On Azéma–Yor processes, their optimal properties and the Bachelier–Drawdown equation*

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Abstract

We study the class of Azéma–Yor processes defined from a general semimartingale with a continuous running supremum process. We show that they arise as unique strong solutions of the Bachelier stochastic differential equation which we prove is equivalent to the Drawdown equation. Solutions of the latter have the drawdown property: they always stay above a given function of their past supremum. We then show that any process which satisfies the drawdown property is in fact an Azéma–Yor process. The proofs exploit group structure of the set of Azéma–Yor processes, indexed by functions, which we introduce.

Secondly we study in detail Azéma–Yor martingales defined from a non-negative local martingale converging to zero at infinity. We establish relations between

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Average Value at Risk, Drawdown function, Hardy-Littlewood transform and its generalised inverse. In particular, we construct Azéma–Yor martingales with a given terminal law and this allows us to rediscover the Azéma–Yor solution to the Skorokhod embedding problem. Finally, we characterise Azéma–Yor martingales showing they are optimal relative to the concave ordering of terminal variables among martingales whose maximum dominates stochastically a given benchmark.

In [2] Azéma and Yor introduced a family of simple local martingales, associated with Brownian motion or more generally with a continuous martingale, which they exploited to solve the Skorokhod embedding problem. These processes, called Azéma–Yor processes, are simply functions of the underlying process X and its running maximum $\overline{X}_t = \sup_{s\leq t} X_s$. They proved to be very useful especially in describing laws of the maximum or of the last passage times of a martingale and were applied in problems ranging from Skorokhod embeddings, through optimal inequalities, to Brownian penalisations (cf. Azéma and Yor [1], Obłój and Yor [26], Roynette, Vallois and Yor [28]). The appearance of Azéma–Yor martingales in all these problems was partially explained with a characterisation in Obłój [25] as the only local martingales which can be written as a function of the couple $(X, \overline{X}_t)$.

Recently these processes have seen a revived interest with applications in mathematical finance including re-interpration of classical pricing formulae (see Madan, Roynette and Yor [22]) and portfolio optimisation under pathwise constraints (see El Karoui and Meziou [9, 10]). In this paper we uncover a more general structure of these processes and present new characterisations. We explore in depth their properties and present some further applications of Azéma–Yor processes. We work in a general setup and extend the concept of Azéma–Yor processes $M^U(X)$, as defined in (2) below, to the context of an arbitrary semimartingale (X_t) with a continuous running supremum process $\overline{X}_t$.

We start by studying the (sub)set of Azéma–Yor processes $M^U(X)$, indexed by increasing absolutely continuous functions U, and show that it has a simple group structure. This allows to see any semimartingale with continuous running supremum as an Azéma–Yor process. The main contribution of the Section 2 is to study how such representation arise naturally. We show that Azéma–Yor processes allow to solve explicitly the Bachelier equation, which we also identify with the Drawdown equation. The solutions to the latter satisfy the Drawdown constraint $Y_t \geq w(\overline{Y}_t)$. Conversely, if (Y_t) satisfies Drawdown constraint up to time ζ then it can be written as $M^U_{t\wedge\zeta}(X)$ for some non-negative X. Further, if $Y_\zeta = w(\overline{Y}_\zeta)$ a.s., then the inverse process $(X_{t\wedge\zeta})$

is stopped upon hitting 0 or b. We provide explicit relation between function U which generates Azéma–Yor process and functions w and φ which feature in the Drawdown constraint and in the SDEs. This characterises the processes both in a pathwise manner and differential manner.

Then in Section 3 we specialise further and investigate Azéma–Yor processes defined from $X = N$ a non-negative local martingale with continuous supremum process and with $N_t \to 0$ as $t \to \infty$. We show how one can identify explicitly Azéma–Yor processes from their terminal values. In Section 3.3 we discuss the Average Value at Risk and the Hardy-Littlewood transform in a unified manner using tail quantiles of probability measures. Then we construct Azéma–Yor martingales with a prescribed terminal law. This allows us to re-discover, in Section 3.4, the Azéma–Yor [2] solution to the Skorokhod embedding problem and give it a new interpretation.

Finally, in the last section, we apply the previous results to uncover optimal properties of Azéma–Yor martingales. More precisely, we show that all uniformly integrable martingales whose maximum dominates stochastically a given floor distribution are dominated by an Azéma–Yor martingale in the concave ordering of terminal values. This problem is an extension of the more intuitive problem, motivated by finance, to find an optimal martingale for the concave order dominating (pathwise) a given floor process. It is rather surprising to find that the two problems have the same solution. We recover in this way the Δ operator of Kertz and Rösler [20] and give a direct way to compute it. These dual results are compared with the classical primal result stating that among all uniformly integrable martingales with a fixed terminal law the Azéma–Yor martingale has the largest maximum (relative to the stochastic order). Furthermore, in both problems we can show that any optimal martingale is necessarily an appropriate Azéma–Yor martingale.

1 The set of Azéma–Yor processes

Throughout, all processes are defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ a filtered probability space satisfying the usual hypothesis and assumed to be taken right-continuous with left limits (*càdlàg*), up to ∞ included if needed. All functions are assumed to be Borel measurable. Given a process (X_t) we denote its running supremum $\overline{X}_t = \sup_{s \le t} X_s$. In what follows, we are essentially concerned with semimartingales with continuous running supremum, that we call *max-continuous semimartingales*. Observe that under this assumption, the process $\overline{X}_t = \sup_{s \le t} X_s$ only increases when $\overline{X}_t = X_t$ or equivalently

$$\int_0^T (\overline{X}_t - X_t)\mathrm{d}\overline{X}_t = 0\ . \tag{1}$$

We let $\tau^b(X) = \tau^b_X = \inf\{t \geq 0 : X_t \geq b\}$ be the first up-crossing time of the level b by process X, with the standard convention that $\inf\{\emptyset\} = \infty$. Note that by max-continuity $X_{\tau^b(X)} = b$, if $0 < \tau^b(X) < \infty$. With a slight abuse of notation, τ^∞_X denotes the explosion time of X.

1.1 Definition and Properties

There are two different ways to introduce Azéma–Yor processes, and their equivalence has been proven by several authors (see the comments below).

Definition 1.1. *Let (X_t) be a max-continuous semimartingale starting from $X_0 = a$, and $\overline{X}_t$ its (continuous) running supremum.*
With any locally bounded Borel function u we associate the primitive function $U(x) = a^* + \int_a^x u(s)\mathrm{d}s$ defined on $[a, +\infty)$. The Azéma–Yor process associated with U and X is defined by one of these two equations,

$$M^U_t(X) \quad := \quad U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t) \tag{2}$$

$$\textit{or} \quad = \quad a^* + \int_0^t u(\overline{X}_s)\mathrm{d}X_s. \tag{3}$$

In consequence, $M^U(X)$ is a semimartingale and it is a local martingale when X is a local martingale.

Observe that the process $M^U(X)$ is càdlàg, since $U(\overline{X})$ is continuous and $u(\overline{X}_t)(\overline{X}_t - X_t)$ is nonzero only on the intervals of constancy of $\overline{X}_t$, where the non regular process $u(\overline{X}_t)$ is constant. Moreover the jumps of $M^U_t(X)$ are given explicitly by $-u(\overline{X}_t)(X_t - X_{t-})$ and $M^U_{t-}(X) = M^U(X_{t-})$.
We note also that when u is defined only on some interval $[a, b)$ but $U(b) \leq \infty$ is well defined then we can still define $M^U_t(X)$ for $t \leq \tau^b(X)$ and $M^U_{\tau^b(X)}(X) = U(b) \leq \infty$. Further, using regularity of paths of (X_t) we have that (2)-(3) hold with $t \wedge \tau^b(X)$ instead of t and $u(b) := 0$.

The symbol $M^U(X)$ is a slight abuse of notation since this process depends explicitly on the derivative u rather than the function U. Azéma and Yor [2] were the first to introduce these processes when (X_t) is a continuous local martingale. The equivalence between (2) and (3) is easy to establish when u is smooth enough to apply Itô's formula, since the continuity of the running supremum implies from (1) that $\int_0^t (\overline{X}_t - X_t) \mathrm{d}u(\overline{X}_t) \equiv 0$. This results may be extended to all bounded functions u via monotone class theorem and to all locally bounded functions u via a localisation argument. Alternatively, the equivalence can be argued using the general balayage formula, see Nikeghbali and Yor [23]. The case of locally integrable function u can be attained for continuous local martingale X, as shown in Obłój and Yor [26].

The importance of the family of Azéma–Yor martingales is well exhibited by Obłój [25] who proves that in the case of a continuous local martingale (X_t) all local martingales which are functions of the couple $(X_t, \overline{X}_t)$, $M_t = H(X_t, \overline{X}_t)$ can be represented as a $M = M^U(X)$ local martingale associated with a locally integrable function u. We note that such processes are sometimes called *max-martingales*.

1.2 Monotonic transformations and Azéma–Yor processes

We want to investigate further the structure of the set of Azéma–Yor processes associated with a max-continuous semimartingale (X_t). One of the most remarkable properties of these processes is that their running supremum can be easily computed, when the function U is non decreasing ($u \geq 0$).

We denote by $\mathcal{U}_{\mathrm{m}}$ the set of such functions, that is absolutely continuous functions defined on an appropriate interval with a locally bounded and non negative derivative. This set is stable by composition, that is if U and F are in $\mathcal{U}_{\mathrm{m}}$, and defined on appropriate intervals then $U \circ F(x) = U(F(x))$ is in $\mathcal{U}_{\mathrm{m}}$. We let $\mathcal{U}_{\mathrm{m}}^+$ be the set of increasing functions $U \in \mathcal{U}_{\mathrm{m}}$, with inverse function $V \in \mathcal{U}_{\mathrm{m}}$, or equivalently of functions U such that $u = U' > 0$ and both u and $1/u$ are locally bounded. Throughout, when we consider an inverse function V of $U \in \mathcal{U}_{\mathrm{m}}^+$ then we choose $v(y) = V'(y) = 1/u(V(y))$.

In light of (2), then we have

Proposition 1.2. *a) Let $U \in \mathcal{U}_{\mathrm{m}}$, X be a max-continuous semimartingale and $M^U(X)$ be the (U,X)-Azéma–Yor process in* (2). *Then*

$$\overline{M_t^U(X)} = U(\overline{X}_t), \tag{4}$$

and $M^U(X)$ is a max-continuous semimartingale.
b) Let $F \in \mathcal{U}_{\mathrm{m}}$ defined on an appropriate interval, so that $U \circ F$ is well defined. Then

$$M_t^U\big(M^F(X)\big) = M_t^{U \circ F}(X).$$

Remark 1.3. It follows from point b) above that the set of Azéma–Yor processes indexed by $U \in \mathcal{U}_{\mathrm{m}}^{+}$ defined on whole $\mathbb{R}$ with $U(\mathbb{R}) = \mathbb{R}$, is a group under the operation $\otimes$ defined by

$$M^U \otimes M^F := M^{U \circ F}.$$

Proof. a) In light of (2), when u is non negative, the Azéma–Yor process $M_t^U(X)$ is dominated by $U(\overline{X}_t)$, with equality if t is a point of increase of $\overline{X}_t$. Since U is non decreasing we obtain (4). Moreover, since $U(\overline{X})$ is a continuous process, $M^U(X)$ is a max-continuous semimartingale and we may take an Azéma–Yor process of it.
b) Let F be in $\mathcal{U}_{\mathrm{m}}$, $f = F' > 0$, such that $U \circ F$ is well defined. We have from (4)

$$\begin{aligned} M_t^U\big(M^F(X)\big) &= U\big(F(\overline{X}_t)\big) - u\big(F(\overline{X}_t)\big) f(\overline{X}_t)\big(\overline{X}_t - X_t\big) \\ &= M_t^{U \circ F}(X), \end{aligned} \tag{5}$$

where we used $\big(U(F(x))\big)' = u(F(x)) f(x)$. ☐

The two properties described in Proposition 1.2 are rather simple but at the same time extremely useful. As we will see, they will be crucial tools in most of the proofs in the paper. We phrase part b) above for stopped processes and for $F = V = U^{-1}$ as a separate corollary.

Corollary 1.4. *Let $a < b \leq \infty$, $U \in \mathcal{U}_{\mathrm{m}}^{+}$ a primitive function of a locally bounded $u : [a, b) \to (0, \infty)$ such that $U(a) = a^*$. Let $V : [a^*, U(b)) \to [a, b)$ be the inverse of U with locally bounded derivative $v(y) = 1/u(V(y))$.*
Then for any max-continuous semimartingale (X_t), $X_0 = a$, stopped at the time $\tau^b = \tau^b(X) = \inf\{t : X_t \geq b\}$ we have

$$X_{t \wedge \tau^b} = M_{t \wedge \tau^b}^V(M^U(X)). \tag{6}$$

From the differential point of view, on $[0, \tau^b)$,

$$\mathrm{d}Y_t := \mathrm{d}M_t^U(X) = u\big(\overline{X}_t\big)\mathrm{d}X_t, \quad dX_t = v\big(\overline{Y}_t\big)\mathrm{d}Y_t. \tag{7}$$

Consider u as above with $b = U(b) = \infty$. As a consequence of the above, any max-continuous semimartingale (X_t) can be seen as an Azéma–Yor process associated with U. Indeed, $X_t = M_t^U(Y)$ with $Y_t = M_t^V(X)$. In the following section we study how such representations arise in a natural way.

2 The Bachelier–Drawdown equation

In his paper ”Théorie des probabilités continues”, published in 1906, French mathematician Louis Bachelier [3] was the first to consider and study stochastic differential equations. Obviously, he didn't prove in his paper existence and uniqueness results but focused his attention on some particular types of SDE's. In this way, he obtained the general structure of processes with independent increments and continuous paths, the definition of diffusions (in particular, he solved the Langevin equation), and generalized these concepts to higher dimensions.

2.1 The Bachelier equation

In particular, Bachelier [3, pp.287–290] considered and ”solved” an SDE depending on the supremum of the solution, $\mathrm{d}Y_t = \varphi(\overline{Y}_t)\mathrm{d}X_t$ which we call the Bachelier equation. Let $U \in \mathcal{U}_{\mathrm{m}}^{+}$ and $V \in \mathcal{U}_{\mathrm{m}}^{+}$ its inverse function with derivative v. From (3) and (4) we see that the Azéma–Yor process $Y = M^U(X)$ verifies the Bachelier equation for $\varphi(y) = 1/v(y)$. Now, we can solve the Bachelier equation as an inverse problem. We present a rigorous and explicit solution to this equation which proves to be surprisingly simple. We note that a similar approach is developed in Revuz and Yor [27, Ex.VI.4.21].

Theorem 2.1. *Let $(X_t : t \geq 0)$, $X_0 = a$, be a max-continuous semimartingale. Consider a positive Borel function $\varphi : [a^*, \infty) \to (0, \infty)$ such that φ and $1/\varphi$ are locally bounded. Let $V(y) = a + \int_{a^*}^{y} \frac{\mathrm{d}s}{\varphi(s)}$, and U its inverse defined on $(a, V(\infty))$. The Bachelier equation*

$$\mathrm{d}Y_t = \varphi(\overline{Y}_t)\mathrm{d}X_t, \quad Y_0 = a^* \tag{8}$$

has a strong, pathwise unique, max-continuous solution defined up to its explosion time $\tau_Y^{\infty} = \tau_X^{V(\infty)}$ given by $Y_t = M_t^U(X), \quad t < \tau_X^{V(\infty)}$.
When X is a continuous local martingale it suffices to assume that $1/\varphi$ is a locally integrable function.

Proof. The assumptions on φ imply that V and therefore U are in $\mathcal{U}_{\mathrm{m}}^{+}$ with $U(a) = a^*$. With the version of $u = \varphi(V)$ we choose, Definition 1.1 gives that the Azéma–Yor process $M^U(X)$ verifies

$$\mathrm{d}M_t^U(X) := u(\overline{X}_t)\mathrm{d}X_t = \varphi(\overline{M_t^U(X)})\mathrm{d}X_t, \quad t < \tau_X^{V(\infty)}.$$

Furthermore, on $\tau^{V(\infty)}(X) < \infty$, $M^U_{\tau^{V(n)}}(X) = U(V(n)) = n$ and we see that if $V(\infty) < \infty$ then $\tau_X^{V(\infty)}$ is the explosion time of $M^U(X)$. So, M^U is a solution of (8). Now let Y be a max-continuous solution to the equation (8). Definition 1.1 and (8) imply that $\mathrm{d}M_t^V(Y) = \mathrm{d}X_t$ on $[0, \tau_Y^\infty)$. It follows from Corollary 1.4 that $Y_t = M_t^U(X)$ and $\tau_X^{V(\infty)}$ is the explosion time τ_Y^∞ of Y.
The above extends to more general φ whenever U, V and $M^U(X)$ are well defined. When X is a continuous local martingale, to define V and U it is sufficient (and necessary) to assume $1/\varphi$ is locally integrable. That $M^U(X)$ is then well defined follows from Obłój and Yor [26]. $\square$

The above extends naturally to the case when a and a^* are some $\mathcal{F}_0$-measurable random variables. It suffices to assume that φ is well defined on $[l, \infty)$ where $-\infty \le l$ is the lower bound of the support of a^*. We could also consider X which is only defined up to its explosion time τ_X^∞ which would induce $\tau_Y^\infty = \tau_X^\infty \wedge \tau_X^{V(\infty)}$.
In Section 3 we will also consider the case when $\varphi \equiv 0$ on (r, ∞) and then (Y_t) is stopped upon hitting r.
Finally note that under a stronger assumption that X has no positive jumps, *any* solution of the Bachelier equation has no positive jumps and hence is a max-continuous semimartingale.

2.2 Drawdown constraint and Drawdown equation

In various applications, in particular in financial mathematics one is interested in processes which remain above a (given) function w of their running maximum. The purpose of this section is to show that Azéma–Yor processes provide a direct answer to this problem when the underlying process X is positive. The following notion will be central throughout the rest of the paper.

Definition 2.2. *Given a function w, we say that a càdlàg process (M_t) satisfies w-*drawdown *(w-DD) constraint up to the (stopping) time ζ, if* $\min\{M_{t-}, M_t\} > w(\overline{M}_t)$ *for all* $0 \le t < \zeta$ *a.s.*

We will see in Section 3 that for a local martingale M it suffices to impose $M_t > w(\overline{M}_t)$ in the above definition.

Azéma–Yor processes, $Y = M^U(X)$ defined from a *positive* max-continuous semimartingale X and function $U \in \mathcal{U}_{\mathrm{m}}^+$ provide an example of such processes with

DD-constraint function w defined from U and $V = U^{-1}$ by:

$$w(y) = h(V(y)) = y - V(y)/v(y), \quad \text{where} \quad h(x) = U(x) - xu(x). \tag{9}$$

Indeed, thanks to the positivity of X and u, and to the characterisation of the left-continuous process Y_{t-} in D have:

$$Y_{t-} = U(\overline{X}_t) - u(\overline{X}_t)\overline{X}_t + u(\overline{X}_t)X_{t-} > U(\overline{X}_t) - u(\overline{X}_t)\overline{X}_t = h(\overline{X}_t) = w(\overline{Y}_t)\ . \tag{10}$$

The converse is possibly more interesting. We show below that if we start with a given w then $M^U(X)$, where $U = V^{-1}$ and V is given in (11), satisfies the w-DD constraint. Furthermore, it turns out that all processes which satisfy a drawdown constraint are of this type. More precisely, given a max continuous semimartingale Y satisfying the w-DD constraint we can find explicitly X such that Y is the Azéma–Yor process $M^U(X)$. Moreover, the first instant Y violates the drawdown constraint is precisely the first hitting time to zero of X. For a precise statement we need to introduce the set of admissible functions w:

Definition 2.3. *We say that* $w : [a^*, \infty] \to \mathbb{R}$ *is a* drawdown function *if it is non-decreasing and there exists* $r_w \leq \infty$ *such that* $y - w(y) > 0$ *is locally bounded and locally bounded away from zero on* $[a^*, r_w)$ *and* $w(y) = y$ *for* $y \geq r_w$*.*

We impose w non-decreasing as it is intuitive for applications. It will also arise naturally in Section 3. We introduced here r_w as it gives a convenient way to stop the process upon hitting a given level and again it will be used in Section 3. If a drawdown function w is defined on $[a^*, \infty)$ then we put $w(\infty) = \lim_{y\uparrow\infty} w(y)$ and the above definition requires that $w(\infty) = \infty$. In fact for the results in this section it is not necessary to require any monotonicity from w or that $w(\infty) = \infty$, we comment this below.

We let $\tau_0(X) = \tau_0^X = \inf\{t : \min\{X_{t-}, X_t\} \leq 0\}$ and note that when X is non-negative then $X_{\tau_0^X} \geq 0$. Further let $\zeta_w(Y) = \zeta_w^Y = \inf\{t : \min\{Y_{t-}, Y_t\} \leq w(\overline{Y}_t)\}$. As mentioned before, definitions of both τ_0 and ζ_w simplify for local martingales, see Lemma 3.1 and Corollary 3.2 in Section 3.

Theorem 2.4. *Consider a drawdown function* w *of Definition 2.3 and* V *solution of the ODE* (9)*,* $V(a^*) = a > 0$*, given as*

$$V(y) = a \exp\left(\int_{a^*}^{y} \frac{1}{u - w(u)} \mathrm{d}u\right), \quad y \geq a^*. \tag{11}$$

For (X_t)*,* $X_0 = a$*, a non-negative max-continuous semimartingale and* $\zeta := \tau_0(X) \wedge \tau^{V(r_w-)}(X)$ *the Drawdown equation*

$$\mathrm{d}Y_t = \big(Y_{t_-} - w(\overline{Y}_t)\big)\frac{\mathrm{d}X_t}{X_{t_-}}, \quad t < \zeta, \tag{12}$$

has a pathwise unique max-continuous solution, $Y_0 = a^*$*, which satisfies* w*-DD constraint on* $[0, \zeta)$*, given by* $Y_t = M_t^U(X)$*, where* U *is the inverse of* V*. We have* $\zeta_w(Y) = \zeta$ *and further* $Y_{\zeta_w^Y} = w(\overline{Y}_{\zeta_w^Y})$ *on* $\{X_\zeta \in \{0, V(r_w-)\}\}$*.*
Conversely, given (Y_t)*,* $Y_0 = a^*$*, a max-continuous semimartingale satisfying* w*-DD constraint up to* $\zeta := \zeta_w(Y)$*, there exists a pathwise unique max-continuous semimartingale* $(X_t : t < \zeta)$*,* $X_0 = a$*, which solves* (12)*.* X *may be deduced from* Y *by the Azéma–Yor bijection* $X_t = M_t^V(Y)$ *and* $\zeta = \tau_0(X) \wedge \tau^{V(r_w-)}(X)$*.*

Remark 2.5. Naturally $V(y) \equiv \infty$ for $y \geq r_w$. However $V(r_w-)$ could be both finite or infinite and consequently $\tau^{V(r_w-)}(X)$ can be both a hitting time of a finite level or the explosion time for X.
Observe that $\{X_\zeta \in \{0, V(r_w-)\}\}$ could be larger than $\{\zeta < \infty\}$. This will be the case in Section 3 where $X_t \to 0$ as $t \to \infty$ and in fact $X_\zeta \in \{0, V(r_w-)\}$ a.s. Naturally, we also have $Y_{\zeta_w^Y-} = w(\overline{Y}_{\zeta_w^Y})$ on $\{X_{\zeta-} \in \{0, V(r_w-)\}\}$. Note also that in the converse part of the theorem we could have $Y_\zeta < w(\overline{Y}_\zeta)$ which would correspond to $X_\zeta < 0$.

Remark 2.6. It will be clear from the proof that the theorem holds without assuming any monotonicity on w or that $w(\infty)$ is defined and equal to ∞. The only change is that $Y_{\zeta_w^Y} = w(\overline{Y}_{\zeta_w^Y})$ on $\{X_\zeta = V(r_w-)\}$ if and only if $w(r_w) = r_w$ and if $V(\infty) < \infty$, for which w would have to decrease faster than linear, then Y explodes at $\tau_X^{V(\infty)}$.

Proof. Expression for V in terms of w follows as $v(y) = V(y)/(y - w(y))$. Note that $V(\infty) = \infty$. Hence, for $t < \zeta$, $Y_t = M_t^U(X)$ is well defined and recall from Corollary 1.4 that $X_t = M_t^V(Y)$ and $\overline{X}_t = V(\overline{Y}_t)$. Direct computation yields

$$Y_{t-} - w(\overline{Y}_t) = Y_{t-} - U(\overline{X}_t) + u(\overline{X}_t)\overline{X}_t = u(\overline{X}_t)X_{t-}\ .$$

Thanks to the positivity of u and X and X_- on $t < \zeta$, we have that $Y_{t-}, Y_t > w(\overline{Y}_t)$ and it follows from (3) that $Y = M^U(X)$ solves (12).
Now consider any Y, a max-continuous solution of (12), $\min\{Y_{t-}, Y_t\} > w(Y_t)$ for $t < \zeta$. Then, using (2) and (3), we have

$$\frac{\mathrm{d}Y_t}{Y_{t_-} - w(\overline{Y}_{t_-})} = \frac{v(\overline{Y}_t)}{M_{t_-}^V(Y)}\mathrm{d}Y_t = \frac{\mathrm{d}M_t^V(Y)}{M_{t_-}^V(Y)}\ .$$

Since Y is solution of (12), X and $M^V(Y)$ have the same relative stochastic differential, and the same initial condition. Then, there are undistinguable processes and Corollary 1.4 yields $Y_t = M^U_t(X)$.
Finally, when $X_\zeta = 0$ (resp. $X_\zeta = V(r_w-)$) we have $Y_\zeta = U(\overline{X}_\zeta) - u(\overline{X}_\zeta)\overline{X}_\zeta = w(V(\overline{X}_\zeta)) = w(\overline{Y}_\zeta)$ (resp. $Y_\zeta = r_w = \overline{Y}_\zeta = w(\overline{Y}_\zeta)$) and $\zeta = \zeta_w(Y)$. If $X_\zeta \notin \{0, V(r_w-)\}$ then $X_{\zeta-} = 0$ or $\zeta = \infty$ and in both cases $\zeta = \zeta_w(Y)$. that $\zeta = \zeta_w(Y)$.

Consider now the second part of the theorem. We can rewrite (12) as

$$\frac{\mathrm{d}Y_t}{Y_{t_-} - w(\overline{Y}_t)} = \frac{\mathrm{d}X_t}{X_{t_-}}, \qquad t < \zeta. \tag{13}$$

This equation defines without ambiguity a positive process X starting from $X_0 = a > 0$. By assumption on w, the solution V of (9) is a positive finite increasing function on $[a^*, r_w)$, $V(y)/v(y) = y - w(y)$. Put $\widehat{X}_t = M^V_t(Y)$, and observe that the differential properties of V imply that $\widehat{X}_t = v(\overline{Y}_t)(Y_t - w(\overline{Y}_t)) > 0$, for $t < \zeta$. Then, the stochastic differential of $M^V_t(Y) = \widehat{X}_t$ is

$$d\widehat{X}_t = v(\overline{Y}_t)dY_t = \widehat{X}_{t-}(Y_{t-} - w(\overline{Y}_t))^{-1}dY_t,$$

and hence both $\widehat{X}$ and X are solutions of the same stochastic differential equation and have the same initial conditions. So, they are undistinguishable processes. Identification of ζ follows as previously. □

Naturally, since $Y = M^U(X)$ solves both the Bachelier equation (8) and the Drawdown equation (12) these equations are equivalent. We phrase this as a corollary in the case $\zeta = \infty$ a.s. in (12).

Corollary 2.7. *Let $(X_t : t \geq 0)$, $X_0 = a$, be a positive max-continuous semimartingale, $\tau_0^X = \infty$ a.s., and φ, V as in Theorem 2.1 with $V(\infty) = \infty$. Then, the Bachelier equation* (8) *is equivalent to the Drawdown equation* (12) *where w and V are linked via* (9) *or equivalently via* (11).

The Drawdown equation (12) was solved previously by Cvitanić and Karatzas [7] for $w(y) = \gamma y$, $\gamma \in (0, 1)$ and recently by Elie and Touzi [11]. The use of Azéma–Yor processes simplifies considerably the proof and allows for a general w and X. We have shown that this equation has a unique strong solution and is equivalent to the Bachelier equation.

Note that we assumed X is positive. The quantity $\mathrm{d}X_t/X_{t-}$ has a natural interpretation as the differential of the stochastic logarithm of X. In various applications, such as financial mathematics, this logarithm process is often given directly since X is defined as a stochastic exponential in the first place.

An Illustrative Example. Let X be a positive max-continuous semimartingale such that $X_0 = 1$. Let U be the power utility function defined on $\mathbb{R}^+$ by $U(x) = \frac{1}{1-\gamma} x^{1-\gamma}$ with $0 < \gamma < 1$ and $u(x) = x^{-\gamma}$ its derivative. The inverse function V of U is $V(y) = ((1-\gamma)y)^{1/(1-\gamma)}$ and its derivative is $v(y) = ((1-\gamma)y)^{\gamma/(1-\gamma)}$.
Then the (power) Azéma–Yor process is

$$M_t^U(X) = Y_t = \frac{1}{1-\gamma}(\overline{X}_t)^{1-\gamma}\Big(\gamma + (1-\gamma)\frac{X_t}{\overline{X}_t}\Big) = \overline{Y}_t\Big(\gamma + (1-\gamma)\frac{X_t}{\overline{X}_t}\Big).$$

Since X is positive, $Y_t > w(\overline{Y}_t) = \gamma\overline{Y}_t$. The drawdown function w is the linear one, $w(y) = \gamma y$.
The process (Y_t) is a semimartingale (local martingale if X is a local martingale) starting from $Y_0 = 1$, and staying in the interval $[\gamma\overline{Y}_t, \overline{Y}_t]$. Since the power function U is concave, we also have an other floor process $Z_t = U(X_t)$. Both processes Z_t and $\gamma\overline{Y}_t = \gamma\overline{Z}_t$ are dominated by Y_t. They are not comparable in the sense that in general at time t either of them can be greater. We study floor process Z in more detail in Section 4.2.
The Bachelier-Drawdown equation (8)-(12) becomes here

$$\begin{aligned} \mathrm{d}Y_t &= \overline{X}_t^{-\gamma}\mathrm{d}X_t = \big((1-\gamma)\overline{Y}_t\big)^{-\frac{\gamma}{1-\gamma}}\,\mathrm{d}X_t \\ &= (Y_{t-} - \gamma\overline{Y}_t)\frac{\mathrm{d}X_t}{X_{t-}}\,. \end{aligned} \tag{14}$$

As noted above, this equation, for a class of processes X, was studied in Cvitanić and Karatzas [7]. Furthermore, in [7] authors in fact introduced processes $M^U(X)$ where U is the a power utility function, and used them to solve the portfolio optimisation problem with drawdown constraint of Grossman and Zhou [15] (see also [11]). Using our methods we can simplify and generalize their results and show that the portfolio optimisation problem with drawdown constraint, for a general utility function and a general drawdown function, is equivalent to an unconstrained portfolio optimisation problem with a modified utility function. We develop these ideas in a separate paper.

3 Setup driven by a non-negative local martingale converging to zero

In previous section we investigated Azéma–Yor processes build from a non-negative semimartingale as solutions to the Drawdown equation (12). We specialise now further and study in detail Azéma–Yor processes associated to $X = N$ a non-negative local martingale converging to zero at infinity. The maximum of N has a universal law which, together with $N_\infty = 0$, allows to write Azéma–Yor martingales explicitly from terminal values, see Sections 3.1-3.2. Our study exploits tail quantiles of probability measures and is intimately linked with the Average Value at Risk and the Hardy–Littlewood transform of a measure, as explored in Section 3.3. Finally, combining these results with Theorem 2.4, we construct Azéma–Yor martingales with prescribed terminal distributions and in particular obtain the Azéma–Yor [2] solution of the Skorokhod embedding problem.

3.1 Universal properties of $X = N$

A non-negative local martingale (N_t) is a supermartingale and it is a (true) martingale if and only if $\mathbb{E}\, N_t = \mathbb{E}\, N_0$, $t \geq 0$. We also have that if N_t or N_{t-} touch zero then N_t remains in zero (see e.g. Dellacherie and Meyer [8, Thm VI.17]).

Lemma 3.1. *Consider a non-negative local martingale (N_t) with $N_{0-} := N_0 > 0$. Then*

$$\tau_0(N) = \inf\{t : N_t = 0 \text{ or } N_{t-} = 0\} = \inf\{t : N_t = 0\} \tag{15}$$

and $N_u \equiv 0$, $u \geq \tau_0(N)$.

This yields an immediate simplification of the w-DD condition. In fact in Definition 2.2 and definition of $\zeta_w(Y)$ on page 9 it suffices to compare $w(\overline{Y}_t)$ with Y_t instead of Y_t *and* Y_{t-}.

Corollary 3.2. *Let w be a drawdown function of Definition 2.2 and (Y_t) a max-continuous local martingale with $Y_\zeta = w(\overline{Y}_\zeta)$ a.s. on $\{\zeta < \infty\}$, where $\zeta = \inf\{t : Y_t \leq w(\overline{Y}_t)\}$. Then Y satisfies w-DD condition up to time $\zeta_w(Y) = \zeta$.*

Proof. Assume $r_w = \infty$ and let $N_t = M_t^V(Y)$ where V is given via (11). Using (9)–(10), similarly as in the proof of Theorem 2.4, and Definition 1.1, $(N_t : t \leq \zeta)$ is a non-negative max-continuous local martingale and $\zeta = \inf\{t : N_t = 0\}$. Using (15) we have $\zeta = \tau_0(N)$ and our assumptions also give $N_\zeta = 0$ on $\{\zeta < \infty\}$. It follows from Theorem 2.4 that $Y_t = M^U(X)_t$, $U = V^{-1}$ satisfies the w-DD constraint up to ζ and $\zeta = \tau_0(N) = \zeta_w(Y)$. For the case $r_w < \infty$ it suffices to note that all processes are max-continuous and hence the first hitting times for $\overline{N}_t$ and $\overline{N}_{t-}$ are equal. $\square$

Throughout this and following sections, we assume that

$$(N_t : t \geq 0) \text{ is a non-negative max-continuous local martingale, } N_t \xrightarrow[t\to\infty]{} 0 \text{ a.s.} \tag{16}$$

We recall some well known results on the distribution of the maximum of N (see Exercice III.3.12 in Revuz-Yor [27]). We assume that N_0 is a constant. If N_0 is random all results should be read conditionally on $\mathcal{F}_0$.

Proposition 3.3. *Consider (N_t) in* (16) *with $N_0 > 0$ a constant.*
a) The random variable $N_0/\overline{N}_\infty$ is uniformly distributed on $[0,1]$.
b) The same result holds for the conditional distribution in the following sense: let $\overline{N}_{t,\infty} = \sup_{t\leq u\leq\infty} N_u$ then

$$\mathbb{P}(K > \overline{N}_{t,\infty} | \mathcal{F}_t) = (1 - N_t/K)^+$$

i.e. $\overline{N}_{t,\infty}$ has the same $\mathcal{F}_t$-conditional distribution as N_t/ξ where ξ is an independent uniform variable on [0,1].
c) Let $\zeta = \tau_0(N) \wedge \tau^b(N) = \inf\{t : N_t \notin (0,b)\}$, $b > N_0$. $(N_{t\wedge\zeta})$ is a bounded martingale and $N_\zeta \in \{0,b\}$. Furthermore, $\overline{N}_\zeta = \overline{N}_\infty \wedge b$ is distributed as $(N_0/\xi)\wedge b$, where ξ is uniformly distributed on $[0,1]$.

Remark. a) Given the event $\{\overline{N}_\zeta < b\} = \{N_\zeta = 0\}$, $N_0/\overline{N}_\zeta$ is uniformly distributed on $(N_0/b, 1]$. The probability of the event $\{\overline{N}_\zeta = b\}$ is N_0/b.
b) Any non-negative martingale N stopped at ζ, with $N_\zeta \in \{0,b\}$ a.s., may be extended into a local martingale (still denoted by N) as in (16), by putting $N_t := N_\zeta + \mathbf{1}_{\{\overline{N}_\zeta = b\}}(N'_t - N'_\zeta)$, $t > \zeta$, where N' is another local martingale as in (16).

Proof. a) Let us consider the Azéma–Yor martingale associated with (N_t) and the function $U(x) = (K-x)^+$, where K is a fixed real ≥ 1. Thanks to the positivity of (N_t), the martingale $M^U(N)$ is bounded by K,

$$0 \leq M_t^U(N) = (K - \overline{N}_t)^+ + \mathbf{1}_{\{K > \overline{N}_t\}}(\overline{N}_t - N_t) = \mathbf{1}_{\{K > \overline{N}_t\}}(K - N_t) \leq K.$$

So $M_t^U(N)$ is a uniformly integrable martingale, and $\mathbb{E}\, M_\infty^U(N) = M_0^U(N)$. In other terms, for $K > N_0$, $K\,\mathbb{P}(K > \overline{N}_\infty) = K - N_0$, or equivalently $\mathbb{P}\left(\frac{N_0}{\overline{N}_\infty} \leq \frac{N_0}{K}\right) = \frac{N_0}{K}$, for any $K \geq N_0$. That is exactly the desired result.

b) This result is the conditional version of the previous one. The reference process is now the process $(N_{t+h} : h \geq 0)$ adapted to the filtration $\mathcal{F}_{t+h}$, local martingale for the conditional probability measure $\mathbb{P}(.| \mathcal{F}_t)$.

c) From (15) and since N is non-negative and max-continuous it follows that $\tau_0(N) \wedge \tau^b(N) = \inf\{t : N_t \notin (0,b)\}$ and that $N_\zeta = b$, or 0. Then, we have that $\overline{N}_\zeta = \overline{N}_\infty \wedge b$ since when $N_\zeta = b$, the maximum $\overline{N}_\zeta$ is also equal to b. □

Remark about last passage times. Recently, for a continuous local martingale N, Madan, Roynette and Yor [22] have interpreted the event $\{K > \overline{N}_{T,\zeta}\}$ in terms of the last passage time $g_K(N)$ over the level K, as $\{K > \overline{N}_{T,\zeta}\} = \{g_K(N) < T\}$. Our last Proposition yields immediately their result: the normalized Put pay-off is the conditional probability of $\{g_K(N) < T\}$: $(1 - N_T/K)^+ = \mathbb{P}(g_K(N) < T| \mathcal{F}_T)$. In particular we obtain the whole dynamics of the put prices:

$$\mathbb{E}\left[(K - N_T)^+ | \mathcal{F}_t\right] = K\, \mathbb{P}(g_K(N) < T| \mathcal{F}_t), \quad t \leq T,$$

and the initial prices $(t = 0)$ are deduced from the distribution of g_K. In the geometrical Brownian motion framework with $N_0 = 1$, the Black-Scholes formula just computes the distribution of $g_1(N)$ as $\mathbb{P}(g_1 < t) = \mathcal{N}(\sqrt{t}/2) - \mathcal{N}(-\sqrt{t}/2) = \mathbb{P}(4B_1^2 \leq t)$, where B_1 is a standard Gaussian random variable and $\mathcal{N}(x) = \mathbb{P}(B_1 \leq x)$ the Gaussian distribution function (See also Madan, Roynette and Yor [22]).

Financial framework. Assume S to be a max-continuous non negative submartingale whose instantaneous return by time unit is an adapted process $\lambda_t \geq 0$ defined on a filtered probability spaced $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. For instance, S is the current price of a stock under the risk neutral probability in a financial market with short rate λ_t. Put differently, $\tilde{S}_t = \exp(-\int_0^t \lambda_s \mathrm{d}s) S_t$ is an $(\mathcal{F}_t)$-martingale. We assume that $\int_0^\infty \lambda_s \mathrm{d}s = \infty$ a.s. Let ζ be an additional r.v. with conditional tail function $\mathbb{P}(\zeta \geq t | \mathcal{F}_\infty) = \exp(-\int_0^t \lambda_s \mathrm{d}s)$. Then $X_t = S_t \mathbf{1}_{t<\zeta}$ is a positive martingale with negative jump to zero at time ζ with respect to the enlarged filtration $\mathcal{G}_t = \sigma(\mathcal{F}_t, \zeta \wedge t)$. Since the $\mathbb{G}$-martingale X goes to zero at ∞, the random variable $\overline{X}_\zeta = \overline{S}_\zeta$ is distributed as $1/\xi$, where ξ is uniformly distributed on $[0,1]$. In particular, for any bounded function h

$$\mathbb{E}[h(\overline{S}_\zeta)] = \mathbb{E}\left[\int_0^\infty e^{-\int_0^\alpha \lambda_s \mathrm{d}s} h(\overline{S}_\alpha) \lambda_\alpha \mathrm{d}\alpha\right] = \int_0^1 h(S_0/u)\mathrm{d}u.$$

In consequence we have access to the law of the properly discounted maximum of the positive submartingale S. We stress that this is contrast to the more usual setting when one only has access to the maximum of the discounted price process, cf. Grossman and Zhou [15]. We could also derive a conditional version of the equation above representing $U(S_t)$ as a potential of the future supremum $\overline{S}_{t,u}$. Such representation find natural applications in financial mathematics, see Bank and El Karoui [4].

3.2 Azéma–Yor martingales with given terminal values

We describe now all local martingales whose terminal values are Borel functions of the maximum of some non-negative local martingale. This will be used in subsequent sections, in particular to construct Azéma–Yor martingales with given terminal distribution and solve the Skorokhod embedding problem. We start with a simple lemma about solutions to a particular ODE.

Lemma 3.4. *Let h be a locally bounded Borel function defined on $\mathbb{R}^+$, such that $h(x)/x^2$ is integrable away from zero. Let U be the solution of the ordinary differential equation (ODE)*

$$\forall x > 0 \quad U(x) - xU'(x) = h(x), \text{ such that } \lim_{x\to\infty} U(x)/x = 0\,. \tag{17}$$

a) The solution U is given by

$$U(x) = x \int_x^\infty \frac{h(s)}{s^2}\mathrm{d}s = \int_0^1 h(\frac{x}{s})\mathrm{d}s, \quad x > 0. \tag{18}$$

b) Let $h_b(x) := h(x \wedge b)$ be constant on (b,∞). The associated solution $U_b(x) = \int_0^1 h(\frac{x}{s} \wedge b)\mathrm{d}s = U_b(x \wedge b)$ is constant on (b,∞), and $U_b(x) = h_b(x) = h(b)$.
c) Let $h(m,x) = h(x \vee m)$ be constant on $(0,m)$. The associated solution $U(m,x)$ is affine $U(m,x) = U(m) - U'(m)(m-x)$ for $x \in (0,m)$.

Remark 3.5. We considered here U on $(0,\infty)$ but naturally if h is only defined for $x > a > 0$ then we consider U also only for $x > a > 0$. Note that to define U_b it suffices to have a locally integrable h defined on $(0,b]$. We then put $h(x) = h(b)$, $x > b$.

Proof. Formula (18) is easy to obtain using the transformation $(U(x)/x)' = -h(x)/x^2$ and the asymptotic condition in (17). Both b) and c) follow simply from general solution (18). □

This analytical lemma allow us to characterize Azéma–Yor martingales from their terminal values. This extends in more details the ideas presented in El Karoui and Meziou [10, Propositon 5.8].

Proposition 3.6. *Consider* (N_t) *in* (16) *with* $N_0 > 0$ *a constant.*
a) Let h *be a Borel function such that* $h(x)/x^2$ *is integrable away from* 0*, and* U *the solution of the ODE* (17) *given via* (18)*. Then* $h(\overline{N}_\infty)$ *is an integrable random variable and the closed martingale* $\mathbb{E}\left(h(\overline{N}_\infty)|\,\mathcal{F}_t\,\right)$*,* $t \geq 0$*, is the Azéma–Yor martingale* $M^U(N)$*.*

b) For a function U *with locally bounded derivative* U' *and with* $U(x)/x \to 0$ *as* $x \to \infty$*, the Azéma–Yor local martingale* $M^U(N)$ *is a uniformly integrable martingale if and only if* $h(x)/x^2$ *is integrable away from zero, where* h *is now defined via* (17)*.*

Proof. We start with the proof of a). We have

$$\mathbb{E}\,|h(\overline{N}_\infty)| = \int_0^1 |h(N_0/s)|\mathrm{d}s = N_0 \int_{N_0}^\infty |h(s)|/s^2 \,\mathrm{d}s < \infty,$$

since we assumed that $h(x)/x^2$ is integrable away from 0. To study the martingale $H_t = \mathbb{E}\left(h(\overline{N}_\infty)|\,\mathcal{F}_t\right)$, we use that $\overline{N}_\infty = \overline{N}_t \vee \overline{N}_{t,\infty}$. From Proposition 3.3, the running supremum $\overline{N}_{t,\infty}$ is distributed as N_t/ξ, for an independent r.v. ξ uniform on $[0,1]$. The martingale H_t is given by the following closed formula $H_t = \mathbb{E}\left(h\big(\overline{N}_t \vee (N_t/\xi)\big)|\,\mathcal{F}_t\,\right)$ that is

$$H_t = \int_0^1 h(\overline{N}_t \vee (N_t/s))\mathrm{d}s = U(\overline{N}_t, N_t) = U(\overline{N}_t) - U'(\overline{N}_t)(\overline{N}_t - N_t),$$

where in the last equalities, we have used Lemma 3.4.
To prove part $b)$ it suffices to observe that $M_t^U(N) \to M_{\infty_-}^U(N) = M_\infty^U(N) = h(\overline{N}_\infty)$ a.s. and hence integrability of $h(\overline{N}_\infty)$, i.e. integrability of $h(x)/x^2$ away from zero, is necessary for uniform integrability of $M^U(N)$. That it is sufficient we proved in part $a)$. ☐

Remark 3.7. It is not necessary to assume that N_0 is a constant in Proposition 3.6. However if N_0 is random we have to further assume that $\mathbb{E}\int_0^1 |h(N_0/s)|\mathrm{d}s = \int_1^\infty \mathbb{E}\,|h(xN_0)|\frac{\mathrm{d}x}{x^2} < \infty$. This holds for example if N_0 is integrable and $N_0 > \epsilon > 0$ a.s. We can apply the same reasoning to the process $(N_{t+u} : u \geq 0)$ to see that if $\mathbb{E}\int_0^1 |h(N_t/s)|\mathrm{d}s < \infty$ then $U(N_t) = \mathbb{E}\left(h(\overline{N}_{t,\infty})|\,\mathcal{F}_t\,\right)$.
Finally, we note that similar consideration as in $a)$ above were independently made in Nikeghbali and Yor [23].

We stress that the boundary condition $U(x)/x \to 0$ as $x \to \infty$ for (17) is essential in part $a)$. Indeed, consider $N_t = 1/Z_t$ the inverse of a three dimensional Bessel process. Note that N_t satisfies our hypothesis and it is well known that N_t is a strict local martingale (cf. Exercise V.2.13 in Revuz and Yor [27]). Then for $U(x) = x$ we have $M_t^U(N) = N_t$ is also a strict local martingale but obviously we have $U(x) - U'(x)x = 0$.

Observe that $\mathbb{P}(N_\zeta \in \{0, b\}) = 1$ if $\zeta = \inf\{t \geq 0 : N_t \notin (0, b)\}$. Then, if h is constant on $[b, \infty)$ then $h(\overline{N}_\infty) = h(\overline{N}_\zeta)$ and the closed martingale $\mathbb{E}\left(h(\overline{N}_\zeta)|\,\mathcal{F}_{t\wedge\zeta}\right)$, $t \geq 0$, is the Azéma–Yor martingale $M_t^{U_b}(N) = M_{t\wedge\zeta}^{U_b}(N)$, where U_b the solution of the ODE (17) given in point b) of Lemma 3.4.

As shown in Sections 1 and 2, Azéma-Yor processes $Y = M^U(N)$ generated by an increasing function U have very nice properties based on the characterisation of their maximum as $\overline{Y} = U(\overline{N})$. In particular, from Theorem 2.4, the process Y satisfies a DD-constraint and can also be characterized from its terminal value. Recall Definitions 2.2, 2.3 and the stopping time $\zeta_w(Y)$ from page 9.

Proposition 3.8. *Let h be a right-continuous non-decreasing function such that $h(x)/x^2$ is integrable away from 0 and put $b = \inf\{x : h(y) = h(x)\ \forall y \geq x\}$.*
a) The solution U of the ODE (17) *is then a strictly increasing concave function on $(0, b)$ and constant and equal to $h(b)$ on (b, ∞).*
b) Let V be the inverse of U. Function $w(y) = h(V(y))$ given for $y < U(b)$ by (9)*, or equivalently* (11)*, and by $w(y) = y$ for $y \geq U(b)$, is a right-continuous drawdown function and $r_w = U(b) = h(b)$.*
c) Consider (N_t) in (16) *with $N_0 > 0$ a constant. The uniformly integrable martingale $Y_t = M_t^U(N) = \mathbb{E}[h(\overline{N}_\infty)|\,\mathcal{F}_t]$ satisfies w-DD constraint. Furthermore, $Y_t = Y_{t\wedge\zeta_w(Y)}$, $Y_{\zeta_w(Y)} = w(\overline{Y}_{\zeta_w(Y)})$ a.s. and $\zeta_w^Y = \inf\{t : N_t \notin (0, b)\}$.*
Conversely, let w be a right-continuous drawdown function, with functions V, U, h satisfying a) and b). Then any uniformly integrable martingale Y, satisfying the w-DD constraint and $Y_{\zeta_w(Y)} = w(\overline{Y}_{\zeta_w(Y)})$ a.s., is an Azéma–Yor martingale $M^U(N)$ for some (N_t) as in (16) *with $N_0 = V(Y_0) > 0$ and such that $N_{t\wedge\zeta_w(Y)} = M_{t\wedge\zeta_w(Y)}^V(Y)$ and $\zeta_w^Y = \inf\{t : N_t \notin (0, V(r_w-)\}$.*

Remark 3.9. Note that h, and in consequence U, need to be defined only for $x \geq N_0$. Then $V(y)$ is defined for $U(N_0) \leq y \leq U(b)$ with $V(U(b)) = b \leq \infty$ and the drawdown function $w(y)$ is defined for $y \geq U(N_0)$.
A solution U of the ODE (17) is strictly increasing if and only if $U > h$. But only increasing and concave solutions are easy to characterize.

Proof. a) When h is non-decreasing, from (17) and (18) it is clear that U is strictly increasing until that h becomes constant, and constant after that. If h is differentiable, concavity of U follows since $-xU_b''(x) = h'(x)$. The general case follows by regularisation or can be checked directly using (18) which yields to $U'(x) = \int_x^\infty (h(s) - h(x))/s^2 ds = \int_0^\infty (h(s) - h(x))^+/s^2 ds$.

b) In consequence, V is increasing and convex on $[U(0), U(b))$ and hence by (9) $w(y)$ is increasing and $w(y) < y$ for $y \in (U(0), U(b))$. We thus have $r_w = U(b)$ but note that we could have $w(U(b)-)$ both less then or equal to $U(b)$. Integrability properties of w in Definition 2.3 follow since V and U are well defined and we conclude that w is a drawdown function. Right-continuity of w follows from right-continuity of h. More precisely, from (17), $u = U'$ is right continuous and hence also $V'(y) = 1/u(V(y))$ is right-continuous and non decreasing.

c) Identification of Y is given in part a) of Proposition 3.6. The rest follows from Lemma 3.1, Theorem 2.4 and the fact that $N_\zeta \in \{0, b\}$ a.s. for $\zeta = \inf\{t : N_t \notin (0, b)\}$ upon noting that $V(r_w-) = b$. □

3.3 On relations between AVaR_μ, Hardy-Littlewood transform and tail quantiles

In this section we present results about probability measures, their tail quantile function, the Average Value at Risk and the Hardy-Littlewood transform. The presentation is greatly simplified using tail quantiles of measure.

The notation and quantities now introduced will be used throughout the rest of the paper. For μ a probability measure on $\mathbb{R}$ we denote l_μ, r_μ respectively the lower and upper bound of the support of μ. We let $\overline{\mu}(x) = \mu\big([x, \infty)\big)$ and $\overline{q}_\mu : (0, 1] \to \mathbb{R} \cup \{\infty\}$ be the tail quantile function defined as the left-continuous inverse of $\overline{\mu}$, $\overline{q}_\mu(\lambda) := \inf\{x \in \mathbb{R} : \overline{\mu}(x) < \lambda\}$. When $\overline{q}_\mu(\lambda)$ is a point of continuity of $\overline{\mu}$, then $\overline{\mu}(\overline{q}_\mu(\lambda)) = \lambda$, whereas if not, $\overline{\mu}(\overline{q}_\mu(\lambda)^+) < \lambda \le \overline{\mu}(\overline{q}_\mu(\lambda))$. In particular, if $\overline{\mu}(r_\mu) > 0$, $r_\mu = \overline{q}(0^+)$ is a jump of $\overline{\mu}$ and $r_\mu = \overline{q}(0^+) = \overline{q}(\lambda)$, if $0 < \lambda \le \overline{\mu}(r_\mu)$.

We write $X \sim \mu$ to denote that X has distribution μ and recall that $\overline{q}_\mu(\xi) \sim \mu$ for ξ uniformly distributed on $[0, 1]$.

Assume $\int_{\mathbb{R}} |s|\mu(\mathrm{d}s) < \infty$ and let $m_\mu = \int_{\mathbb{R}} s\mu(\mathrm{d}s)$. We define Call function[1] C_μ and barycentre function ψ_μ by

$$C_\mu(K) = \int_{[K,\infty)} (s - K)^+ \mu(ds), \quad \psi_\mu(x) = \frac{1}{\overline{\mu}(x)} \int_{[x,\infty)} s\mu(ds), \qquad (19)$$

[1]This denomination is used in financial literature while the actuarial literature uses rather the notion of stop-loss function, cf. [18].

where $K \in \mathbb{R}, x < r_\mu$. We put $\psi_\mu(x) = x$ for $x \geq r_\mu$.
Finally, we also introduce the *Average Value at Risk* at the level $\lambda \in (0,1)$, given by

$$\mathrm{AVaR}_\mu(\lambda) = \frac{1}{\lambda} \int_0^\lambda \overline{q}_\mu(u)\,\mathrm{d}u \tag{20}$$

which is strictly decreasing on $(\overline{\mu}(r_\mu), 1)$, equal to $r_\mu = \overline{q}_\mu(0^+)$ on $(0, \overline{\mu}(r_\mu))$ and $\mathrm{AVaR}_\mu(1) = m_\mu$.
The average value at risk AVaR_μ is thus a quantile function of some probability measure μ^{HL} with support (m_μ, r_μ), which can be defined by

$$\mu^{HL} \sim \mathrm{AVaR}_\mu(\xi), \quad \xi \text{ uniform on } [0,1]. \tag{21}$$

This distribution, called the *Hardy–Littlewood transform* of μ has been intensively studied by many authors, from the famous paper of Hardy and Littlewod [16]. We will describe its prominent rôle in the study of distributions of maxima of martingales in Section 4 below. Recently Föellmer and Schied [12] studied properties of AVaR_μ as a coherent risk measure. Finally note that here μ is the law of losses (i.e. negative of gains) and some authors refer to $\mathrm{AVaR}_\mu(\lambda)$ as $\mathrm{AVaR}_\mu(1-\lambda)$.

As in [12, pp 179–182, p 408, Lemma A.22], it is easy to characterize the Fenchel transform of the concave function $\lambda\mathrm{AVaR}_\mu(\lambda)$ as the Call function. From this property, we infer a non classical representation of the tail function $\overline{\mu}^{HL}(y)$ as an infimum.

Proposition 3.10. *Let μ be a probability measure on $\mathbb{R}$, $\int |s|\mu(\mathrm{d}s) < \infty$.*
i) The Average Value at Risk $\mathrm{AVaR}_\mu(\lambda)$ can be described as, $\lambda \in (0,1)$,

$$\mathrm{AVaR}_\mu(\lambda) \ = \frac{1}{\lambda}\mathrm{C}_\mu(\overline{\mathrm{q}}_\mu(\lambda)) + \overline{\mathrm{q}}_\mu(\lambda) = \frac{1}{\lambda}\inf_{\mathrm{K}\in\mathbb{R}}(\mathrm{C}_\mu(\mathrm{K}) + \lambda\mathrm{K}). \tag{22}$$

ii) The Call function is the Fenchel transform of $\lambda\mathrm{AVaR}_\mu(\lambda)$, so that

$$C_\mu(K) = \sup_{\lambda\in(0,1)} (\lambda\mathrm{AVaR}_\mu(\lambda) - \lambda K), \quad K \in \mathbb{R}\,. \tag{23}$$

iii) The Hardy-Littlewood tail function $\overline{\mu}^{HL}$ is given for any $y \in (m_\mu, r_\mu)$ by

$$\overline{\mu}^{HL}(y) = \inf_{z>0} \frac{1}{z} C_\mu(y-z). \tag{24}$$

iv) The barycentre function and its right continuous inverse are related to the Average Value at Risk and Hardy-Littlewood tail function by

$$\psi_\mu(x) = \mathrm{AVaR}_\mu(\overline{\mu}(x)),\ x \leq r_\mu, \qquad \psi_\mu^{-1}(y) = \overline{q}_\mu(\,\overline{\mu}^{HL}(y)),\ y \in [m_\mu, r_\mu]. \tag{25}$$

Remark 3.11. From (19) we have $\psi_\mu(x) = \mathbb{E}[X|X \geq x]$, where $X \sim \mu$. Then (20) gives $\mathrm{AVaR}_\mu(\lambda) = \mathbb{E}[X|X \geq \overline{q}_\mu(\lambda)]$, d$\overline{q}_\mu(\lambda)$-a.e., which justifies names *expected shortfall*, or *Conditional Value at Risk* used for AVaR_μ.

Proof. We write $\overline{q} = \overline{q}_\mu$.
i) The proof is based on the classical property, $\overline{q}(\xi) \sim \mu$ for ξ uniformly distributed on $[0, 1]$. Then

$$C_\mu(\overline{q}(\lambda)) = \int_0^1 (\overline{q}(u) - \overline{q}(\lambda))^+ \,\mathrm{d}u = \int_0^\lambda (\overline{q}(u) - \overline{q}(\lambda))\,\mathrm{d}u = \lambda(\mathrm{AVaR}_\mu(\lambda) - \overline{q}(\lambda)).$$

Moreover, the convex function $G_\lambda(K) := C_\mu(K) + \lambda K$ attains its minimum in K_λ such that $\overline{\mu}(K_\lambda) = \lambda$.
When $\overline{\mu}(\overline{q}(\lambda)) = \lambda$, $\overline{q}(\lambda)$ is a minimum of the function $G_\lambda(K)$, $\lambda\mathrm{AVaR}_\mu(\lambda) = G_\lambda(\overline{q}(\lambda))$, and (22) holds true.
If $\overline{\mu}(\overline{q}(\lambda)) > \lambda > \overline{\mu}(\overline{q}(\lambda)+)$ then μ has an atom in $x := \overline{q}(\lambda)$. G_λ has a minimum in x and G'_λ changes sign discontinuously in x. Then we see that $G_\lambda(\overline{q}(\lambda) = G_\lambda(x)$ is linear in $\lambda \in (\overline{\mu}(x+), \overline{\mu}(x))$.
ii) Convex duality for Fenchel transforms yields (23) from (22).
iii) Using (22) we have, for any $y > m_\mu$ and $\lambda \in (0, 1)$:

$$\begin{aligned} \mathrm{AVaR}(\lambda) < y \;&\Leftrightarrow \exists\, K \text{ such that } y > \frac{C_\mu(K)}{\lambda} + K \\ &\Leftrightarrow \exists\, K < y \text{ such that } \lambda > \frac{C_\mu(K)}{y-K} \Leftrightarrow \lambda > \inf_{K<y} \frac{C_\mu(K)}{y-K} \end{aligned} \tag{26}$$

The function $\inf_{K<y} \frac{C_\mu(K)}{y-K} = \inf_{z>0} \frac{1}{z} C_\mu(y-z)$ is decreasing and left-continuous. We conclude that it is the left-continuous inverse function of $\mathrm{AVaR}_\mu(\lambda)$ which is $\overline{\mu}^{HL}$.
iv) By definition, $\overline{\mu}(x)\mathrm{AVaR}_\mu(\overline{\mu}(x)) = \int_0^{\overline{\mu}(x)} \overline{q}_\mu(u)\,\mathrm{d}u = \int_{[x,\infty)} s\mu(ds)$.
The right-continuous inverse $\psi_\mu^{-1}(y)$ of the non decreasing left-continuous function ψ_μ is defined by $\psi_\mu^{-1}(y) = \sup\{x : \psi_\mu(x) \leq y\} = \sup\{x : \mathrm{AVaR}_\mu(\overline{\mu}(x)) \leq y\}$. Since, $\overline{\mu}^{HL}$ is the left continuous inverse of AVaR_μ, the following inequalities hold true for $y \in [m_\mu, r_\mu]$: $\psi_\mu^{-1}(y) = \sup\{x : \overline{\mu}(x) \geq \overline{\mu}^{HL}(y)\} = \sup\{x :\ x \leq \overline{q}(\overline{\mu}^{HL}(y))\} = \overline{q}(\overline{\mu}^{HL}(y))$. ☐

We now describe the relationship between μ, AVaR_μ, ψ_μ and μ^{HL} on one hand, and w_μ, solutions U_μ of (17) when $h(x) = \overline{q}_\mu(1/x)$ and the associated Azéma–Yor martingales $M^{U_\mu}(N)$ on the other hand. It turns out all these objects are intimately linked together in a rather elegant manner. Some of our descriptions below, in particular characterisation of AVaR in a), appear to be different from classical forms in the

literature. We note that we start with μ and define h but equivalently we could start with a non-decresing right-cotninuous h and use $h(x) = \overline{q}_\mu(1/x)$ to define μ.
Recall Definitions 2.2, 2.3 and the stopping time $\zeta_w(Y)$ from page 9.

Proposition 3.12. *Let μ be a probability measure on $\mathbb{R}$, $\int |s|\mu(\mathrm{d}s) < \infty$.*
a) $U_\mu(x) := \mathrm{AVaR}_\mu(1/x)$ solves (17) *with $h_\mu(x) = \overline{q}_\mu(1/x)$, $x \geq 1$, and $h_\mu(x)/x^2$ is integrable away from zero. In particular U_μ is given by* (18) *and $U_\mu(x) = U_\mu(x \wedge b_\mu)$ with $b_\mu = 1/\overline{\mu}(r_\mu)$. U_μ is concave and $V_\mu(y) = 1/\overline{\mu}^{HL}(y)$ is the inverse function of U_μ.*
b) Let $w_\mu(y) = h_\mu(V_\mu(y)) = q_\mu(\overline{\mu}^{HL}(y))$ be the function associated with μ by (9) *or equivalently* (11), *for $y \in (m_\mu, r_\mu)$, and extended via $w_\mu(y) = y$ for $y \geq r_\mu$. Then w_μ is a drawdown function, $r_w = r_\mu$ and w_μ is the right-continuous inverse of the barycentre function ψ_μ. Furthermore, w_μ is the hyperbolic derivative of V_μ as defined by Kertz and Rösler [21].*
c) Let N be as in (16) *with $N_0 = 1$ and $Y_t = M_t^{U_\mu}(N)$. Then $Y_t \geq U_\mu(N_t)$, $Y_\infty = Y_{\zeta_{w_\mu}(Y)} = \overline{q}_\mu\big(1/\overline{N}_\zeta\big)$ is distributed according to μ and $\overline{Y}_\infty = U_\mu(\overline{N}_\zeta) = \mathrm{AVaR}_\mu(1/\overline{N}_\zeta)$ is distributed according to μ^{HL}. Furthermore, the process (Y_t) is a uniformly integrable martingale which satisfies w_μ-DD constraint and $\zeta^Y_{w_\mu} = \inf\{t : N_t \notin (0, b_\mu)\} = \inf\{t : \psi_\mu(Y_t) \leq \overline{Y}_t\}$.*
The same properties hold true for any max-continuous uniformly integrable martingale Y satisfying the w_μ-DD constraint up to $\zeta = \zeta^Y_{w_\mu}$ and $Y_\zeta = w(\overline{Y}_\zeta)$ a.s.

Proof. We write $\overline{q} = \overline{q}_\mu$, $r = r_\mu$, $b = b_\mu = 1/\overline{\mu}(r_\mu) = V_\mu(r_\mu-)$.
a) From definition (20) we have $\mathrm{AVaR}_\mu(\lambda) = \int_0^1 \overline{q}(\lambda s)\mathrm{d}s$ which is exactly the formula (18). Note that in the case $\overline{\mu}(r) > 0$ we have $h(x) = h(b)$, $x \geq b$ with $b = 1/\overline{\mu}(r)$. U_μ is concave by Proposition 3.8. The rest follows since AVaR_μ is the tail quantile of μ^{HL}, see (21).
b) This follows by part iv) in Proposition 3.10 and the last statement follows from (11) and Theorem 4.3 in [21].
c) We have $Y_t \geq U(N_t)$ from concavity of U_μ. The rest follows easily from points a) and b) above together with Proposition 3.8, properties of $\overline{q}$, universal law of $\overline{N}_\zeta$ given in point c) of Proposition 3.3 and the definition of μ^{HL} in (21). □

An illustrative example (continued from page 12) We come back to the example with linear DD-constraint $w(y) = \gamma y$, $0 < \gamma < 1$, resulting from function $U(x) = \frac{1}{1-\gamma}x^{1-\gamma}$, $x \geq 1$. Using Proposition 3.12 we have $Y_\infty \sim \mu$ and $\overline{Y}_\infty \sim \mu^{HL}$ which we can now easily describe. We have $\overline{\mu}^{HL}(y) = 1/V(y) = ((1-\gamma)y)^{1/(\gamma-1)}$ for

$y \geq m_\mu = \mathrm{AVaR}_\mu(1) = U(1) = \frac{1}{1-\gamma}$. In consequence, the random variable $\overline{Y}_\infty$ is distributed according to a *Pareto distribution*, with shape parameter $a = m_\mu = \frac{1}{1-\gamma}$ and location parameter $m = m_\mu$. The mean of $\overline{Y}_\infty$ is $am/(a-1) = (\gamma(1-\gamma))^{-1}$. Since $Y_\infty = w(\overline{Y}_\infty) = \gamma\overline{Y}_\infty$ we see that Y_∞ is still distributed according to a Pareto distribution, with same shape parameter, and location parameter $m_1 = m\gamma = \frac{\gamma}{1-\gamma}$. Naturally, we could also describe μ using $\overline{q}_\mu(\lambda) = h(1/\lambda) = \frac{\gamma}{1-\gamma}\lambda^{\gamma-1}$ which, taking inverses, gives $\overline{\mu}(x) = \left(\frac{\gamma}{1-\gamma}\frac{1}{x}\right)^{\frac{1}{1-\gamma}}$ as required.
As a consequence we see that if (Y_t) is a max-continuous martingale which satisfies a linear drawdown constraint $Y_t > \gamma\overline{Y}_t$ until $\zeta = \zeta_w(Y) < \infty$ a.s. then necessarily $Y_\zeta = \gamma\overline{Y}_\zeta$ has a Pareto distribution.

3.4 The Skorohod embedding problem revisited

The Skorokhod embedding problem can be phrased as follows: given a probability measure μ on $\mathbb{R}$ find a stopping time T such that X_T has the law μ, $X_T \sim \mu$. One further requires T to be *small* in some sense, typically saying that T is minimal. We refer the reader to Obłój [24] for further details and the history of the problem.
In [2] Azéma and Yor introduced the family of martingales described in Definition 1.1 and used them to give an elegant solution to the Skorokhod embedding problem for X a continuous local martingale (and μ centered). Namely, they proved that

$$T_\psi(X) = \inf\{t \geq 0 : \psi_\mu(X_t) \leq \overline{X}_t\}, \tag{27}$$

where ψ_μ in the barycentre function (19), solves the embedding problem.

We propose to rediscover their solution in a natural way using our methods, based on the observation that the process X satisfies the w_μ-DD constraint up to $T_\psi(X)$. If we show the equality $X_\zeta = w_\mu(\overline{X}_\zeta)$ at time $\zeta = T_\psi(X)$, Proposition 3.12 gives us the result.

Theorem 3.13 (Azéma and Yor [2])**.** *Let (X_t) be a continuous local martingale, $X_0 = 0$, $\langle X\rangle_\infty = \infty$ a.s. and μ a centered probability measure on $\mathbb{R}$: $\int |x|\mu(\mathrm{d}x) < \infty$, $\int x\mu(\mathrm{d}x) = 0$. Then $T_\psi < \infty$ a.s., $(X_{t\wedge T_\psi})$ is a uniformly integrable martingale and $X_{T_\psi} \sim \mu, \quad \overline{X}_{T_\psi} \sim \mu^{HL}$, where T_ψ is defined via* (27)*.*
With notation of Proposition 3.12, define $N_t = M^{V_\mu}_{t\wedge\tau^{r_\mu}(X)}(X)$. Then

$$T_\psi = \inf\{t \geq 0 : X_t \leq w_\mu(\overline{X}_t)\} = \inf\{t \geq 0 : N_t \leq 0\} \wedge \tau^{b_\mu}(N) \tag{28}$$

and $X_{t\wedge T_\psi} = M^{U_\mu}_{t\wedge T_\psi}(N)$.

Proof. Let $\tau = \tau^{r_\mu}(X)$. $(N_t : t < \tau)$ is a continuous local martingale with $N_0 = 1$ since $U_\mu(1) = 0$ thanks to μ being centred. If $b_\mu < \infty$ then $r_\mu < \infty$ and $(N_t : t \le \tau)$ is a local martingale stopped at $\inf\{t : N_t = b_\mu\} = \tau < \infty$ a.s. Suppose $b_\mu = \infty$. Then $\overline{N}_{\tau-} = \lim_{x \to r_\mu} V(x) = \infty$. This readily implies that $\langle N \rangle_{\tau-} = \infty$ a.s. and in particular $\tau_0(N) < \tau$ a.s. (cf. Proposition V.1.8 in Revuz and Yor [27]). Note that this applies both for the case r_μ finite and infinite. We conclude that $N_{t \wedge \tau_0(N)}$ is as in (16) and furthermore that $\tau_0(N) \wedge \tau^{b_\mu}(N) < \infty$ a.s. Theorem now follows from part c) in Proposition 3.12. ☐

Remark 3.14. Note that in general only max-continuity of (X_t) would not be enough. More precisely we need to have $X_{T_\psi} = w_\mu(\overline{X}_{T_\psi})$ a.s. or equivalently that the process N_t crosses zero continuously. Also, unlike in Propostion 3.12, we need to assume that μ is centred to ensure that $N_0 = M_0^{V_\mu}(X) = 1$. Finally note that we do *not* necessarily have that $\psi_\mu(X_{T_\psi}) = \overline{X}_{T_\psi}$.

4 On optimal properties of AY martingales related to HL transform and its inverse

In this final section we investigate the optimal properties of Azéma–Yor processes and of the Hardy-Littlewood transform $\mu \to \mu^{HL}$ and its (generalised) inverse operator Δ. We use two orderings of probability measures. We say that μ dominates ν in the *stochastic order* (or stochastically) if $\overline{\mu}(y) \ge \overline{\nu}(y)$, $y \in \mathbb{R}$. We say that μ dominates ν in the *increasing convex order* if $\int g(y)\mu(\mathrm{d}y) \ge \int g(y)\nu(\mathrm{d}y)$ for any increasing convex function g whenever the expectations are defined. Observe that the latter order is equivalent to $C_\mu(K) \ge C_\nu(K)$, $K \in \mathbb{R}$ (cf. Shaked and Shanthikumar [29, Thm. 3.A.1]).

From (23) we deduce instantly that if μ, ρ are probability measures on $\mathbb{R}$ which admit first moments, then

$$\mathrm{AVaR}_\mu(\lambda) \le \mathrm{AVaR}_\rho(\lambda),\ \lambda \in (0,1) \Leftrightarrow C_\mu(K) \le C_\rho(K),\ K \in \mathbb{R}\,. \qquad (29)$$

Using $\mathrm{AVaR}_\mu(\lambda) = \overline{q}_{\mu^{HL}}(\lambda)$ we then obtain

$$\begin{aligned} \overline{\mu}^{HL}(y) \le \overline{\rho}^{HL}(y),\ y \in \mathbb{R} \Leftrightarrow\ & \overline{q}_{\mu^{HL}}(\lambda) \le \overline{q}_{\rho^{HL}}(\lambda),\ \lambda \in [0,1] \\ \Leftrightarrow\ & C_\mu(K) \le C_\rho(K),\ K \in \mathbb{R}, \end{aligned} \qquad (30)$$

so that ρ^{HL} dominates μ^{HL} stochastically if and only if ρ dominates μ in the convex order.

4.1 Optimality of Azéma–Yor stopping time and Hardy-Littlewood transformation

The Azéma–Yor stopping time has a remarkable property that the distribution of maximum of the martingale stopped at this time is known, as a Hardy-Littlewood maximum r.v. associated with μ (cf. Proposition 3.12). The importance of this result comes from the result of Blackwell and Dubins [5] (see also the concise version of Gilat and Meljison [14]) showing that:

Theorem 4.1 (Blackwell-Dubins(63))**.** *Let (P_t) be a uniformly integrable martingale and μ the distribution of P_∞. Then,*

$$\mathbb{P}(\overline{P}_\infty \geq y) \leq \mu^{HL}([y,\infty)), \quad y \in \mathbb{R}\,. \tag{31}$$

In other words, any Hardy-Littlewood maximal r.v. associated with P_∞ dominates stochastically $\overline{P}_\infty$.

In fact μ^{HL} is sometimes defined as the smallest measure which satisfies (31). One then proves the representation (21).

Azéma–Yor martingales, stopped appropriately, are examples of martingales which achieve equality in (31). We can reformulate this result in terms of optimality of the Azéma–Yor stopping time, which has been studied by several authors ([1], [14], Kertz and Rösler [19] and Hobson [17]).

Corollary 4.2 (Azéma–Yor [1])**.** *In the setup and notation of Theorem 3.13, the distribution of $\overline{X}_{T_\psi}$ is μ^{HL}. In consequence, $\overline{X}_{T_\psi}$ dominates stochastically the maximum of any other uniformly integrable martingale with terminal distribution μ.*

The result is a corollary of Theorem 4.1 and the fact that the maximum $\overline{X}_{T_\psi}$ is a Hardy-Littlewood maximal r.v. associated with μ, which follows from Proposition 3.12. We present however an independent proof based on arguments in Brown, Hobson and Rogers [6].

Proof. Let (P_t) be a uniformly martingale with terminal distribution μ and chose $y \in (0, r_\mu)$. Observe that for any $K < y$ the following inequality holds a.s.

$$\mathbf{1}_{\overline{P}_\infty \geq y} \leq \frac{(P_\infty - K)^+}{y-K} + \frac{y - P_\infty}{y-K}\mathbf{1}_{\overline{P}_\infty \geq y}\,. \tag{32}$$

If P is max-continuous then the last term on the RHS is simply $-M^F_\infty(P)$ for $F(z) = \frac{(z-y)^+}{y-K}$ and has zero expectation. In general, we can substitute the last term with a greater term $\frac{P_{\tau^{y}(P)}-P_\infty}{y-K}\mathbf{1}_{\overline{P}_\infty\geq y}$ which has zero expectation. Hence, taking expectations in (32) we find

$$\mathbb{P}(\overline{P}_\infty \geq y) \leq \frac{1}{y-K}\int_K^\infty (x-K)\mu(\mathrm{d}x).$$

Taking infimum in $K<y$ and using (24) we conclude that $\mathbb{P}(\overline{P}_\infty \geq y) \leq \overline{\mu}^{HL}(y)$. To end the proof it suffices to observe from the definition of T_ψ that $X_{T_\psi} = w_\mu(\overline{X}_{T_\psi})$ and hence, with $P_t = X_{t\wedge T_\psi}$, we have a.s. equality in (32) for $K = w_\mu(y)$ and in consequence $\mathbb{P}(\overline{X}_{T_\psi} \geq y) = \overline{\mu}^{HL}(y)$. □

We identified so far μ^{HL} as the maximal, relative to stochastic order, possible distribution of supremum of a uniformly integrable martingale with a fixed terminal law μ. We look now at the dual problem: we look for a maximal terminal distribution of a uniformly integrable martingale with a fixed law of supremum. We saw in (30) that stochastic order of HL transforms translates into increasing convex ordering of the underlying distributions, and we expect the solution to the dual problem to be optimal relative to increasing convex order.

Let us fix a distribution ν and look at measures ρ, $\int |x|\rho(\mathrm{d}x)<\infty$, such that ρ^{HL} stochastically dominates ν: $\overline{\nu}(x) \leq \overline{\rho}^{HL}(x)$, $x\in\mathbb{R}$. We note $\mathcal{S}_\nu$ the set of such measures. Passing to the inverses, we can express the condition on $\rho\in\mathcal{S}_\nu$ in terms of tail quantiles:

$$\rho\in\mathcal{S}_\nu \Leftrightarrow \overline{q}_\nu(\lambda) \leq \overline{q}_{\rho^{HL}}(\lambda) = \mathrm{AVaR}_\rho(\lambda) = \frac{1}{\lambda}\int_0^\lambda \overline{q}_\rho(u)\mathrm{d}u, \quad \lambda\in[0,1], \tag{33}$$

and where we used (20)-(21). Note that for existence of $\rho\in\mathcal{S}_\nu$ it is necessary that

$$\lambda\overline{q}_\nu(\lambda) \xrightarrow[\lambda\to 0]{} 0 \text{ which is equivalent to } x\overline{\nu}(x) \xrightarrow[x\to\infty]{} 0\,. \tag{34}$$

We have the following theorem which synthesis several results from Kertz and Rösler [20, 21] as well as adds new interpretation of Δ operator as the inverse of $\mu\to\mu^{HL}$ and gives a construction of ν_Δ. The proof is greatly simplified using tail quantiles.

Theorem 4.3. *Let ν be a probability measure on $\mathbb{R}$. The set $\mathcal{S}_\nu$ is non-empty if and only if ν satisfies* (34)*. Under* (34)*, $\mathcal{S}_\nu$ admits a minimal element ν_Δ relative to the increasing convex order, which is characterised by*

$$\lambda\overline{q}_{\nu_\Delta^{HL}}(\lambda) = \int_0^\lambda \overline{q}_{\nu_\Delta}(u)\mathrm{d}u \text{ is the concave envelope of } \lambda\overline{q}_\nu(\lambda).$$

Furthermore, if $\nu=\mu^{HL}$ for an integrable probability measure μ, then $\nu_\Delta=\mu$.

Proof. Assume (34) and let $G(\lambda)$ be the concave envelope (i.e. the smallest concave majorant) of $\lambda\overline{q}_\nu(\lambda)$. If there exists a measure ν_Δ such that $G(\lambda) = \int_0^\lambda \overline{q}_{\nu_\Delta}(u)\mathrm{d}u$ then clearly $\nu_\Delta \in \mathcal{S}_\nu$ by definition in (33). Furthermore, since $\int_0^\lambda \overline{q}_\rho(u)\mathrm{d}u$ is a concave function, we have that

$$\int_0^\lambda \overline{q}_{\nu_\Delta}(u)\mathrm{d}u \le \int_0^\lambda \overline{q}_\rho(u)\mathrm{d}u, \quad \lambda \in [0,1],\ \forall\, \rho \in \mathcal{S}_\nu\ . \tag{35}$$

This in turn, using (29), is equivalent to ν_Δ being the infimum of $\rho \in \mathcal{S}_\nu$ relative to increasing convex ordering of measures and thus being a solution to our dual problem.

It remains to argue that ν_Δ exists. Recall that $-\infty \le l_\nu < r_\nu \le \infty$ are respectively the lower and the upper bounds of the support of ν. Let $\tilde{G}(x)$ be the (formal) Fenchel transform of $\lambda\overline{q}_\nu(\lambda)$:

$$\tilde{G}(x) = \sup_{\lambda\in(0,1)} \left(\lambda\overline{q}_\nu(\lambda) - \lambda x\right), \quad x \in [l_\nu, r_\nu]. \tag{36}$$

Observe that $\tilde{G}(x) \ge 0$ thanks to assumption (34) and by definition $\tilde{G}(x)$ is convex, decreasing and $\tilde{G}'(x) \in [-1,0]$. This implies that there exists a probability measure ν_Δ such that $\tilde{G}(x) = \int (y-x)^+\nu_\Delta(\mathrm{d}y) = C_{\nu_\Delta}(y)$. In fact we simply have $\overline{\nu}_\Delta(x) := -\tilde{G}'(x-)$. Since G was the concave envelope of $\lambda\overline{q}_\nu(\lambda)$ we can recover it as the dual Fenchel transform of $\tilde{G}$ and, using 22, we have

$$G(\lambda) = \inf_{x\in[l_\nu,r_\nu]} \left(\tilde{G}(x) + x\lambda\right) = \int_0^\lambda \overline{q}_{\nu_\Delta}(u)\mathrm{d}u, \quad \lambda \in [0,1], \tag{37}$$

as required. Note that we could also take $x \in \mathbb{R}$ above since the infimum is always attained for $x \in [l_\mu, r_\mu]$. Finally, if $\nu = \mu^{HL}$ then $\lambda\overline{q}_\nu(\lambda)$ is concave and equal to $\int_0^\lambda \overline{q}_\mu(u)\mathrm{d}u$ and hence $\nu_\Delta = \mu$. □

We stress that in the proof we obtain in fact a rather explicit representation which can be used to construct ν_Δ. Namely we have $\overline{\nu}_\Delta(x) = -\tilde{G}'(x)$ with $\tilde{G}$ defined in (36).

We find it is useful to rephrase conclusions of Theorem 4.3 in martingale terms. Furthermore, we also show that any max–continuous martingale (P_t), $P_\infty \sim \mu$, which achieves the upper bound on the law of supremum, $\overline{P}_\infty \sim \mu^{HL}$, is of the form $P_t = X_{t\wedge T_\psi} = M_t^{U_\mu}(N)$ for some N as in (16).

Theorem 4.4. *Let ν be a distribution satisfying* (34)*. For any uniformly integrable martingale (P_t) such that $\overline{P}_\infty$ dominates ν for the stochastic order, P_∞ dominates $Y_\infty \sim \nu_\Delta$ for the increasing convex order, where $Y_t = M^{U_{\nu_\Delta}}(N)_t$ is the Azéma–Yor*

martingale associated with ν_Δ by Proposition 3.12.
Furthermore, if $\nu = \mu^{HL}$ and (P_t) as above is max–continuous with $P_\infty \sim \mu$ then P is the Azéma–Yor martingale $M^{U_\mu}(N)$ for some (N_t) as in (16).

Proof. Let $\mu \sim P_\infty$. By Corollary 4.2 the distribution of $\overline{P}_\infty$ is dominated stochastically by μ^{HL}. Hence μ^{HL} dominates stochastically ν and $\mu \in \mathcal{S}_\nu$. The first part of Theorem is then a corollary of Theorem 4.3.
It remains to argue the last statement of the Theorem. Since $P_\infty \sim \mu$ and the distribution of $\overline{P}_\infty$ dominates stochastically μ^{HL} it follows from Theorem 4.1 that $\overline{P}_\infty \sim \mu^{HL}$. We deduce from the proof of Corollary 4.2 that we have an a.s. equality in (32) for any $y > 0$ and $K = w_\mu(y)$ and hence

$$\{P_\infty \geq w_\mu(y)\} \supseteq \{\overline{P}_\infty > y\} \supseteq \{P_\infty > w_\mu(y)\}.$$

It follows that $P_\infty = w_\mu(\overline{P}_\infty)$. Further, from uniform integrability of (P_t),

$$\mathbb{E}\, P_\infty = \mathbb{E}\, P_{\zeta_{w_\mu}(P)} \leq \mathbb{E}\, w_\mu(\overline{P}_{\zeta_{w_\mu}(P)}) \leq \mathbb{E}\, w_\mu(\overline{P}_\infty).$$

In consequence $P_t = P_{t \wedge \zeta_{w_\mu}(P)}$ and the statement follows with $N_t = M^{V_\mu}_{t \wedge \tau^{r_\mu}(P)}(P)$, see Theorem 3.13 and Remark 3.14. □

4.2 Floor Constraint and concave order

In this final section we study how Theorem 4.4 can be used to solve different optimization problems motivated by portfolio insurance. Our insight comes in particular from constrained portfolio optimization problems discussed by El Karoui and Meziou [9]. In such problems it is more natural to consider conditions of pathwise domination. We note that it is quite remarkable that these turn out to be equivalent to, potentially weaker, conditions of ordering of distributions. Finally we remark that in financial context we often use the increasing concave order between two variables (rather then convex). This is simply a consequence of the fact that utility functions are typically concave.

Consider g an increasing function on $\mathbb{R}_+$ whose increasing concave envelope U is finite and such that $\lim_{x\to\infty} U(x)/x = 0$. Let N_t be as in (16) with $N_0 = 1$. In the financial context, the floor underlying is modelled by $F_t = g(N_t)$. Financial positions can be modelled with uniformly integrable martingales and we are interested in choosing the optimal one, among all which dominate F_t for all $t \geq 0$.

Proposition 4.5. *Let $F_t = g(N_t)$ be the floor process and $\mathcal{M}_F^s$ denote the set of uniformly integrable martingales (P_t), with $P_0 = U(N_0)$ and $P_t \geq F_t$, $t \geq 0$. Then the Azéma–Yor martingale $M_t^U(N)$ belongs to $\mathcal{M}_F^s$ and is optimal for the concave order of the terminal values, i.e. for any increasing concave function G and $P \in \mathcal{M}_F^s$, $\mathbb{E}\, G(M_\infty^U(N)) \geq \mathbb{E}\, G(P_\infty)$.*
In fact the same result holds in the larger set $\mathcal{M}_F^w$ of uniformly integrable martingales (P_t) with $P_0 = U(N_0)$ and $\mathbb{P}(\overline{P}_\infty \geq x) \geq \mathbb{P}(\overline{F}_\infty \geq x)$, for all $x \in \mathbb{R}$.

Remark 4.6. The process $(U(N_t))$ is the Snell envelop of $(g(N_t))$, that is the smallest supermartingale dominating $g(N)$, as shown in Galtchouk and Mirochnitchenko [13] using that U is an affine function on $\{x : U(x) > g(x)\}$.

Proof. From concavity of U we have $M_t^U(N) \geq U(N_t) \geq F_t$ which shows that $M_t^U(N)$ belongs to $\mathcal{M}_F^s$. Naturally, it suffices to prove the statement for the larger set $\mathcal{M}_F^w$. Observe that $\lambda U(\frac{1}{\lambda})$ is the concave envelope of $\lambda g(\frac{1}{\lambda})$ on $\lambda \in (0,1)$. If we define a probability measure ν by $\nu \sim g(\overline{N}_\infty)$ then $\overline{\nu}(x) = \frac{1}{g^{-1}(x)}$, $x \in [g(1), g(\infty))$. In consequence $\lambda U(\frac{1}{\lambda})$ is the concave envelope of $\lambda \overline{q}_\nu(\lambda)$. Theorem 4.3 and properties of AVaR in (20)-(21) imply that

$$U(\frac{1}{\lambda}) = \overline{q}_{\nu_\Delta^{HL}}(\lambda) = \mathrm{AVaR}_{\nu_\Delta}(\lambda).$$

Using Proposition 3.12 we have $U = U_{\nu_\Delta}$ and h defined via (17) is given by $h(x) = \overline{q}_{\nu_\Delta}(1/x)$. Finally note that, since $\mathbb{E}\, P_\infty = F_0 = \mathbb{E}\, M_\infty^U(N)$, increasing convex order, increasing concave order and convex order on P_∞ and $M_\infty^U(N)$ are all equivalent (cf. Shaked and Shanthikumar [29, Thms 3.A.15 and 3.A.16]). The rest now follows from Theorem 4.4. $\square$

If we want show the above statement only for the smaller set $\mathcal{M}_F^s$ then we can give a direct proof as in [10]. Any martingale X which dominates F_t dominates also the smallest supermartingale Z_t which dominates F_t and it is easy to see that $Z_t = U(N_t)$. From Propositon 3.6 we know that $M_t = \mathbb{E}[h(\overline{N}_\infty)|\mathcal{F}_t]$ is a uniformly integrable martingale and we also have $\overline{M}_t = U(\overline{N}_t) = \overline{Z}_t$ (cf. Proposition 1.2). We assume G is twice continuously differentiable, the general case following via a limiting argument. Since h is concave, $G(y) - G(x) \leq G'(x)(y-x)$ for all $x, y \geq 0$. In consequence

$$\begin{aligned}\mathbb{E}\left[G(P_\infty) - G(M_\infty)\right] &\leq \mathbb{E}\left[G'(M_\infty)(P_\infty - M_\infty)\right] = \mathbb{E}\left[G'(h(\overline{N}_\infty))(P_\infty - M_\infty)\right] \\ &\leq \mathbb{E}\int_0^\infty G'(h(\overline{N}_t))d(P_t - M_t) + \mathbb{E}\int_0^\infty (P_t - M_t)G''(h(\overline{N}_t))d(h(\overline{N}_t)).\end{aligned}$$

The first integral is a difference of two uniformly integrable martingales (note that $\overline{N}_0 > 0$) and its expectation is zero. For the second integral, recall that h is increasing and the support of $d(h(\overline{N}_t))$ is contained in the support of $d\overline{N}_t$ on which $M_t = \overline{M}_t = \overline{Z}_t = Z_t \leq P_t$. As G is concave we see that the integral is a.s. negative which yields the desired inequality.

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