingly, frames J_1, J_2, J_3 which are parallel with respect to $\overset{\circ}{D}$ (i.e., $\overset{\circ}{D} J_\alpha = 0$) can be constructed. Namely, if V is a local transversal submanifold of $\mathcal{F}$, we fix J_α along V then, translate them parallely along the local slices of $\mathcal{F}$, with respect to an arbitrary Bott connection. Q.e.d.

If a decomposition $TM = E \oplus L$ is chosen, $\nu\mathcal{F}$ is isomorphic with the subundle E of TM, and the following tensorial projectability criterion of an almost complex structure J of E (i. e., $\overset{\circ}{D}\ J = 0$) holds. Let $\tilde{J}$ be the endomorphism of TM defined by

$$(3.1) \qquad \tilde{J}(X) = \begin{cases} JX, & \text{for } X \in \Gamma E, \\ 0, & \text{for } X \in \Gamma L, \end{cases}$$

and consider the *Nijenhuis tensor*

$$(3.2) \quad N_{\tilde{J}}(X_1, X_2) = [\tilde{J}X_1, \tilde{J}X_2] - \tilde{J}[\tilde{J}X_1, X_2] - \tilde{J}[X_1, \tilde{J}X_2] + \tilde{J}^2[X_1, X_2],$$

where $X_1, X_2 \in \Gamma\ TM$. Then the almost complex structure J is projectable iff

$$N_{\tilde{J}}|_{\Gamma L \times \Gamma TM} \equiv 0.$$

This is easily checked, by using the fact that $N_{\tilde{J}}$ is a tensor. Indeed, consider the vector fields $Y \in \Gamma L, X \in \Gamma TM$, extensions of $Y_p \in T_p\mathcal{F}$ and $X_p \in T_pM$ $(p \in M)$; generality is not affected if we assume X projectable, which, hereafter, we will denote by $X \in \Gamma_{pr}TM$, and which means that $\forall Y \in \Gamma L$, $[Y, X] \in \Gamma L$. Then,

$$(3.3) \qquad N_{\tilde{J}}(Y_p, X_p) = N_{\tilde{J}}(Y, X)/_p = -\{\tilde{J}([Y, \tilde{J}X] - \tilde{J}[Y, X])\}/_p$$

$$= -J\pi[Y, \tilde{J}X]_p = -J(\overset{\circ}{D}_Y\ J)\pi X/_p.$$

Hence, $N_{\tilde{J}}(Y, X) = 0$ if and only if $(\overset{\circ}{D}_Y\ J)X = 0$, as stated.

From (3.3), we also notice that, $\forall Y \in \Gamma L, \forall X \in \Gamma E$, $N_{\tilde{J}}(Y, X)$ takes values in E.

It follows that an almost hypercomplex structure $H = (I_1, I_2, I_3)$ on $E \approx \nu\mathcal{F}$ is projectable iff one has $N_{\tilde{I}_\alpha}(Y, X) = 0$ for $Y \in \Gamma L, X \in \Gamma TM$, $\alpha = 1, 2, 3$.

This assertion can be rephrased by using a unique tensor $T^{\tilde{H}} : TM \times TM \to TM$ defined as follows. Recall that for an almost hypercomplex structure $H = (I_1, I_2, I_3)$ on a manifold M^{4q}, a *structure tensor* is defined by

$$(3.4) \qquad T^H = \frac{1}{6} \sum_{\alpha=1}^{3} N_{I_\alpha},$$

the torsion of the *Obata connection* on M ([1], pp. 239-241). In our case, the almost hypercomplex structure $H = (I_1, I_2, I_3)$ is only defined on a complementary distribution E of the tangent bundle L of the $4q$-codimensional foliation $\mathcal{F}$. But, we may take the triple $\tilde{H} = (\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ defined as in (3.1), and define the *structure tensor*

$$T^{\tilde{H}} = \frac{1}{6}\sum_{\alpha=1}^{3} N_{\tilde{I}_\alpha}. \tag{3.5}$$

The following formula, where $Y \in \Gamma L, X \in \Gamma TM$, and $\alpha = 1, 2, 3$, is a consequence of (3.5)

$$N_{\tilde{I}_\alpha}(Y, X) = \frac{3}{2}\{T^{\tilde{H}}(Y, X) + \tilde{I}_\alpha T^{\tilde{H}}(Y, \tilde{I}_\alpha X)\}. \tag{3.6}$$

It follows:

3.2 Proposition. *The almost hypercomplex structure $H = (I_1, I_2, I_3)$ defined on the transversal bundle $\nu\mathcal{F}$ of the foliation $\mathcal{F}$ of M^{p+4q} is projectable iff $T^{\tilde{H}}|_{T\mathcal{F}\times TM}$ is zero.*

Using the tensor (3.5), we can also show another interesting fact namely,

3.3 Proposition. *If the foliation $\mathcal{F}$ has a projectable, transversal, almost hypercomplex structure H, there exists a projectable connection of $\nu\mathcal{F}$ which preserves the structure H.*

Proof. We recall that a Bott connection ∇ of $\nu\mathcal{F}$ is projectable if $\nabla_{X_1}X_2$ is projectable $\forall X_1, X_2 \in \Gamma_{pr}E$. The stated result will be proven by writing down analogs of connections defined by Oproiu and Obata. First, let us define a Bott connection ∇^H i.e., $\nabla^H_Y = \overset{\circ}{D}_Y$ given by (1.1′) ($Y \in \Gamma L$), by adding the equation [16, 1]

$$\begin{aligned}\nabla^H_{X_1}X_2 = {} & \frac{1}{12}\pi\sum_{(\alpha,\beta,\gamma)}\Big(\tilde{I}_\alpha[I_\beta X_1, I_\gamma X_2] + \tilde{I}_\alpha[I_\beta X_2, I_\gamma X_1]\Big) \\ & + \frac{1}{6}\,\pi\sum_{\alpha}\Big(\tilde{I}_\alpha[I_\alpha X_1, X_2] + \tilde{I}_\alpha[I_\alpha X_2, X_1]\Big) + \frac{1}{2}\pi[X_1, X_2],\end{aligned} \tag{3.7}$$

where $X_1, X_2 \in \Gamma E$, $\sum_{(\alpha,\beta,\gamma)}$ denotes the sum over the cyclic permutations of $(1, 2, 3)$, and $\pi : TM \to E$ is the natural projection. ∇^H is a projectable connection of $\nu\mathcal{F}$. It does not preserve H but, if we correct (3.7) by defining [13, 1]

$$D^H_{X_1}X_2 = \nabla^H_{X_1}X_2 + \frac{1}{2}\pi T^{\tilde{H}}(X_1, X_2), \quad X_1, X_2 \in \Gamma E, \tag{3.8}$$

we get a connection as required by the proposition. The projectability of the additional term of (3.8) follows from (3.2) since, if J of (3.2) is projectable then $\forall X_1, X_2 \in \Gamma_{pr}E$, $N_{\tilde{J}}(X_1, X_2)$ has a projectable transversal part. Q.e.d.

The connection D^H of (3.8) will be called the *Bott-Obata connection*, and for its torsion we get

$$T_{D^H}(X_1, X_2) := D^H_{X_1}X_2 - D^H_{X_2}X_1 - \pi[X_1, X_2] = \pi T^{\tilde{H}}(X_1, X_2),$$

$\forall X_1, X_2 \in \Gamma E$.

Proposition 3.3 shows that, generally, there are obstructions to the existence of a projectable, transversal, hypercomplex structure of a foliation $\mathcal{F}$. One such obstruction is, of course, the Atiyah class of $\mathcal{F}$, since the Atiyah class is the obstruction to the existence of a projectable, transversal connection [11]. Sometimes, it is also possible to detect secondary characteristic classes.

Let $\mathcal{F}$ be a foliation of codimension $4q$ on M^{p+4q}, which has a projectable, transversal almost complex structure I_1. Then there exist Bott connections ∇ which preserve I_1. Indeed, for any Bott connection $\nabla_Y I_1 = 0$ for all $Y \in \Gamma L$, and the existence of ∇ with $\nabla_X I_1 = 0$ for all $X \in \Gamma E$ follows in the same way as the existence of, say, an almost complex connection on an almost complex manifold. Moreover, if we also choose a Riemannian metric g on M such that $g/_E$ is I_1-Hermitian, we can get ∇ as above which also satisfies $\nabla_X(g/_E) = 0$, $\forall X \in \Gamma E$.

Accordingly, as in the classical Bott vanishing theorem [4], we have:

$$Chern_{2k}(E, I_1) = 0 \quad \text{if } k > 4q,$$

where $Chern_{2k}$ denotes elements of cohomological degree $2k$ in the ring generated by the real Chern classes. More exactly the representative differential forms of these classes in terms of the curvature forms of ∇ vanish.

Now, assume that I_1 can be completed by I_2, I_3 to a (not necessarily projectable) transversal almost hypercomplex structure, with an almost hyperhermitian metric g. Then, the odd dimensional Chern classes $c_{2h+1}(E, I_1)$ vanish, since their representative differential forms in terms of the curvature of an almost hyperhermitian (not necessarily Bott) connection D are $\mathcal{C}_{2h+1}(D) = 0$ (cf. [10], vol. II, p. 304).

Thus, if $2h + 1 > 4q$, and if the connections ∇, D are as above, we have

$$\mathcal{C}_{2h+1}(\nabla) - \mathcal{C}_{2h+1}(D) = d(\Delta_{(h)}(\nabla, D)) = 0,$$

where $\Delta_{(h)}$ are the *Bott comparison forms* [4], and we get cohomology classes

$$[\Delta_{(h)}(\nabla, D)] \in H^{4h+1}(M, \mathbf{R}) \quad (h \geq 2q)$$

which are well defined and independent of the choice of the connections ∇, D. These precisely are the secondary classes that we mentioned. They are obstructions to the existence of a projectable almost hyperhermitian transversal structure with the given almost complex component I_1 since, if such a structure exists, we may use equal connections $D = \nabla$, in which case $\Delta_{(h)}(\nabla, D) = 0$.

Next, assume that $Q \subset End\,E$ is a transversal almost quaternionic structure of a foliation $\mathcal{F}$. In order to get a tensorial criterion for the projectability of Q, we look at the extension $\tilde{Q} \subset End(TM)$ of Q defined by extending each $S \in Q$ to $\tilde{S} \in End\,TM$ by $\tilde{S}/_L = 0$. Recall that the structure tensor T^Q of an almost quaternionic structure Q of a manifold M^{4q} is defined by

$$T^Q(X_1, X_2) = T^H(X_1, X_2) + \sum_{\alpha=1}^{3} [(\tau_\alpha X_1\ )I_\alpha X_2 - (\tau_\alpha X_2\ )I_\alpha X_1], \tag{3.9}$$

where $X_1, X_2 \in \Gamma TM$, $H = (I_1, I_2, I_3)$ is any local basis of Q, and

$$\tau_\alpha X = \frac{1}{4q-2}\ tr\ [(I_\alpha\ T^H)(X, -)], \quad X \in \Gamma TM$$

(cf. [1], p. 244). Both T^H and T^Q are invariant by a change of the local basis H since such a change is via an $SO(3)$-matrix and the sums on α which enter in the expressions of T^H, T^Q behave like scalar products in $\mathbf{R}^3$.

In our situation, Q is defined only on the complementary distribution E of $L = T\mathcal{F}$, and a suitable extension $T^{\tilde{Q}}$ of T^Q (i.e., $T^{\tilde{Q}}(X_1, X_2)$ is given by (3.9) with T^H replaced by $\pi T^{\tilde{H}}$, if $X_1, X_2 \in \Gamma E$) will result from

3.4 Proposition. *The transversal almost quaternionic structure Q of the foliation $\mathcal{F}$ on M^{p+4q} is projectable iff there exists a local basis H of Q such that*

$$T^{\tilde{H}}(Y, X) = \sum_{\alpha=1}^{3} \kappa_\alpha(Y)\tilde{I}_\alpha X \quad (Y \in \Gamma L,\ X \in \Gamma TM) \tag{3.10}$$

for some leafwise 1-forms κ_α on M.

Proof. If (3.10) holds, then, $\forall X \in \Gamma E$, $\forall Y \in \Gamma L$, (3.3) and (3.6) yield

$$-I_\lambda(\overset{\circ}{D}_Y\ I_\lambda)X = N_{\tilde{I}_\lambda}(Y,X) = \frac{3}{2}\{T^{\tilde{H}}(Y,X) + \tilde{I}_\lambda T^{\tilde{H}}(Y,\tilde{I}_\lambda X)\} =$$

$$= \frac{3}{2}(\sum_{\alpha=1}^{3} \kappa_\alpha(Y)\ I_\alpha X + I_\lambda \sum_{\alpha=1}^{3} \kappa_\alpha(Y)\ I_\alpha(I_\lambda X))$$

$(\alpha, \lambda = 1,2,3)$, whence

$$(\overset{\circ}{D}_Y\ I_\lambda)X = \frac{3}{2}\sum_{\alpha=1}^{3} \kappa_\alpha(Y)(I_\lambda\ I_\alpha X - I_\alpha\ I_\lambda X). \tag{3.11}$$

Since the structures I_α satisfy the quaternionic identities, (3.11) shows that Q is a projectable structure (compare with (1.2)).

Conversely, if Q is projectable then, $\forall Y \in \Gamma L$, $\forall X \in \Gamma_{pr}E$, (1.2) and (3.3) imply

$$N_{\tilde{I}_1}(Y,X) = -\alpha(Y)I_3X + \beta(Y)I_2X,$$

and similarly:

$$N_{\tilde{I}_2}(Y,X) = -\alpha(Y)I_3X - \gamma(Y)I_1X,$$
$$N_{\tilde{I}_3}(Y,X) = \beta(Y)I_2X - \gamma(Y)I_1X.$$

Accordingly, (3.5) yields

$$T^{\tilde{H}}(Y,X) = -\frac{1}{3}[\gamma(Y)I_1 - \beta(Y)I_2 + \alpha(Y)I_3]X,$$

which is (3.10) for $X \in \Gamma E$. For $X \in \Gamma L$, (3.10) is just $0 = 0$. Q.e.d.

Moreover, by taking into account that $tr\ I_\alpha = 0$, we get

$$\alpha(Y) = \frac{3}{4q}\ tr\ \{I_3 T^{\tilde{H}}(Y,-)\}, \quad \beta(Y) = -\frac{3}{4q}\ tr\ \{I_2 T^{\tilde{H}}(Y,-)\},$$

$$\gamma(Y) = \frac{3}{4q}\ tr\ \{I_1 T^{\tilde{H}}(Y,-)\},$$

where the missing argument is in ΓE.

Therefore, the coefficients of (3.10) must be α, β, γ, and we have to define the extension of T^Q by asking $T^{\tilde{Q}}(Y_1,Y_2) = 0$ for $Y_1, Y_2 \in \Gamma L$, and

$$T^{\tilde{Q}}(Y,X) = T^{\tilde{H}}(Y,X) + \sum_{\alpha=1}^{3} \rho_\alpha(Y) I_\alpha X, \tag{3.12}$$

for $Y \in \Gamma L, X \in \Gamma E$, where

$$\rho_\alpha(Y) = (1/4q)tr\ [(I_\alpha\ T^{\tilde{H}})(Y,-)]. \tag{3.13}$$

This $T^{\tilde{Q}}$ is independent of the choice of the local basis H for the same reason T^Q was.

Accordingly, we see that Proposition 3.4 is equivalent to

3.5 Proposition. *The almost quaternionic structure Q, transversal to the foliation $\mathcal{F}$ of M^{p+4q}, is projectable iff $T^{\tilde{Q}}|_{T\mathcal{F}\times TM}$ vanishes.*

Formula (3.11) gives a geometric meaning to the 1-forms ρ_α of (3.13) in the case of a projectable structure Q. Namely, they are local connection forms of $\overset{\circ}{D}$ restricted to Q. In particular, if the triple (I_1, I_2, I_3) consists of projectable structures, one has $\rho_\alpha = 0$.

It is also interesting to notice that $T^{\tilde{Q}}(X_1, X_2)$ $(X_1, X_2 \in \Gamma E)$ can be related with the torsion of some well chosen Bott connections. First, all the Q-preserving Bott connections on E are given by

$$\nabla^Q_{X_1} X_2 = \begin{cases} \pi[X_1, X_2], & \text{for } X_1 \in \Gamma L, \\ \nabla^{Op}_{X_1} X_2 & \text{for } X_1 \in \Gamma E, \end{cases} \tag{3.14}$$

where ∇^{Op} denotes the connection defined by Oproiu's formula ([15], p. 295)

$$\nabla^{Op}_{X_1} X_2 = \nabla_{X_1} X_2 + \sum_{\alpha=1}^{3} \{\frac{1}{4}(\nabla_{X_1} I_\alpha) I_\alpha + \frac{1}{2}\eta_\alpha(X_1) I_\alpha\} X_2 \tag{3.15}$$

$$+\frac{1}{4}\{A_{X_1} X_2 - \sum_\alpha I_\alpha A_{X_1}(I_\alpha X_2)\} \quad (X_1, X_2 \in \Gamma E).$$

In (3.15) ∇ is an arbitrary Bott connection on E, $H = (I_1, I_2, I_3)$ is a local compatible almost hypercomplex structure, A_{X_1} is an arbitrary endomorphism of E, and η_α $(\alpha = 1, 2, 3)$ are arbitrary 1-forms on M.

Now, let us fix a connection $\overset{1}{\nabla}$ among those given by (3.14), (3.15). Following [1], p. 244, $\overset{1}{\nabla}$ has an *associated Bott-Oproiu connection*

$$\overset{Op}{}\overset{1}{\nabla}_X = \overset{1}{\nabla}_X + \sum_{\alpha=1}^{3}(\varphi_\alpha + \frac{1}{3}\varphi \circ I_\alpha)(X) I_\alpha - \frac{1}{4}(A_X - \sum_{\alpha=1}^{3} I_\alpha A_X I_\alpha), \tag{3.16}$$

where

$$\varphi_\alpha(X) = \frac{1}{4q-2} tr(I_\alpha T_X),\ \varphi = \sum_{\alpha=1}^{3} \varphi_\alpha \circ I_\alpha,\ A_X = T_X + \frac{1}{3} \sum_{\alpha=1}^{3} T_{I_\alpha X} \circ I_\alpha,\ X \in \Gamma M,$$

T being the torsion of $\overset{1}{\nabla}$, and T_X the endomorphism of E obtained by fixing the first argument of the torsion as X.

Then, the same computations as in [1] show that the tensor $T^{\tilde{Q}}$ and the torsion of the Bott-Oproiu connections are related by the formula

$$T_{{}^{O}p\overset{1}{\nabla}}(X_1, X_2) = \pi T^{\tilde{Q}}(X_1, X_2), \quad X_1, X_2 \in \Gamma E,$$

which is the result we wanted to mention.

Finally, let us also note that, as a consequence of (3.9), if Q is a projectable structure, $\pi T^{\tilde{Q}}$ is a projectable tensor field.

4 Integrability

Consider an almost hypercomplex structure $H = (I_1, I_2, I_3)$, respectively an almost quaternionic structure Q transversal to a foliation $\mathcal{F}$ of codimension $4q$ on a manifold M^{p+4q}. By Definition 1.2, the integrability of H and Q includes projectability. It is natural to ask whether integrability can be recognized by means of the structure tensors $T^{\tilde{H}}$ and $T^{\tilde{Q}}$, defined by formulas (3.5) and (3.9).

4.1 Proposition. *H, respectively Q, is integrable iff its structure tensor $T^{\tilde{H}}$, respectively $T^{\tilde{Q}}$, takes values in the tangent bundle $L = T\mathcal{F}$.*

Proof. If H (respectively Q)is projectable, as seen in Section 3, $\pi \circ T^{\tilde{H}}$ (respectively $\pi \circ T^{\tilde{Q}}$) projects to the local slice spaces, and, clearly, the projection is the torsion tensor of the corresponding almost hypercomplex (quaternionic) structures of these slice spaces. Accordingly, the statement follows by Definition 1.2 and by the fact that $T^H = 0$ (respectively $T^Q = 0$) is the integrability condition for H (respectively Q) on manifolds. Q.e.d.

Now, we will discuss another aspect concerning transversal quaternionic (i.e., integrable, almost quaternionic) structures Q of a foliation. The integrability of Q is equivalent to the existence of an open covering $M = \cup_{a \in \mathcal{A}} U_a$

($\mathcal{A}$ is an arbitrary set) such that one has local, torsionless, projectable connections D^a of $E/_{U_a}$ which preserve $Q/_{U_a}$. These connections can be glued together by means of a partition of unity. The resulting global connection is then a torsionless, Q-preserving, Bott connection but, generally, it is not projectable. As a matter of fact, we already have explicit expressions of such connections namely, the Bott-Oproiu connection of any Q-preserving Bott connection has a vanishing torsion because of Proposition 4.1. We write this result as

4.2 Proposition. *If the foliation $\mathcal{F}$ admits a transversal quaternionic structure Q, then $\mathcal{F}$ admits a Q-preserving, torsionless, Bott connection D on its transversal bundle.*

On the other hand, from the system of local connections D^a above, we can build a Čech 1-cocycle, as follows. The differences

$$\tau^{ab}(X,Y) = D^a_X Y - D^b_X Y \quad (X, Y \in \Gamma E,\ a, b \in \mathcal{A}) \tag{4.1}$$

are projectable cross sections of the foliated vector bundle $Hom(E \odot E, E)$, where $\odot$ denotes the symmetrized tensor product. Symmetry comes from the fact that the connections D^a have no torsion. Furthermore, if we denote

$$\tau^{ab}_X = D^a_X - D^b_X, \quad X \in \Gamma TM, \tag{4.2}$$

we obtain $(End\,E)$-valued 1-forms $\tau^{ab} \in \Lambda^1(U_a \cap U_b, End\,E)$, and $\tau^{ab}_X(Q) \subseteq Q$.

Let us denote by $End_Q E \subseteq End\,E$ the subbundle of Q-preserving endomorphisms, and notice the injections of vector bundles

$$i : Hom(E \odot E, E) \to Hom(TM \otimes TM, E),$$

$$j : \Lambda^1(M, End\,E) \to Hom(TM \otimes TM, E),$$

where i extends a tensor defined on arguments in E to one with arguments in TM by giving it the value 0 if an argument is in L, and

$$j(\lambda)(X_1, X_2) := \lambda(X_1)\pi X_2, \quad X_1, X_2 \in \Gamma TM.$$

The integrability of Q implies that the $(End\,E)$-valued 1-forms τ defined by (4.2) are projectable cross sections of the vector bundle $j^{-1}(i(Hom(E \odot E, E))$ over $U_a \cap U_b$. Thus, the forms τ^{ab} may be seen as a 1-cocycle with values in the sheaf $\mathcal{S}$ of germs of projectable cross sections of the vector bundle $j^{-1}(i(Hom(E \odot E, E))$ on M. Of course, $\mathcal{S}$ is a subsheaf of germs of projectable $(End\, E)$-valued 1-forms on M.

Correspondingly, we have a cohomology class $[\tau]_{\mathcal{S}} \in H^1(M, \mathcal{S})$ associated with the structure Q, which we call the *integrability class* of Q.

The integrability class can be handled as follows. The splitting $TM = E \oplus L$ yields a natural bigrading, called $\mathcal{F}$-type, of the spaces of vector fields and differential forms (our convention is to write the E-degree first), and a decomposition of the exterior differential

$$d = d'_{(1,0)} + d''_{(0,1)} + \partial_{(2,-1)}, \tag{4.3}$$

where the indices denote the type of the operators, and d'' is differentiation along the leaves of $\mathcal{F}$ [21, 22]. Following the de Rham type Theorem 4 of [22], p.217, $[\tau]_{\mathcal{S}}$ is the d''-cohomology class of a $(1,0)$-form with values in $j^{-1}(i(Hom(E \odot E, E))$. Namely, put

$$\tau^{ab} = \tau^a - \tau^b, \tag{4.4}$$

where

$$\tau^a \in \Lambda^{1,0}(U^a, End\, E) \cap \Gamma(j^{-1}(i(Hom(E \odot E, E))/_{U^a}).$$

Then, since τ^{ab} are projectable forms, the local forms $d''\tau^a$ glue up to a global d''-closed form $\mathcal{T}$, and this is the required representative form of $[\tau]_{\mathcal{S}}$.

4.3 Proposition. *Let Q be a transversal quaternionic structure of the foliation $\mathcal{F}$. Then, a torsionless, projectable, transversal connection of $\mathcal{F}$ which preserves Q exists iff $[\tau]_{\mathcal{S}} = 0$ i.e., iff $\mathcal{T}$ is d''-exact.*

Proof. $[\tau] = 0$ iff one can get relations (4.4) where the local forms τ^a, τ^b are projectable. If this happens, the operator

$$D = D^a - \tau^a \quad (a \in \mathcal{A}) \tag{4.5}$$

yield a global projectable connection on E which preserves Q. Conversely, if D exists, $\tau^a = D^a - D$ are projectable, and satisfy (4.4). Q.e.d.

Notice that the (possibly non projectable) connection D of (4.5) exists for any integrable structure Q. From (4.5) and the projectability of D^a it follows that $\mathcal{T}$ is the $(1,1)$-part of the curvature form of D hence, $\mathcal{T}$ also represents the Atiyah class of $\mathcal{F}$ [11]. This proves

4.4 Proposition. *The Atiyah class of a transversally quaternionic foliation belongs to* $\iota^*(H^1(M,\mathcal{S}))$*, where* ι *is the inclusion of* $\mathcal{S}$ *into the sheaf of germs of projectable* 1*-forms with values in* $End\,E$*.*

In view of the above results, the following terminology is natural. A projectable, almost quaternionic transversal structure of a foliation will be called *semi-integrable* if it is preserved by a global, torsionless Bott connection, and it will be called *strongly integrable* if it is preserved by a global, torsionless, projectable, Bott connection.

5 The transversal twistor space of $(\mathcal{F}, Q)$

Let $\mathcal{F}$ be a foliation of codimension $4q$ on the manifold M^{p+4q}, endowed with a projectable almost quaternionic structure Q with local bases (I_1, I_2, I_3) on the transversal bundle $\nu\mathcal{F} = TM/T\mathcal{F}$, and let $TM = E \oplus L$ ($L = T\mathcal{F}$) be a chosen splitting, allowing us to transfer structures between $\nu\mathcal{F}$ and E.

Similarly to the case of quaternionic manifolds, we define the *transversal twistor space* of $\mathcal{F}$ by:

(5.1) $$Z\mathcal{F} = \{J \in Q, \quad J = \alpha_1 I_1 + \alpha_2 I_2 + \alpha_3 I_3, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1\},$$

i.e., $Z\mathcal{F}$ is the sphere bundle associated with the Euclidean vector bundle Q, where the metric of Q is that which makes the compatible almost hypercomplex structures (I_1, I_2, I_3) orthonormal bases.

The quaternionic structure Q reduces the structure group of E to $Gl(q,\mathbf{H})\cdot Sp(1)$, and there exists a corresponding principal bundle $\pi : \mathcal{B}(E,Q) \to M$ of *quaternionic frames (bases).* A frame $b \in \mathcal{B}(E,Q)$ may be identified with an isomorphism $B : (\mathbf{R}^{4q}, I_1^o, I_2^o, I_3^o) \to E$, where the left hand side is equivalent to the left quaternionic space $\mathbf{H}^q$, such that:

(5.2) $$B^{-1} \circ H \circ B = H^0 \cdot A, \quad H = (\,I_1 I_2 I_3\,)\,, \;\; H^0 = (\,I_1^o I_2^o I_3^o\,)\,.$$

In (5.2), H is an arbitrary almost hypercomplex local basis of Q seen as a line matrix, H^0 is the canonical basis of $\mathbf{H}^q$ seen as a line matrix, the composition $\circ$ is for each element of the line, dot is matrix multiplication, and $A \in SO(3)$. Accordingly, we may see a quaternionic frame as

(5.3) $$b = (b_i, b_{i'} = I_1 b_i, b_{i*} = I_2 b_i, b_{i'*} = I_3 b_i)_{i=1}^q$$

where (b_i) is the image by B of the canonical basis of $\mathbf{H}^q$ over $\mathbf{H}$.

From formula (5.2) we see that the structure group of the principal bundle $\mathcal{B}(E,Q)$ appears as

$$(5.4)\quad Gl(q,\mathbf{H})\cdot Sp(1) \approx \{\phi \in Aut(\mathbf{R}^{4q})/\ \phi^{-1}\circ H^0\circ\phi = H^0\cdot A,\ A\in SO(3)\},$$

and the corresponding Lie algebra $gl(q,\mathbf{H})\oplus sp(1)$ is isomorphic to

$$(5.5)\qquad \{\chi \in End(\mathbf{R}^{4q})/\ H^0\circ\chi - \chi\circ H^0 = H^0\cdot\alpha,\ \alpha\in so(3)\},$$

(cf. [19], p. 595).

For further use, we notice that the dual coframe of b is of the form

$$(5.6)\qquad \beta = (\beta^i, \beta^{i'} = -\beta^i\circ I_1, \beta^{i*} = -\beta^i\circ I_2, \beta^{i'*} = -\beta^i\circ I_3)_{i=1}^q$$

where $\beta^i(b_j) = \delta^i_j$.

Then, b provides the complex frame (b_i, b_{i*}) of (E, I_1) with the dual coframe (β^i, β^{i*}). As a complex vector bundle, (E, I_1) is isomorphic to the holomorphic part of $E\otimes\mathbf{C}$, and it is well known that the corresponding basis of this holomorphic part is

$$(5.7)\qquad c_i = I_1^+ b_i,\quad c_{i*} = I_1^+ b_{i*}$$

where:

$$(5.8)\qquad I_1^+ = \frac{1}{2}(Id - \sqrt{-1}I_1).$$

The dual complex cobasis is:

$$(5.9)\qquad \gamma^i = \beta^i + \sqrt{-1}\beta^{i'},\quad \gamma^{i*} = \beta^{i*} + \sqrt{-1}\beta^{i'*}.$$

5.1 Proposition. *$\mathcal{B}(E,Q)$ is a foliated principal bundle over $(M,\mathcal{F})$.*

Proof. A foliated structure on a principal bundle is a maximal local trivialization atlas with projectable transition functions e.g., [11, 25]. Consider real local bases of E which have projectable transition functions. Then, there exists local bases of E over $\mathbf{H}$ which consist of some of the vectors of the given bases, and their images by the operators (I_1, I_2, I_3) which span Q. Clearly, if we choose a projectable triple (I_1, I_2, I_3) (which is possible because of the projectability of Q) the corresponding $\mathbf{H}$-bases will also have projectable transition functions. Q.e.d.

Now, from formulas (5.3) and (5.6) it follows that

$$I_1 = \sum_{i=1}^{q}(b_{i'} \otimes \beta^i - b_i \otimes \beta^{i'} - b_{i*} \otimes \beta^{i'*} + b_{i'*} \otimes \beta^{i*}),$$

$$(5.10) \qquad I_2 = \sum_{i=1}^{q}(b_{i*} \otimes \beta^i - b_{i'*} \otimes \beta^{i'} - b_i \otimes \beta^{i*} + b_{i'} \otimes \beta^{i'*}),$$

$$I_3 = \sum_{i=1}^{q}(b_{i'*} \otimes \beta^i + b_{i*} \otimes \beta^{i'} - b_{i'} \otimes \beta^{i*} - b_i \otimes \beta^{i'*}),$$

and these formulas define a projection:

$$(5.11) \qquad \pi_Q : \mathcal{B}(E,Q) \to \mathcal{B}(Q),$$

where $\mathcal{B}(Q)$ is the $SO(3)$ principal bundle of the positive orthonormal bases of Q.

Furthermore, we may also consider the projection

$$(5.12) \qquad \pi_Z : \mathcal{B}(Q) \to Z\mathcal{F}$$

defined by $\pi_Z(I_1, I_2, I_3) = I_1$.

Clearly, π_Q is a principal fibration with structure group $GL(q, \mathbf{H})$, π_Z is a principal circle bundle, and $\pi_M : Z\mathcal{F} \to M$ is an associated bundle of $\mathcal{B}(Q) \to M$ with group $SO(3)$, and fiber $SO(3)/SO(2) = S^2$.

Furthermore, as a consequence of Proposition 5.1 we have

5.2 Corollary. *There exists a lift $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $\mathcal{B}(E,Q)$, and $\pi_Z \circ \pi_Q$ maps $\tilde{\mathcal{F}}$ onto a foliation $\hat{\mathcal{F}}$ of the twistor space $Z\mathcal{F}$. The leaves of $\tilde{\mathcal{F}}$ and $\hat{\mathcal{F}}$ are covering spaces of the leaves of $\mathcal{F}$.*

Proof. A slice of $\tilde{\mathcal{F}}$ through $b \in \mathcal{B}(E,Q)$ appears as the result of the translation of b along a slice of $\mathcal{F}$ through $\pi(b) \in M$ by the linear holonomy of $\mathcal{F}$. And, a slice of $\hat{\mathcal{F}}$ is the result of the projection of the previous slice of $\tilde{\mathcal{F}}$ by $\pi_Z \circ \pi_Q$ [11]. Q.e.d.

In what follows we will derive local tangent cobases of the manifold $Z\mathcal{F}$. We begin by looking at the $(GL(p, \mathbf{R}) \times (GL(q, \mathbf{H}) \cdot Sp(1)))$- principal bundle $\mathcal{B}(M,Q)$, consisting of all the tangent bases of M which are of the form (a, b),

a being a frame of L, and b a frame of the form (5.3) in E. The mapping $(a,b)\rightarrow b$ is a $GL(p,\mathbf{R})$-principal fibration

$$\pi_{\mathcal{B}}:\mathcal{B}(M,Q)\rightarrow\mathcal{B}(E,Q).$$

On $\mathcal{B}(M,Q)$, there exists the canonical 1-form [10] which, in our case, has the scalar components, say

$$\alpha^u,\beta^i,\beta^{i'},\beta^{i*},\beta^{i'*}, \tag{5.13}$$

where $u=1,...,p$; $i=1,...,q$, and the forms β are as in formula (5.6). From the known condition [10]:

$$R_g^*\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=g^{-1}\circ\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \tag{5.14}$$

where α,β are the columns with the entries defined by (5.13), and $g\in GL(p,\mathbf{R})\times(GL(q,\mathbf{H})\cdot Sp(1))$, it easily follows that the pullbacks of the forms β by local cross sections of $\pi_{\mathcal{B}}$ are global 1-forms on $\mathcal{B}(E,Q)$ (the transversal canonical 1-form, see [11]), while the pullbacks of α^u yield some local 1-forms. In this paper, the pulling back sections will not be written explicitly. Overall, we get $p+4q$ independent horizontal (i.e., vanishing on the fibers) 1-forms on $\mathcal{B}(E,Q)$

Formula (5.14) implies that for any $g\in GL(q,\mathbf{H})\cdot Sp(1)$, and for the corresponding right translation of the principal bundle $\mathcal{B}(E,Q)$, one has

$$R_g^*\beta=g^{-1}\circ\beta. \tag{5.15}$$

In particular, if $g\in GL(q,\mathbf{H})$, the $\mathbf{H}$-version of formula (5.15) yields right translation formulas of (β^i), $(\beta^{i'})$, (β^{i*}), $(\beta^{i'*})$ separately. Accordingly, if the forms β are pulled back by local cross sections of π_Q, one gets local 1-forms on $\mathcal{B}(Q)$ such that each of the four sets of forms above has transition relations of its own i.e., the annihilator of each set is invariant. These pullbacks, and those of (α^u) yield $p+4q$ independent *horizontal* [10] local 1-forms on $\mathcal{B}(Q)$.

Then, the same forms will be pulled back to $Z\mathcal{F}$ by local cross sections of π_Z. Since the composition $\pi_Z\circ\pi_Q$ has right translations which only preserve the complex structure I_1, the 1-forms obtained in the end on $Z\mathcal{F}$ have right translation equations which only preserve the annihilator of the sets $\{\gamma^i,\gamma^{i*}\}$, $\{\bar{\gamma}^i,\bar{\gamma}^{i*}\}$, defined by formula (5.9).

Finally, after we make a choice of E, the column of the forms α^u also has an invariant annihilator.

The continuation of the building of nice cobases on $Z\mathcal{F}$ is by fixing a Q-preserving Bott connection D defined by a 1-form ϖ with values in the Lie algebra (5.5) on $\mathcal{B}(E,Q)$. Then, ϖ induces an $so(3)$-valued connection form ω on $\mathcal{B}(Q)$ by means of the relation:

$$H^0 \circ \varpi - \varpi \circ H^0 = H^0 \cdot \omega. \tag{5.16}$$

Of course, both ϖ and ω vanish on the leaves of the lifted foliations of $\mathcal{F}$ to $\mathcal{B}(E,Q)$ and $\mathcal{B}(Q)$, respectively. Since we see $Z\mathcal{F}$ as a quotient of $\mathcal{B}(Q)$, it is the form ω which will be of interest.

The 1-forms α, β, ω provide local tangent cobases on $\mathcal{B}(Q)$, and if we look at the symmetric decomposition

$$so(3) = so(2) + m, \qquad \omega = \phi + \psi, \tag{5.17}$$

where

$$\omega = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}, \ \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix}, \ \psi = \begin{pmatrix} 0 & a & b \\ -a & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}, \tag{5.18}$$

we see that ψ is a horizontal form on the principal fibration $\mathcal{B}(Q) \to Z\mathcal{F}$. Thus:

5.3 Proposition. *The pullbacks of the local 1-forms*

$$\alpha^u, \beta^i, \beta^{i'}, \beta^{i*}, \beta^{i'*}, a, b$$

to $Z\mathcal{F}$ by local cross sections of π_Z are local tangent cobases of the manifold $Z\mathcal{F}$. Except for α^u, all these forms are of the $\widehat{\mathcal{F}}$-type $(1,0)$ and the system of equations $\alpha^u = 0$ is invariant, and it defines a complementary subbundle $\widehat{E}$ of $\widehat{L} = T\widehat{\mathcal{F}}$ in the tangent bundle $TZ\mathcal{F}$. Moreover, the following system of equations also are invariant by the transition functions of these cobases, and define subbundles of $TZ\mathcal{F} \otimes_{\mathbf{R}} \mathbf{C}$:

$$\alpha^u = 0, \quad \gamma^i = 0, \quad \gamma^{i*} = 0, \quad \xi := a + \sqrt{-1}\, b = 0, \tag{$\mathcal{C}_1$}$$

$$\alpha^u = 0, \quad \gamma^i = 0, \quad \gamma^{i*} = 0, \quad \bar{\xi} = 0. \tag{$\mathcal{C}_2$}$$

Proof. The only thing which has not yet been proven is the invariance of the equation $\xi = 0$. For this, we recall the formula [10]

$$R_\gamma^*\omega = \gamma^{-1}\omega\gamma \quad \gamma \in SO(3). \tag{5.19}$$

In particular, if

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \in SO(2),$$

we get

$$R_\gamma^*(a,b) = (a\cos\phi - b\sin\phi, a\sin\phi + b\cos\phi)$$

hence,

$$R_\gamma^*\xi = \xi(\cos\phi + \sqrt{-1}\sin\phi). \tag{5.20}$$

Q.e.d.

The last part of Proposition 5.3 means that we have

5.4 Theorem. *The normal bundle $\nu\widehat{\mathcal{F}} \approx \widehat{E}$ is equipped with two almost complex structures J_1, J_2 which have $\mathcal{C}_1$, $\mathcal{C}_2$, respectively, as bundles of anti-holomorphic vectors.*

6 Projectability conditions on $Z\mathcal{F}$

In this section we find the conditions which ensure that the almost complex structures J_1, J_2 are $\widehat{\mathcal{F}}$-projectable structures. It was proven in [25] that the projectability conditions are $d''A^\sigma = 0$ (mod. A^σ) where $A^\sigma = 0$ are the equations of $\mathcal{C}_1$ and $\mathcal{C}_2$, except for $\alpha^u = 0$, respectively, and d'' is the $\widehat{\mathcal{F}}$-leafwise differential as fixed by the complementary subbundle $\widehat{E}$ (see (4.3)).

From the definition of the canonical form [10, 11], and if we use projectable local bases (I_1, I_2, I_3) of Q, it follows that, on $Z\mathcal{F}$, the forms β of (5.13) and the corresponding γ of (5.9), are $\widehat{\mathcal{F}}$-projectable. Hence, $d''\gamma^i = 0$, $d''\gamma^{i*} = 0$, which agrees with the above mentioned projectability condition.

As a matter of fact, we can write down explicit formulas for the differentials $d\gamma^i, d\gamma^{i*}$, and we do so since the formulas will also be needed later on.

The required differentials are given by the torsion-structure equations of ϖ, which may be written on M and, then, lifted to $\mathcal{B}(E,Q)$ or $Z\mathcal{F}$.

Let us start with a local basis (I_1, I_2, I_3) of Q where the induced connection ω has the equations

$$DI_1 = aI_2 + bI_3, \quad DI_2 = -aI_1 + cI_3, \quad DI_3 = -bI_1 - cI_2. \tag{6.1}$$

This basis can be used to define the frames of (5.3) whence, we see that the local equations of ϖ can be written as

$$\begin{aligned} Db_i &= \overset{0}{\varpi}{}_i^j\, b_j + \overset{1}{\varpi}{}_i^j\, b_{j'} + \overset{2}{\varpi}{}_i^j\, b_{j*} + \overset{3}{\varpi}{}_i^j\, b_{j'*}, \\ Db_{i'} &= -\overset{1}{\varpi}{}_i^j\, b_j + \overset{0}{\varpi}{}_i^j\, b_{j'} - (\overset{3}{\varpi}{}_i^j - a\delta_i^j) b_{j*} + (\overset{2}{\varpi}{}_i^j + b\delta_i^j) b_{j'*}, \\ Db_{i*} &= -\overset{2}{\varpi}{}_i^j\, b_j + (\overset{3}{\varpi}{}_i^j - a\delta_i^j) b_{j'} + \overset{0}{\varpi}{}_i^j\, b_{j*} - (\overset{1}{\varpi}{}_i^j - c\delta_i^j) b_{j'*}, \\ Db_{i'*} &= -\overset{3}{\varpi}{}_i^j\, b_j - (\overset{2}{\varpi}{}_i^j + b\delta_i^j) b_{j'} + (\overset{1}{\varpi}{}_i^j - c\delta_i^j) b_{j*} + \overset{0}{\varpi}{}_i^j\, b_{j'*}. \end{aligned} \tag{6.2}$$

Corresponding to these connection equations, there are classical torsion structure equations which provide the differentials $d\beta^i, d\beta^{i'}, d\beta^{i^*}, d\beta^{i'*}$ [10, 25], and these equations give us the required formulas

$$\begin{aligned} d\gamma^i = \gamma^h \wedge (\overset{0}{\varpi}{}_h^i + \sqrt{-1}\,\overset{1}{\varpi}{}_h^i) - \gamma^{h*} \wedge (\overset{2}{\varpi}{}_h^i - \sqrt{-1}\,\overset{3}{\varpi}{}_h^i) \\ + \frac{\sqrt{-1}}{2}(\xi \wedge \bar{\gamma}^{i*} + \bar{\xi} \wedge \gamma^{i*}) + \gamma^i \circ T_D, \end{aligned} \tag{6.3}$$

$$\begin{aligned} d\gamma^{i*} = \gamma^h \wedge (\overset{2}{\varpi}{}_h^i + \sqrt{-1}\,\overset{3}{\varpi}{}_h^i) + \gamma^{h*} \wedge (\overset{0}{\varpi}{}_h^i - \sqrt{-1}\,\overset{1}{\varpi}{}_h^i) \\ + \frac{\sqrt{-1}}{2}\xi \wedge (\gamma^i - \bar{\gamma}^i) - \sqrt{-1}c \wedge \gamma^{i*} + \gamma^{i*} \circ T_D, \end{aligned} \tag{6.4}$$

where T_D is the torsion of the connection ϖ.

Since T_D vanishes if one of its arguments is in L, we again see that $d\gamma^i$, $d\gamma^{i*}$ do not contain terms in α^u. This is another way to justify the equalities $d''\gamma^i = 0$, $d''\gamma^{i*} = 0$ i.e., the fact that the forms γ^i, γ^{i*} are $\widehat{\mathcal{F}}$-projectable 1-forms.

Now, we must also compute $d\xi$. First, the structure equations of ω on Q are:

$$d\omega + \omega \wedge \omega = \Omega, \tag{6.5}$$

where, say,

$$\Omega = \begin{pmatrix} 0 & \mathcal{A} & \mathcal{B} \\ -\mathcal{A} & 0 & \mathcal{C} \\ -\mathcal{B} & -\mathcal{C} & 0 \end{pmatrix}$$

is the curvature matrix of ω. The entries of Ω are defined by

$$da = b \wedge c + \mathcal{A}, \;\; db = -a \wedge c + \mathcal{B}, \;\; dc = a \wedge b + \mathcal{C}, \tag{6.6}$$

whence,

$$d\xi = -\sqrt{-1}\xi \wedge c + (\mathcal{A} + \sqrt{-1}\mathcal{B}). \tag{6.7}$$

The 2-forms $\mathcal{A}$, $\mathcal{B}$ are related to the curvature operator R_D of D. We could obtain this relation by differentiating (5.16), but we prefer to proceed as follows. If $\Phi \in \Gamma End\ E$ is seen as a 0-form with values in $End\ E$, and if we denote by $\mathbf{D}$ the covariant exterior differential associated with the connection ϖ of E, it is easy to get (cf. [9], Section 11.15)

$$\mathbf{D}^2\Phi(X_1, X_2) = [R_D(X_1, X_2), \Phi] := R_D(X_1, X_2) \circ \Phi - \Phi \circ R_D(X_1, X_2). \tag{6.8}$$

By applying this formula to I_1, I_2, I_3 and using (6.1) we get

$$\begin{array}{rcl} \mathcal{A}I_2 + \mathcal{B}I_3 & = & [R_D, I_1], \\ -\mathcal{A}I_1 + \mathcal{C}I_3 & = & [R_D, I_2], \\ -\mathcal{B}I_1 - \mathcal{C}I_2 & = & [R_D, I_3] \end{array} \tag{6.9}$$

In order to solve equations (6.9), we use the canonical Euclidean metric $< \ , \ >_Q$ of the $SO(3)$-vector bundle Q, while identifying the Lie algebra $so(3)$ with the Euclidean space $\mathbf{R}^3$. Then, $< \ , \ >_Q$ corresponds to the scalar product, and composition of endomorphisms, elements of Q, to the vector product of vectors of $\mathbf{R}^3$. The solutions are

$$\begin{array}{rcl} \mathcal{A} & = & < I_2, [R_D, I_1] >_Q, \\ \mathcal{B} & = & < I_3, [R_D, I_1] >_Q = - < I_2, I_1 \circ [R_D, I_1] >_Q, \\ \mathcal{C} & = & < I_3, [R_D, I_2] >_Q . \end{array} \tag{6.10}$$

Accordingly, (6.7) becomes

$$d\xi = -\sqrt{-1}\xi \wedge c + < I_2, [R_D, I_1] + \sqrt{-1}[R_D, I_1] \circ I_1 >_Q . \tag{6.11}$$

Now, the projectability conditions left are

$$(6.12)\qquad \begin{aligned} d''\xi = 0 \quad (\text{mod. } \gamma^i, \gamma^{i*}, \xi) \quad \text{for } J_1, \\ d''\xi = 0 \quad (\text{mod } \bar\gamma^i, \bar\gamma^{i*}, \xi) \quad \text{for } J_2. \end{aligned}$$

With (6.7), the meaning of the projectability conditions (6.12) is

$$(6.13)\qquad (\mathcal{A} + \sqrt{-1}\mathcal{B})(c_i, Y) = 0, \;\; (\mathcal{A} + \sqrt{-1}\mathcal{B})(c_{i^*}, Y) = 0, \;\; \text{for } J_1,$$

$$(6.14)\qquad (\mathcal{A} + \sqrt{-1}\mathcal{B})(\bar c_i, Y) = 0, \;\; (\mathcal{A} + \sqrt{-1}\mathcal{B})(\bar c_{i^*}, Y) = 0, \;\; \text{for } J_2,$$

correspondingly, where (c_i, c_{i^*}) were defined in (5.7), and $Y \in \Gamma L$.

Since for any vector $X \in \Gamma E$, $I_1^+ X$ can play the role of c_i for one frame, and of c_{i^*} for another frame, and $I_1^- X := \overline{I_1^+ X}$ can play the role of $(\bar c_i, \bar c_{i^*})$, respectively, the projectability conditions become

$$(6.15)\qquad (\mathcal{A} + \sqrt{-1}\mathcal{B})(X - \sqrt{-1} I_1 X, Y) = 0,$$

$$(6.16)\qquad (\mathcal{A} + \sqrt{-1}\mathcal{B})(X + \sqrt{-1} I_1 X, Y) = 0,$$

for J_1 and J_2, respectively, and where $X \in \Gamma E$, $Y \in \Gamma L$.

If the real and imaginary parts are separated, this means

$$(6.17)\qquad \begin{aligned} < I_2, [R_D(X,Y), I_1] + [R_D(I_1X, Y), I_1] \circ I_1 >= 0, \\ < I_2, [R_D(I_1X,Y), I_1] - [R_D(X, Y), I_1] \circ I_1 >= 0, \end{aligned}$$

for J_1, and

$$(6.18)\qquad \begin{aligned} < I_2, [R_D(X,Y), I_1] - [R_D(I_1X, Y), I_1] \circ I_1 >= 0, \\ < I_2, [R_D(I_1X,Y), I_1] + [R_D(X, Y), I_1] \circ I_1 >= 0, \end{aligned}$$

for J_2.

Now, if the basis (I_1, I_2, I_3) of Q is changed to $(I_1, -I_3, I_2)$, the same conditions will hold for I_3 instead of I_2, which means that we have to replace the projectability conditions of J_1 by

$$(6.19)\qquad \begin{aligned} [R_D(X,Y), I_1] + [R_D(I_1X, Y), I_1] \circ I_1 = \mu I_1, \\ [R_D(I_1X,Y), I_1] - [R_D(X, Y), I_1] \circ I_1 = \nu I_1, \end{aligned}$$

and those of J_2 by

$$
(6.20)\qquad \begin{aligned}
&[R_D(X,Y), I_1] - [R_D(I_1X,Y), I_1] \circ I_1 = \mu' I_1,\\
&[R_D(I_1X,Y), I_1] + [R_D(X,Y), I_1] \circ I_1 = \nu' I_1,
\end{aligned}
$$

where, in fact, I_1 is any $S \in Q$, $S^2 = -Id$. Furthermore, if we take the trace in (6.19), (6.20), we get $\mu = \nu = \mu' = \nu' = 0$. Then, in both (6.19) and (6.20), the second relation is the first composed by I_1. Therefore, the projectability conditions reduce to

$$
(6.21)\qquad \begin{aligned}
&[R_D(X,Y), S] + [R_D(SX,Y), S] \circ S = 0, \quad \text{for } J_1,\\
&[R_D(X,Y), S] - [R_D(SX,Y), S] \circ S = 0, \quad \text{for } J_2.
\end{aligned}
$$

Since these conditions are tensorial, it suffices to write them for a projectable cross section S of Q, and a projectable vector field X. Using

$$R_D(X,Y) = [D_X, D_Y] - D_{[X,Y]}$$

we get

$$
(6.22)\qquad \begin{aligned}
&D_Y(D_XS) - SD_Y(D_{SX}S) = 0, \qquad \text{for } J_1,\\
&D_Y(D_XS) + SD_Y(D_{SX}S) = 0, \qquad \text{for } J_2.
\end{aligned}
$$

These formulas give us the final form of the projectability conditions:

6.1 Theorem. *(a) the structure J_1 is projectable iff $\forall X \in \Gamma_{pr}E$ and for any projectable cross section S of Q the endomorphism $D_XS - SD_{SX}S$ is projectable.*
(b) the structure J_2 is projectable iff $\forall X \in \Gamma_{pr}E$ and for any projectable cross section S of Q the endomorphism $D_XS + SD_{SX}S$ is projectable.
(c) J_1 and J_2 are both projectable iff the connection induced by D in Q is projectable.

7 Integrability conditions on $Z\mathcal{F}$

Now, let us assume that we are in the case where J_1, J_2 are both projectable, and study the integrability of these structures.

In this case, and if we use projectable local bases (I_1, I_2, I_3) of the projectable, transversal, almost quaternionic structure Q, γ^i, γ^{i^*} and ξ are $\tilde{\mathcal{F}}$-projectable (see (6.3), (6.4) and Theorem 6.1 (c)), and it remains to ask that, for arguments in E, one had

$$d\gamma^i = 0, \qquad d\gamma^{i^*} = 0, \qquad d\xi = 0 \qquad (\text{mod. } \gamma^i, \gamma^{i^*}, \xi) \tag{7.1}$$

for J_1, and

$$d\gamma^i = 0, \qquad d\gamma^{i^*} = 0, \qquad d\bar{\xi} = 0 \qquad (\text{mod. } \gamma^i, \gamma^{i^*}, \bar{\xi}) \tag{7.2}$$

for J_2.

From (6.4), we see that (7.2) never holds. Thus, J_2 is never integrable, and we do not have to worry about it anymore.

Furthermore, (6.3) and (6.4) yield a *torsion integrability condition* of J_1 namely,

$$\gamma^i \circ T_D = 0, \qquad \gamma^{i^*} \circ T_D = 0 \qquad (\text{mod.} \gamma^i, \gamma^{i^*}). \tag{7.3}$$

The forms (7.3) are the holomorphic components of T_D, i. e., of $I_1^+ \circ T_D$, and (7.3) means that $I_1^+ \circ T_D$ must vanish on arguments of the form

$$I_1^- X, \quad I_1^- I_2 X = \frac{1}{2}(I_2 X + \sqrt{-1} I_3 X).$$

If we assume $q \geq 2$, independent arguments $I_1^- X_1, I_1^- X_2$ exist, and the torsion integrability condition reduces to

$$I_1^+(T_D(I_1^- X_1, I_1^- X_2)) = 0, \tag{7.4}$$

$\forall I_1 \in Q$, $\forall X_1, X_2 \in \Gamma E$. The explicit form of (7.4) is

$$\begin{gathered} T_D(X_1 + \sqrt{-1} I_1 X_1, X_2 + \sqrt{-1} I_1 X_2) \\ -\sqrt{-1} I_1 T_D(X_1 + \sqrt{-1} I_1 X_1, X_2 + \sqrt{-1} I_1 X_2) = 0, \end{gathered} \tag{7.5}$$

where, in fact, I_1 is any $S \in Q$, $S^2 = -Id$. Then, after we separate the real and imaginary part of (7.5), we get the integrability conditions

$$T_D(X_1, X_2) - T_D(SX_1, SX_2) + ST_D(SX_1, X_2) + ST_D(X_1, SX_2) = 0, \tag{7.6}$$

$$T_D(SX_1, X_2) + T_D(X_1, SX_2) - ST_D(X_1, X_2) + ST_D(SX_1, SX_2) = 0. \tag{7.7}$$

Since (7.7) is the result of composing (7.6) by S at the left, we get

7.1 Proposition. *If $q \geq 2$, the torsion integrability condition of J_1 is* (7.6) *$\forall S \in Q$, $S^2 = -Id$, and $\forall X_1, X_2 \in \Gamma E$. In particular, this condition holds if $T_D = 0$.*

Furthermore, we also have a curvature integrability condition which follows from (6.7) namely, that on arguments in E one had

$$\mathcal{A} + \sqrt{-1}\mathcal{B} = 0 \pmod{\gamma^i, \gamma^{i^*}}. \tag{7.8}$$

If $q \geq 2$, all we have to ask is that, for all $X_1, X_2 \in \Gamma E$, the following relation holds:

$$(\mathcal{A} + \sqrt{-1}\mathcal{B})(X_1 + \sqrt{-1}I_1X_1, X_2 + \sqrt{-1}I_1X_2) = 0. \tag{7.9}$$

The imaginary part of (7.9) is equivalent to its real part by the transformation $X_1 \mapsto I_1X_1$. Therefore, the only remaining curvature integrability condition is

$$\mathcal{A}(X_1, X_2) - \mathcal{A}(I_1X_1, I_1X_2) - \mathcal{B}(I_1X_1, X_2) - \mathcal{B}(X_1, I_1X_2) = 0. \tag{7.10}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are given by (6.10), which transforms (7.10) into

$$\begin{aligned} &< I_2, [R_D(X_1, X_2), I_1] - [R_D(I_1X_1, I_1X_2), I_1] \\ &+ I_1 \circ [R_D(I_1X_1, X_2), I_1] + I_1 \circ [R_D(X_1, I_1X_2), I_1] >_Q = 0. \end{aligned} \tag{7.11}$$

Now, note that condition (7.11) must be imposed for any basis (I_1, I_2, I_3) of Q hence, if $(I_1, I_2, I_3) \mapsto (I_1, -I_3, I_2)$, we get the same relation (7.11) for I_3 instead of I_2. It follows that the second factor of the scalar product (7.11) must be of the form λI_1. Then, by taking the trace as we did in (6.19), (6.20), we get $\lambda = 0$. Moreover, we may take any $S \in Q$, $S^2 = -Id$ as I_1.

Therefore, we have obtained

7.2 Theorem. *The curvature integrability condition of J_1 is:*

$$\begin{aligned} &[R_D(X_1, X_2), S] - [R_D(SX_1, SX_2), S] \\ &+ S \circ [R_D(SX_1, X_2), S] + S \circ [R_D(X_1, SX_2), S] = 0. \end{aligned} \tag{7.12}$$

$\forall S \in Q$ and $\forall X_1, X_2 \in \Gamma E$.

7.3 Remark. *Lemma* 14.74 *of* [3] *tells us that, if* $T_D = 0$, (7.12) *holds. In particular, if* Q *is strongly integrable (see Section 4), and if* D *is a* Q*-preserving, projectable, torsionless connection,* J_1 *is integrable.*

For $q = 1$, since we have only one independent vector c_1, which can be obtained from an arbitrary X, the torsion integrability condition is

$$I_1^+(T_D(I_1^- X, I_2X + \sqrt{-1}I_3X)) = 0, \tag{7.13}$$

where the real and imaginary parts are equivalent by $X \mapsto I_1X$. Hence, (7.13) reduces to

$$T_D(X, I_2X) - T_D(I_1X, I_3X) + I_1T_D(X, I_3X) + I_1T_D(I_1X, I_2X) = 0, \tag{7.14}$$

which has to hold for any basis (I_1, I_2, I_3) of Q.

Furthermore, for $q = 1$, the curvature integrability condition is

$$(\mathcal{A} + \sqrt{-1}\mathcal{B})(I_1^- X, I_2X + \sqrt{-1}I_3X) = 0. \tag{7.15}$$

and, if we separate the real and imaginary parts, we get

$$\begin{array}{rcl} \mathcal{A}(X, I_2X) - \mathcal{A}(I_1X, I_3X) - \mathcal{B}(I_1X, I_2X) - \mathcal{B}(X, I_3X) & = & 0, \\ \mathcal{A}(I_1X, I_2X) + \mathcal{A}(X, I_3X) + \mathcal{B}(X, I_2X) - \mathcal{B}(I_1X, I_3X) & = & 0. \end{array} \tag{7.16}$$

Now, if $(I_1, I_2, I_3) \mapsto (I_1, -I_3, I_2)$, then $(\mathcal{A}, \mathcal{B}) \mapsto (-\mathcal{B}, \mathcal{A})$, and the first relation (7.16) becomes the second. Hence the only remaining condition is

$$\begin{array}{c} < I_2, [R_D(X, I_2X), I_1] - [R_D(I_1X, I_3X), I_1] \\ +I_1 \circ [R_D(I_1X, I_2X), I_1] + I_1 \circ [R_D(X, I_3X), I_1] >_Q = 0, \end{array} \tag{7.17}$$

for all the local, hypercomplex bases of Q.

If we write equation (7.17) for $(I_1, -I_3, I_2)$, replacing the first factor I_2 by $I_3 \circ I_1$ and using $< \Phi \circ \psi, \chi >_Q = < \Phi, \psi \circ \chi >_Q$, the result is again (7.17), where the first factor is replaced by I_3. Hence, the second factor of the scalar product is proportional to I_1, and using the trace as we already did, this second factor must be zero. Therefore, the curvature integrability condition becomes

$$\begin{array}{c} [R_D(X, I_2X), I_1] - [R_D(I_1X, I_3X), I_1] \\ +I_1 \circ [R_D(I_1X, I_2X), I_1] + I_1 \circ [R_D(X, I_3X), I_1] = 0, \end{array} \tag{7.18}$$

for any orthonormal basis of Q and $\forall X \in \Gamma E$.

The case $q = 1$ is that of a four-dimensional conformal structure. Hence, the obtained conditions must be equivalent with those of classical twistor theory.

Coming back to the torsion integrability condition of J_1, we will notice the following interesting fact

7.4 Proposition. *Two Q-preserving Bott connections ϖ, ϖ' define the same structure J_1 on $Z(\mathcal{F})$ iff their torsions differ by a term which satisfies the torsion integrability condition.*

Proof. From the definition of J_1, it follows that ϖ, ϖ' define the same structure J_1 iff the (horizontal) difference form of the connections induced in Q satisfies

$$\xi' - \xi = 0 \ \ (\mathrm{mod}. \gamma^i, \gamma^{i^*}) \tag{7.19}$$

By subtracting the corresponding structure equations (6.3), (6.4) of the two connection forms ϖ, ϖ', we get

$$\gamma^h \wedge A^i_h - \gamma^{h^*} \wedge B^i_h + \frac{\sqrt{-1}}{2}[(\bar{\xi}' - \bar{\xi}) \wedge \gamma^{i*} + (\xi' - \xi) \wedge \bar{\gamma}^{i^*}] + \gamma^i \circ (T_{D'} - T_D) = 0, \tag{7.20}$$

$$-\gamma^h \wedge C^i_h - \gamma^{h^*} \wedge S^i_h + \frac{\sqrt{-1}}{2}(\xi' - \xi) \wedge (\gamma^i - \bar{\gamma}^i) - \sqrt{-1}(c' - c) \wedge \gamma^{i*} + \gamma^{i^*} \circ (T_{D'} - T_D) = 0, \tag{7.21}$$

where A, B, C, S are the entries of the difference forms of the connections, and D, D' are the corresponding covariant derivatives. Hence, $\xi' - \xi$ may be calculated by applying $i(I_1^- X)$, with $X \in \Gamma E$ to (7.20), (7.21), and the result contains only γ^i, γ^{i^*} iff $(T_{D'} - T_D)$ satisfy (7.4), if $q \geq 2$, and (7.13), if $q = 1$. Q.e.d.

7.5 Corollary. *Any two connections which satisfy the torsion integrability condition define the same structure J_1.*

It is also possible to find the condition for two connections ϖ, ϖ' as in Proposition 7.4 to define the same pair of structures (J_1, J_2). Namely,

7.6 Proposition. *Under the hypotheses of Proposition 7.4, the connections ϖ, ϖ' define the same structures J_1, J_2 iff one of the following equivalent conditions is satisfied:*
(a) ϖ and ϖ' induce the same connection in the vector bundle Q;
(b) $\forall S \in Q$ with $S^2 = -Id$, one has equal commutants $[S, \varpi] = [S, \varpi']$;
(c) $\forall S \in Q$ with $S^2 = -Id$, one has equal traces $tr(S\varpi) = tr(S\varpi')$.

Proof. Now, we ask condition (7.19) and also

$$\xi' - \xi = 0 \quad (\mathrm{mod.}\bar{\gamma}^i, \bar{\gamma}^{i^*}), \tag{7.22}$$

which expresses the fact that the two connections define the same structure J_2. Together, (7.19) and (7.22) yield $\xi' - \xi = 0$, which exactly is condition (a).

Furthermore, let us look at the relation (5.16) between the connection form ϖ and the connection form ω of the connection induced in Q. With the notation (5.18), and like for (6.10), we obtain

$$a =< I_2, [I_1, \varpi] >_Q, \qquad b =< I_3, [I_1, \varpi]_Q > . \tag{7.23}$$

If $t := \varpi' - \varpi$, and in view of condition (a), we will have $(J_1, J_2) = (J'_1, J'_2)$ iff

$$< I_2, [I_1, t] >_Q = 0, \qquad < I_3, [I_1, t] >_Q = 0. \tag{7.24}$$

This implies $[I_1, t] = \alpha I_1$, and the trace yields $\alpha = 0$, which exactly is condition (b).

Finally, we will obtain condition (c) by using the following straightforward solutions of (5.16):

$$\begin{array}{rcl} a \;\; Id & = & \frac{1}{2}\{I_3\varpi + \varpi I_3 + I_2\varpi I_1 - I_1\varpi I_2\}, \\ b \;\; Id & = & -\frac{1}{2}\{I_2\varpi + \varpi I_2 - I_3\varpi I_1 + I_1\varpi I_3\}, \\ c \;\; Id & = & \frac{1}{2}\{I_1\varpi + \varpi I_1 - I_2\varpi I_3 + I_3\varpi I_2\}. \end{array} \tag{7.25}$$

By taking the traces in (7.25), we get

$$a = \frac{1}{2q}\; tr\; (I_3\varpi), \quad b = -\frac{1}{2q}\; tr\; (I_2\varpi), \quad c = \frac{1}{2q}\; tr\; (I_1\varpi). \tag{7.26}$$

Obviously, this proves that (c) is equivalent to (a). Q.e.d.

7.7 Remark. *It is also possible to solve equations (6.9) by formulas of the type (7.25):*

$$
(7.27)\qquad \begin{aligned}
\mathcal{A}\, Id &= \tfrac{1}{2}\{[R_D, I_3] + [R_D, I_2]\circ I_1 - [R_D, I_1]\circ I_2\},\\
\mathcal{B}\, Id &= \tfrac{1}{2}\{[R_D, I_2] - [R_D, I_3]\circ I_1 + [R_D, I_1]\circ I_3\},\\
\mathcal{C}\, Id &= \tfrac{1}{2}\{[R_D, I_1] - [R_D, I_2]\circ I_3 + [R_D, I_3]\circ I_2\},
\end{aligned}
$$

which implies that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are $(1/4q)$ of the traces of the right hand sides of (7.27). These formulas can provide another form of writing the projectability and integrability conditions.

References

[1] D.V. Alekseevsky, S. Marchiafava, *Quaternionic structures on a manifold and subordinated structures*, Ann. di Mat. Pura e Appl., **171** (1996), 205-273.

[2] D.V. Alekseevsky, S. Marchiafava, M. Pontecorvo *Compatible complex structures on almost quaternionic manifolds*, Trans. Am. Math. Soc., **351** (1999), 997–1014.

[3] A. Besse, Einstein Manifolds, Springer-Verlag, Berlin, 1987.

[4] R. Bott, *Lectures on characteristic classes of foliations*, Lecture Notes in Math., 279, Springer Verlag, Berlin, 1972, 1-94.

[5] Ch. P. Boyer, K. Galicki, 3-*Sasakian manifolds*, to appear in "Essays on Einstein Manifolds", M. Wang and C. LeBrun eds., available at http://xxx.lanl.gov/, hep-th 9810250.

[6] Ch. P. Boyer, K. Galicki, B. Mann, *The geometry and topology of 3-Sasakian manifolds*, J. Reine Angew. Math., **455** (1994), 183-220.

[7] P. Gauduchon, *Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$*, J. Reine Angew. Math. **469**, (1995), 1-50.

[8] J. R. Jensen and M. Rigoli, *Twistor and Gauss lifts of surfaces in four-manifolds*, Contemp. Math. AMS, **132** (1989), 197-232.

[9] I. Kolàř, P. W. Michor and J. Slovák, Natural Operations in Differential Geometry, Springer-Verlag, Berlin, 1993.

[10] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I, II, Intersc. Publ., New York, 1963, 1969.

[11] P. Molino, Riemannian Foliations, Progress in Math. Series, 73, Birkhäuser Boston, 1988.

[12] S. Nishikawa, Ph. Tondeur and L. Vanhecke, *Spectral geometry for Riemannian foliations*, Ann. Global Anal. Geom., 10 (1992), 291-304.

[13] M. Obata, *Affine connections on manifolds with almost complex, quaternion or Hermitian structure*, Jap. J. Math., **26** (1956), 43-77.

[14] N. R. O'Brian and J. H. Rawnsley, *Twistor Spaces*, Ann. Global Anal. Geom., 3 (1985), 29-58.

[15] V. Oproiu, *Almost quaternal structures*, Anal. St. Univ. Iaşi, Matematică, **23** (1977), 287-298.

[16] V. Oproiu, *Integrability of almost quaternal structures*, Anal. St. Univ. Iaşi, Matematică, **30** (1984), 75-84.

[17] L. Ornea, P. Piccinni, *Locally conformal Kähler structures in quaternionic geometry*, Trans. Am. Math. Soc. **349** (1997), 641-655.

[18] L. Ornea, P. Piccinni, *Compact hyperhermitian-Weyl and quaternion Hermitian-Weyl manifolds*, Ann. Global Anal. Geom. **16** (1998), 383-398.

[19] P. Piccinni, *On the infinitesimal automorphisms of quaternionic structures*, J. Math. Pures Appl. **72** (1993), 593-605.

[20] S. Salamon, Riemannian Geometry and Holonomy Groups, Longman Sc. and Tech., Essex, 1989.

[21] I. Vaisman, *Variétés Riemanniennes feuilletées*, Czech. Math. J., 21 (1971), 46-75.

[22] I. Vaisman, Cohomology and Differential Forms, M. Dekker Publ. House, New York, 1973.

[23] I. Vaisman, *Exotic characteristic classes of quaternionic bundles*, Israel J. of Math., 69 (1990), 46-58.

[24] I. Vaisman, *On the Twistor spaces of Almost Hermitian Manifolds*, Ann. Global Anal. Geom. 16 (1998), 335-356.

[25] I. Vaisman, *Transversal Twistor Spaces of Foliations*, available at http://xxx.lanl.gov, math.DG/9911038.

Dipartimento di Matematica
Università di Roma "La Sapienza", Italy
e-mail: piccinni@mat.uniroma1.it

Department of Mathematics
University of Haifa, Israel
e-mail: vaisman@math.haifa.ac.il