A BRAIDED SIMPLICIAL GROUP

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ABSTRACT. By studying braid group actions on Milnor's construction of the 1-sphere, we show that the general higher homotopy group of the 3-sphere is the fixed set of the pure braid group action on certain combinatorially described group. We also give certain representation of higher differentials in the Adams spectral sequence for $\pi_*(S^2)$.

1. INTRODUCTION

In this article, we study the homotopy groups by considering the braid group actions on simplicial groups. The point of view here is to establish a relation between the fixed set of braid group actions and the homotopy groups of the 3-sphere. We first recall a combinatorial description of the homotopy groups of the 3-sphere in [11].

Let $F(x_1, \cdots, x_n)$ be the free group generated by the letters $x_1, \cdots, x_n$. Let $w(x_1, \cdots, x_n) = x_{i_1}^{\epsilon_1} \cdots x_{i_t}^{\epsilon_t}$ be a word. Given $a_1, \cdots, a_n \in F(x_1, \cdots, x_n)$, we write $w(a_1, \cdots, a_n) \in F(x_1, \cdots, x_n)$ for $a_{i_1}^{\epsilon_1} \cdots a_{i_t}^{\epsilon_t}$. The *n-th W-group* $G(n)$ is the quotient group of $F(x_1, \cdots, x_n)$ modulo the following relations:

($\mathcal{R}_1$) the product $x_1 \cdots x_n$;
($\mathcal{R}_2$) the words $w(x_1, \cdots, x_n)$ that satisfy: $w(x_1, \cdots, x_{i-1}, 1, x_{i+1}, \cdots, x_n) = 1$ for $1 \leq i \leq n$.

Relations $\mathcal{R}_2$ consist of all of words that will be trivial if one of the generators is replaced by the identity element 1. The smallest normal subgroup of $F(x_1, \cdots, x_n)$ which contains relations $\mathcal{R}_1$ and $\mathcal{R}_2$ was determined as a subgroup of $F(x_1, \cdots, x_n)$ generated by certain systematic and uniform iterated commutators [11].

Theorem 1.1. [11] *For $n \geq 3$, the homotopy group $\pi_n(S^3)$ is isomorphic to the center of $G(n)$.*

A natural question arisen from Theorem 1.1 is how to give a group theoretical approach to the homotopy groups, that is how to understand the center of the group $G(n)$. There is a canonical braid group action on $G(n)$ which is induced by the

canonical braid group action on the free group $F(x_1, \cdots, x_n)$, namely

$$\sigma_i(x_j) = \begin{cases} x_{i+1} & if \quad j = i \\ \\ x_{i+1}^{-1} x_i x_{i+1} & if \quad j = i + 1 \\ \\ x_j & otherwise \end{cases}$$

for $1 \leq i \leq n - 1$. These actions gives a canonical homomorphism from the braid group B_n into the automorphism group of $G(n)$. Since Quillen's plus construction of the classifying space for the stable braid group is (up to homotopy type) the double loop space of the 3-sphere [2], these braid group actions do not seem occasional event. Fred Cohen therefore conjectured that the center of $G(n)$ is the fixed set of the braid group action on $G(n)$. We answer Cohen's question as follows. Let K_n be the pure braid group, that is, K_n is the normal divisor of B_n generated by σ_1^2 (See [6]). Equivalently K_n is the kernel of the canonical homomorphism from B_n to the symmetric group Σ_n. In geometry, the group K_n is the fundamental group of the configuration space $F(\mathbb{R}^2, n)$, where

$$F(M, n) = \{(x_1, \cdots, x_n) \in M^n | x_i \neq x_j \text{ for } i \neq j\}$$

for any manifold M (See [2]). Let $Z(G(n))$ be the center of $G(n)$.

Theorem 1.2. *For $n \geq 4$, then*

1) *the center of $G(n)$ is the fixed set of the pure braid group action on $G(n)$ and so is $\pi_n(S^3)$;*
2) *the fixed set of the braid group action on $G(n)$ is the subgroup*

$$\{x \in Z(G(n)) | 2x = 0\}.$$

We should point out that the determination of the fixed set of K_n-action on $G(n)$ by (combinatorial) group theoretic means seems beyond the reach of current techniques. On the other hand, braid group actions have largely studied in several areas such as group theory and low dimensional topology. Various problems arising from physics are related to braid group actions as well. Theorem 1.2 suggests that the homotopy groups play certain role for braid group actions. In the range in which $\pi_*(S^3)$ is known ($* \leq 55$, see [5, 9]), by homotopy theoretic means, we gain insight into these difficult group theoretic questions.

The article is organized as follows. In section 2, we study the braid group action on the Milnor's construction on the simplicial 1-sphere. A relation between the simplicial structure and the braid group action is given in Proposition 2.1. This relation is based on the direct calculation. Roughly the braid group action interchanges the

faces together with conjugates. It is possible to have a more general theory to study group actions on simplicial groups, particularly simplicial group models for iterated loop spaces. But we only intend to investigate the most important example $F(S^1)$ in this article. After establishing the systematic relations between the braid group actions and the simplicial structure, braided simplicial groups are introduced in this section. Then we show that loop simplicial group and the Moore-Postnikov system of a braided simplicial group are braided. Theorem 2.9 and Proposition 2.10 give a relation between the fixed set of the braid group action and the homotopy groups for a general braided simplicial group. Theorem 1.2 follows from Lemma 2.8 and Theorem 2.13. In section 3, we study a braided representation of the Milnor's construction of the simplicial 1-sphere into a simplicial algebra. Theorem 3.8 gives certain representation of higher differentials in the Adams spectral sequence for $\pi_*(S^2)$.

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2. Braid Group Actions on $F(S^1)$

2.1. **Braided simplicial groups.** Let $F(S^1)$ be Milnor's construction of the simplicial 1-sphere S^1. Then $F(S^1)_{n+1} = F(y_0, \cdots, y_n)$ is a free group generated by letters $y_0, \cdots, y_n$ with the following simplicial structure

$$d_j y_k = \begin{cases} y_{k-1} & \text{for} & j \leq k, \\ 1 & \text{for} & j = k+1, \\ y_k & \text{for} & j > k+1, \end{cases}$$

and

$$s_j y_k = \begin{cases} y_{k+1} & \text{for} & j \leq k, \\ y_k y_{k+1} & \text{for} & j = k+1, \\ y_k & \text{for} & j > k+1, \end{cases}$$

for $0 \leq j \leq n+1$, where $y_{-1} = (y_0 \cdots y_{n-1})^{-1}$ in $F(S^1)_n$ (See [11, Lemma 4.1]). Let the braid group B_{n+1} act on $F(S^1)_{n+1} = F(y_0, \cdots, y_n)$ in the usual way, that is,

$$\sigma_k(y_j) = \begin{cases} y_{k+1} & \text{if} & j = k \\ y_{k+1}^{-1} y_k y_{k+1} & \text{if} & j = k+1 \\ y_j & & \text{otherwise} \end{cases}$$

for $0 \leq k \leq n-1$. Let σ_{-1} be an automorphism of $F(y_0, \cdots, y_n)$ defined by

$$\sigma_{-1}(y_0) = y_0^{-1} y_{-1} y_0 = y_0^{-1} y_n^{-1} \cdots y_1^{-1} \quad \sigma(y_j) = y_j \quad \text{for} \quad j > 0.$$

The subgroup of the automorphism group of $F(y_0, \cdots, y_n)$ generated by σ_j for $-1 \leq j \leq n-1$ is the braid group B_{n+2}. By direct calculation, we have

Proposition 2.1. *The following identities hold for the braid groups action on $F(S^1)$:*

$$(1) \qquad d_j\sigma_k = \begin{cases} \sigma_{k-1}d_j & j \le k \\[2ex] d_{k+2} & j = k+1 \\[2ex] d_{k+1} & j = k+2 \\[2ex] \sigma_k d_j & j > k+2 \end{cases}$$

$$(2) \qquad s_j\sigma_k = \begin{cases} \sigma_{k+1}s_j & j \le k \\[2ex] \sigma_{k+1} \circ \sigma_k \circ s_{k+2} & j = k+1 \\[2ex] \sigma_k \circ \sigma_{k+1} \circ s_{k+1} & j = k+2 \\[2ex] \sigma_k s_j & j > k+2 \end{cases}$$

By this proposition, we give the following definition.

Definition 2.2. A simplicial group G is called *braided* if there is a braid group B_{n+1}-action on G_n for each n such that the identities 1 and 2 are satisfied, where B_{n+1} is considered as the braid group generated by $\sigma_{-1}, \sigma_0, \cdots, \sigma_{n-2}$.

Let G be a simplicial group and let NG be the Moore chain complex of G, that is,

$$NG_n = \{x \in G_n | d_j x = 1 \quad \text{for} \quad j > 0\}$$

. Let $\mathcal{Z}(G)$ and $\mathcal{B}(G)$ be the sets of cycles and boundaries of G, respectively, that is,

$$\mathcal{Z}_n(G) = \{x \in G_n | d_j x = 1 \quad \text{for} \quad \text{all} \quad j\},$$

$$\mathcal{B}_n(G) = \{d_0 x | x \in N_{n+1}(G)\}.$$

By Moore's classical theorem [7], $\pi_n(G) = \mathcal{Z}_n(G)/\mathcal{B}_n(G)$. Let G be a braided simplicial group. A subgroup H of G is called a *braided subgroup* if $\sigma(H) \subseteq H$ for any $\sigma \in B_{n+1}$, that is, H is invariant under B_{n+1}-action.

Proposition 2.3. *Let G be a braided simplicial group. Then the subgroups $\mathcal{Z}(G)_n$ and $\mathcal{B}(G)_n$ of G_n are braided.*

Proof. By Proposition 2.1, $\mathcal{Z}(G)_n$ is a braided subgroup. Now let $x = d_0 y \in \mathcal{B}(G)_n$, where $y \in NG_{n+1}$. By Proposition 2.1, we have

$$\sigma_k x = \sigma_k d_0 y = d_0 \sigma_{k+1} y$$

for each $k \geq -1$ and

$$d_j \sigma_{k+1} y = 1$$

for each $j > 0$ and $k \geq -1$. Thus $\sigma_k x \in \mathcal{B}(G)_n$ for each $k \geq -1$ and so $\mathcal{B}(G)_n$ is a braided subgroup, which is the assertion. $\square$

Note: $N_n(G)$ is invariant under the subgroup of B_{n+1} generated by σ_k with $k \geq 0$. But it is not invariant under σ_{-1}.

Since there is a relation

$$d_0 \sigma_{-1} = d_1, \quad d_1 \sigma_{-1} = d_0, \quad \text{and} \quad d_j \sigma_{-1} = \sigma_{-1} d_j \quad \text{for} \quad j > 1,$$

we have

Proposition 2.4. *Let G be a braided simplicial group. Then*

$$\mathcal{Z}(G)_n = NG_n \cap \sigma_{-1}(NG_n).$$

for each n.

Now we study the braided group actions on the Postnikov systems of a braided simplicial group G. Let $I = (i_1, \cdots, i_k)$ be a sequence of non-negative integers and let d_I denote the composite of face homomorphisms

$$d_I = d_{i_1} \cdots d_{i_k}.$$

Given a simplicial group G, the simplicial sub groups $R_n G$ and $\bar{R}_n(G)$ are defined as follows:

$$R_n G_q = \{x \in G_q | d_I(x) = 1 \quad \text{for any} \quad I = (i_1, \cdots, i_{q-n})\},$$
$$\bar{R}_n G_q = \{x \in G_q | d_I(x) \in \mathcal{B}(G)_n \quad \text{for any} \quad I = (i_1, \cdots, i_{q-n})\}.$$

Let $P_n G = G / R_n G$ and $\bar{P}_n G = G / \bar{R}_n G$. Then $\{P_n G\}$ is the Postnikov system of G (See [4, 7]). The quotient homomorphism $P_n G \to \bar{P}_n G$ is a homotopy equivalence (See [11]). The tower

$$\cdots \to P_n G \to \bar{P}_n G \to P_{n-1} G \to \cdots$$

is called a *modified Postnikov system* of G. One of the important properties of the modified Postnikov system is that the short exact sequence of simplicial groups

$$K(\pi_n(G), n) \to \bar{P}_n G \to P_{n-1} G$$

is a central extension [11, Theorem 2.12].

By Propositions 2.1 and 2.3, we have

Theorem 2.5. *Let G be a braided simplicial group. Then, for each n, the simplicial quotient groups $P_n G$ and $\bar{P}_n G$ are braided. Thus the modified Postnikov system of G are braided. In particular, there is a braided central extension*

$$K(\pi_n(G), n) \to \bar{P}_n(G) \to P_{n-1}(G).$$

Let G be a braided simplicial group. Then $R_0 G$ is a braided simplicial subgroup of G by the theorem above. Recall that the loop simplicial group ΩG of G is defined by $\Omega G_n = \text{Ker}(d_0) \cap R_0 G_{n+1}$ with $d_j(\Omega G) = d_{j+1}(G)$ and $s_j(\Omega G) = s_{j+1}(G)$ (See [4]). By Proposition 2.1, $\text{Ker}(d_0) \cap R_0 G_{n+1}$ is invariant under the action of σ_j for $0 \leq j \leq n-1$. Let B'_{n+1} be the subgroup of B_{n+2} generated by $\sigma_0, \cdots, \sigma_{n-1}$. Then $B'_{n+1} \cong B_{n+1}$ under the canonical isomorphism which sends σ_j to σ_{j-1} for $0 \leq j \leq n-1$. Thus we obtained the following theorem.

Theorem 2.6. *Let G be a braided simplicial group. Then the loop simplicial group ΩG is braided. Thus any iterated loop simplicial groups of G are braided.*

Corollary 2.7. *The loops and the modified Postnikov system of $F(S^1)$ are braided.*

2.2. **Fixed Sets of Braided Actions.** Let G be a simplicial group and let $x, y \in G_n$ be two elements. We call that x is *homotopic* to y, which is denoted by $x \simeq y$, if $xy^{-1} \in \mathcal{B}(G)_n$.

Lemma 2.8. *Let G be a braided simplicial group and let $x \in \mathcal{Z}(G)_n$ be a cycle with $n \geq 1$. Then*

$$\sigma_k(x) \simeq x^{-1}$$

for each $k \geq -1$.

Proof. For each $k \geq -1$, consider the element $\sigma_{k+1} s_{k+1} x$. By the identities 1 and 2, we have

$$d_j \sigma_{k+1} s_{k+1} x = \begin{cases} \sigma_k d_j s_{k+1} x = 1 & \text{for} \quad j < k+1 \\[2ex] \sigma_k x & \text{for} \quad j = k+1 \\[2ex] d_{k+3} s_{k+1} x = 1 & \text{for} \quad j = k+2 \\[2ex] d_{k+2} s_{k+1} x = x & \text{for} \quad j = k+3 \\[2ex] \sigma_{k+1} d_j s_{k+1} x = 1 & \text{for} \quad j > k+3. \end{cases}$$

Thus $\sigma_k x \simeq x^{-1}$, which is the assertion. $\qquad\square$

Let G be a braided simplicial group and let S be a subset of G_n. Define

$$B_{n+1}(S) = \{\sigma \in B_{n+1} | \sigma x \simeq x \quad \text{for} \quad \text{all} \quad x \in S\}.$$

Since $\mathcal{B}(G)$ is invariant under the braided group action, $B_{n+1}(S)$ is a subgroup of B_{n+1}. Let $\tilde{B}_{n+1}$ be the kernel of the composite

$$B_{n+1} \longrightarrow \Sigma_{n+1} \xrightarrow{\text{sign}} \mathbb{Z}/2,$$

that is, $\tilde{B}_{n+1}$ is the pre-image of the alternating group A_{n+1}.

Theorem 2.9. *Let G be a braided simplicial group. Then*

1) $B_{n+1}(\mathcal{Z}(G)_n) = \tilde{B}_{n+1}$ *or* B_{n+1};
2) $B_{n+1}(\mathcal{Z}(G)_n) = B_{n+1}$ *if and only if* $2 \cdot \pi_n(G) = 0$.

Proof. Let $\sigma \in B_{n+1}(\mathcal{Z}(G)_n)$ and let $x \in \mathcal{Z}(G)_n$. By Lemma 2.8, we have

$$(\sigma_k^{-1}\sigma\sigma_k)(x) \simeq (\sigma_k\sigma)(x^{-1}) \simeq \sigma_k(x^{-1}) \simeq x$$

for each $k \geq -1$. Thus $B_{n+1}(\mathcal{Z}(G)_n)$ is a normal subgroup of B_{n+1}. By Lemma 2.8, we have

$$\sigma_s\sigma_t \in B_{n+1}(\mathcal{Z}(G)_n)$$

for any $s, t \geq -1$. It follows that $B_{n+1}(\mathcal{Z}(G)_n) = \tilde{B}_{n+1}$ or B_{n+1}, which is the assertion 1.

If $B_{n+1}(\mathcal{Z}(G)_n) = B_{n+1}$, then by Lemma 2.8

$$x \simeq x^{-1}$$

for any $x \in \mathcal{Z}(G)_n$. Thus $2 \cdot \pi_n(G) = 0$. Conversely, if $2 \cdot \pi_n(G) = 0$, then

$$\sigma_k(x) \simeq x$$

for any $x \in \mathcal{Z}(G)_n$ and $k \geq -1$. Thus $B_{n+1}(\mathcal{Z}(G)_n) = B_{n+1}$. This shows assertion 2. $\square$

Let G be a braided simplicial group and let H be a subgroup of B_{n+1}. Define

$$G_n(H) = \{x \in G_n | \sigma(x) \simeq x \quad \text{for} \quad \text{all} \quad \sigma \in H\},$$

that is, $G_n(H)$ is the "homotopy" fixed set of H. Then $G_n(H)$ is a subgroup of G_n. By Theorem 2.9, we have that

$$G_n(\tilde{B}_{n+1}) \supseteq \mathcal{Z}(G)_n.$$

Proposition 2.10. *Let G be a braided simplicial group and let $x \in G_n(\tilde{B}_{n+1})$ with $n \geq 2$. Then*

1) $d_j(x) = d_{j+2}(x)$ *for each j;*
2) $\sigma_k d_j(x) = d_{j+1}(x)$ *for each j, k;*

3) $d_j(x)$ *is a fixed point of* $\tilde{B}_n$ *for each* j.

In particular, if $d_j x = 1$ *for some* j, *then* $x \in \mathcal{Z}(G)_n$.

Proof. Since $x \in G_n(\tilde{B}_{n+1})$, we have

$$\sigma_s \sigma_t(x) \simeq x$$

for any $s, t \geq -1$. It follows that

$$\sigma_{-1} x \simeq \sigma_0 x \simeq \sigma_1 x \simeq \cdots \simeq \sigma_{n-2} x.$$

Now for each $-1 \leq k \leq n - 3$, we have

$$d_{k+1} x = d_{k+2} \sigma_k x = d_{k+2} \sigma_{k+1} x = d_{k+3} x$$

and so assertion 1 follows.

Now for each $0 \leq s \leq n - 2$, we have

$$d_{s+1}(x) = d_s \sigma_{s-1}(x) = d_s \sigma_s(x) = \sigma_{s-1} d_s(x).$$

Assertion 2 follows.

For any $s, t \geq -1$, we have

$$\sigma_s(\sigma_t d_0(x)) = \sigma_s(d_0(\sigma_{t+1} x)) = d_0((\sigma_{t+1} \sigma_{s+1}) x) = d_0 x.$$

Thus

$$\sigma_t \sigma_s(d_0(x)) = d_0(x)$$

for any $s, t \geq -1$ and so $d_0(x)$ is a fixed point of $\tilde{B}_n$. Since

$$\sigma_t \sigma_s(d_1(x)) = \sigma_s((\sigma_{-1} \sigma_t)(d_0(x))) = \sigma_s(d_0(x) = d_1(x),$$

$d_1(x)$ is a fixed point of $\tilde{B}_{n+1}$ and hence assertion 3. $\square$

Let B_n be the subgroup of B_{n+1} generated by σ_j with $j \geq 0$. By inspecting the proof, we have

Proposition 2.11. *Let* G *be a braided simplicial group and let* $x \in G_n(\tilde{B}_n)$ *with* $n \geq 2$. *Then*

1) $d_j(x) = d_{j+2}(x)$ *for each* $j \geq 1$;
2) $\sigma_k d_j(x) = d_{j+1}(x)$ *for each* $j, k \geq 1$;
3) $d_j(x)$ *is a fixed point of* $\tilde{B}_{n-1}$ *for each* j;
4) $d_0(x)$ *is a fixed point of* $\tilde{B}_n$.

In particular, if $d_0 x = 1$ *and* $d_j x = 1$ *for some* $j \geq 1$, *then* $x \in \mathcal{Z}(G)_n$.

The following lemma is well-known. We give an elementary proof. Let S be a subset of a group G. We write $\langle S \rangle$ for the subgroup of G generated by S.

Lemma 2.12. *Let $w \in F(y_0, \cdots, y_n)$ with $n \geq 0$. Suppose that there is a positive integer k such that $\sigma_j^k(w) = w$ for $0 \leq j \leq n - 1$. Then w lies in the subgroup generated by $y_0 y_1 \cdots y_n$. In addition, if $\sigma_{-1}^k w = w$ and $n \geq 1$, then $w = 1$.*

Proof. The proof is given by induction on n. The assertion is trivial for $n = 0$. Let $n = 1$. We may assume that $k = 2t$ is an even integer. Let $x_0 = y_1^{-1}$ and let $x_1 = y_0 y_1$. Then $F(y_0, y_1) = F(x_0, x_1)$. Since $\sigma_0(y_0) = y_1$ and $\sigma_0(y_1) = y_1^{-1} y_0 y_1$, we have $\sigma_0 = \chi_{x_1}$ and so

$$\sigma_0^k = \chi_{x_1}^t = \chi_{x_1^t}.$$

We can write w as a reduced word in $F(y_0, y_1) = F(x_0) * F(x_1)$. Then

$$w = x_0^{n_1} x_1^{l_1} \cdots x_0^{n_s} x_1^{l_s},$$

where $n_j \neq 0$ for $2 \leq j \leq s$ and $l_j \neq 0$ for $1 \leq j \leq s - 1$. Suppose that $w \notin \langle x_1 \rangle$. There are two cases: $n_1 \neq 0$ or $n_1 = 0$. If $n_1 \neq 0$, then $x_1^t w \neq w x_1^t$. This contradicts to that $\chi_{x_1^t}(w) = w$. Otherwise, $n_1 = 0$ and $s > 1$. Then $w = x_1^{l_1} x_0^{n_2} w'$ and $x_1^{t+l_1} x_0^{n_2} w' \neq x_1^{l_1} x_0^{l_2} w' x_1^t$. This contradicts to that $\chi_{x_1^t}(w) = w$. Hence $w \in \langle x_1 \rangle = \langle y_0 y_1 \rangle$.

Now suppose that the assertion holds for $n - 1$ with $n > 1$. Since

$$F(y_0, \cdots, y_n) = F(y_0, \cdots, y_{n-1}) * F(y_n)$$

is a free product, we can write w as a word

$$w = y_n^{l_0} w_1 y_n^{l_1} \cdots w_t y_n^{l_t},$$

where $w_j \neq 1 \in F(y_0, \cdots, y_{n-1})$ and $l_j \neq 0$ for $1 \leq j \leq t - 1$. Because $\sigma_j(y_n) = y_n$ for $j < n - 1$, we have

$$\sigma_j^k(w_i) = w_i$$

for $1 \leq i \leq t$ and $0 \leq j \leq n - 2$. Let $x = y_0 y_1 \cdots y_{n-1}$. By induction, we have

$$w_i \in \langle x \rangle$$

for $1 \leq i \leq t$ and so

$$w \in \langle x, y_n \rangle.$$

Let $q \colon F(y_0, y_1, \cdots, y_n) \to F(y_{n-1}, y_n)$ be the projection defined by $q(y_j) = 1$ for $j < n - 1$ and $q(y_j) = y_j$ for $j \geq n - 1$. Then

$$q \circ \sigma_{n-1} = \sigma_{n-1} \circ q.$$

Since $\sigma_{n-1} w = w$, we have $\sigma_{n-1}(q(w)) = q(w)$ and so

$$q(w) \in \langle y_{n-1} \cdot y_n \rangle.$$

Because the restriction

$$q|_{\langle x, y_n \rangle} \colon \langle x, y_n \rangle \to F(y_{n-1}, y_n)$$

is an isomorphism, we have

$$w \in \langle x \cdot y_n \rangle = \langle y_0 y_1 \cdots y_n \rangle$$

and hence the result. $\square$

Let K_n be the pure braided group, that is, K_n is that kernel of the canonical epimorphism $B_n \to \Sigma_n$. Let B_{n+1} be the subgroup of B_{n+2} generated by σ_j for $j \geq 0$. Recall that

$$\pi_{n+1}(F(S^1)) = \mathcal{Z}(F(S^1))_{n+1} / \mathcal{B}(F(S^1))_{n+1}.$$

Consider the actions of two braided groups B_{n+2} and B_{n+1} on $F(S^1)_{n+1} / \mathcal{B}(F(S^1))_{n+1}$. We have

Theorem 2.13. *If $n \geq 2$, then in $F(S^1)_{n+1} / \mathcal{B}(F(S^1))_{n+1}$,*

1) *the fixed set of the pure braided group K_{n+1}-action is*

$$\mathbb{Z} \oplus \pi_{n+1}(F(S^1));$$

2) *the fixed set of K_{n+2}-action is*

$$\pi_{n+1}(F(S^1)).$$

Proof. We show that the homotopy fixed set of K_{n+1} on $F(S^1)_{n+1}$ is generated by y_{-1} and $\mathcal{Z}(F(S^1))_{n+1}$. Assertions 1 and 2 will follow from this statement. Let w be a homotopy fixed point of K_{n+1}-action on $F(S^1)_{n+1}$. Since

$$\sigma_k^2 d_0 = d_0 \sigma_{k+1}^2$$

for $k \geq -1$, we have

$$\sigma_k^2 d_0(w) = d_0(w)$$

for each $k \geq -1$. By Lemma 2.12, we have

$$d_0(w) = 1.$$

Now for each $1 \leq j \leq n+1$, we have

$$\sigma_k^2 d_j = \begin{cases} d_j \sigma_{k+1}^2 & \text{if} \quad j \leq k+1; \\ d_j \circ \sigma_{j-1}^{-1} \circ \sigma_{j-2}^2 \circ \sigma_{j-1} & \text{if} \quad j = k+2; \\ d_j \sigma_k^2 & \text{if} \quad j > k+2. \end{cases}$$

By Lemma 2.12, there exists integers $k_1, k_2, \cdots, k_{n+1}$ such that

$$d_j(w) = y_{-1}^{k_j}$$

for $1 \leq j \leq n+1$. Since $d_k y_{-1} = y_{-1}$ for $k > 0$, we have

$$y_{-1}^{k_j} = d_j(y_{-1}^{k_j}) = d_j d_j w = d_j d_{j+1} w = d_j(y_{-1}^{k_{j+1}}) = y_{-1}^{k_{j+1}}$$

for $1 \leq j \leq n$ and so

$$k_1 = k_2 = \cdots = k_{n+1}.$$

Let $w' = y_{-1}^{-k_1} w$. Then

$$d_j(w') = 1$$

for each $0 \le j \le n+1$ and $w' \in \mathcal{Z}(F(S^1))_{n+1}$. This shows that w lies in the subgroup generated by y_{-1} and cycles $\mathcal{Z}(F(S^1))_{n+1}$ and hence the result. $\square$

Note: In $F(S^1)_{n+1}/\mathcal{B}_{n+1}$, since any element in the homotopy group is a homotopy fixed point of $\tilde{B}_{n+2}$, the fixed set of $\tilde{B}_{n+1}$ is $\mathbb{Z} \oplus \pi_{n+1}(F(S^1))$ and the fixed set of $\tilde{B}_{n+2}$ is $\pi_{n+1}(F(S^1))$.

3. Braided Representation of $F(S^1)$

3.1. **A Representation of $F(S^1)$.** In this subsection, the ground ring R is $\mathbb{Z}$ or $\mathbb{Z}/p$. Let X be a pointed set. Let $A(X)$ be the algebra of non-commutative formal power series with variables in any finite subset of X over R modulo the single relation that $* = 1$, where $*$ is the base point of X. Let X be a pointed simplicial set. The simplicial algebra $A(X)$ is defined by applying the functor A to X. Let $X = S^1$ be the simplicial circle. By using the methods in [11], we have

1) There is a choice of generators in $A(S^1)_{n+1}$ such that

$$A(S^1)_{n+1} = A(x_0, x_1, \cdots, x_n)$$

is the associated algebra of the non-commutative formal power series in variables $x_0, x_1, \cdots, x_n$ over R.

2) The simplicial structure on $A(S^1)$ is given by

$$d_j x_k = \begin{cases} x_{k-1} & \text{for} & j \le k, \\ 0 & \text{for} & j = k+1, \\ x_k & \text{for} & j > k+1, \end{cases}$$

and

$$s_j x_k = \begin{cases} x_{k+1} & \text{for} & j \le k, \\ x_k + x_{k+1} + x_k x_{k+1} & \text{for} & j = k+1, \\ x_k & \text{for} & j > k+1, \end{cases}$$

for $0 \le j \le n+1$, where

$$x_{-1} = (1 + x_{n-1})^{-1}(1 + x_{n-2})^{-1} \cdots (1 + x_0)^{-1} - 1$$

in $A(S^1)_n$.

Let the Braided group B_{n+1} act on $A(S^1)_{n+1}$ as follows.

For each $0 \le k \le n - 1$, $\sigma_k \colon A(S^1)_{n+1} \to A(S^1)_{n+1}$ is an automorphism of algebras with

$$\sigma_k(x_j) = \begin{cases} x_{k+1} & \text{if} & j = k \\ (1 + x_{k+1})^{-1}(1 + x_k)(1 + x_{k+1}) - 1 & \text{if} & j = k + 1 \\ x_j & & \text{otherwise} \end{cases}$$

for $0 \le k \le n - 1$.

Let σ_{-1} be an automorphism of $A(S^1)_{n+1}$ defined by

$$\sigma_{-1}(x_0) = (1 + x_0)^{-1}(1 + x_{-1})(1 + x_0) - 1 = (1 + x_0)^{-1}(1 + x_n)^{-1} \cdots (1 + x_1)^{-1} - 1$$

and $\sigma(x_j) = x_j$ for $j > 0$. The subgroup of the automorphism group of $A(S^1)_{n+1}$ generated by σ_j for $-1 \le j \le n - 1$ is the braid group B_{n+2}. Let

$$e \colon F(S^1) \to A(S^1)$$

be the canonical representation, that is

$$e(y_j) = 1 + x_j$$

for each j.

Proposition 3.1. *The simplicial algebra $A(S^1)$ is a braided simplicial algebra, that is, the braided action satisfies the identities 1 and 2 in Proposition 2.1. Furthermore, the representation $e \colon F(S^1) \to A(S^1)$ is a braided representation, that is e commutes with the braid group action and the simplicial structure.*

The proof is straight forward.

Since the representation $e \colon F(S^1) \to A(S^1)$ is faithful [6], we have

Proposition 3.2. *Let $w \in F(S^1)_{n+1}$. Then*

1) $w \in \mathcal{Z}(F(S^1))_{n+1}$ *if and only if* $e(w) - 1 \in \mathcal{Z}(A(S^1))_{n+1}$;
2) $w \in NF(S^1)_{n+1}$ *if and only if* $e(w) - 1 \in NA(S^1)_{n+1}$.

Let $w = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_t}^{n_t}$ be a monomial in $A(x_0, \cdots, x_n)$. We call w is *non-degenerate* if the set

$$\{i_1, \cdots, i_t\} = \{0, \cdots, n\},$$

that is each letter x_j with $0 \le j \le n$ appears in w at least once.

Theorem 3.3. *Let f be a series in $A(S^1)_{n+1}$. Then $f \in NA(S^1)_{n+1}$ if and only if f is a formal series of non-degenerate monomials.*

The assertion follows from the following lemma.

Lemma 3.4. *Let f be a series in $A(S^1)_{n+1}$. Then*

$$f \in \bigcap_{j=1}^{i+1} \mathrm{Ker}(d_j)$$

if and only if f is a linear summation of monomials in which each x_j appears at least once for $0 \le j \le i$.

Proof. The proof is given by induction on i. Let $i = 0$. Since d_1 is the projection, $\mathrm{Ker}(d_0)$ is a two sided ideal generated by x_0. Suppose that the assertion holds for $i - 1$ with $i > 0$. Then there is a decomposition

$$\bigcap_{j=1}^{i} \mathrm{Ker}\, d_j = C \oplus D,$$

where C is the set of series of monomials in which each x_j appears at least once for $0 \le j \le i$. Since d_{i+1} is the projection, we find that $d_{i+1}|_C = 0$ and $d_{i+1}|_D$ is a monomorphism from D to $A(S^1)_n$. This shows that

$$\bigcap_{j=1}^{i+1} \mathrm{Ker}\, d_j = C$$

and hence the result. $\qquad\square$

3.2. **Formal Steenrod Operations and Higher Differentials.** In this subsection, the ground ring R is $\mathbb{Z}$ or $\mathbb{Z}/p$. Let $A(x_0, \cdots, x_n)_i$ be the sub R-module of $A(x_0, \cdots, x_n)$ generated by monomial of degree i. Let $f \colon A(x_0, \cdots, x_n) \to A(x_0, \cdots, x_{n-1})$ be an R-linear map. We call f is a homogenous map of degree t if

$$f(A(x_0, \cdots, x_n)_i) \subseteq A(x_0, \cdots, x_{n-1})_{i+t}$$

for each q. Let V be the free R-module generated by $x_0, \cdots, x_n$.

Lemma 3.5. *Let $f_t \colon V \to A(x_0, \cdots x_{n-1})$ be a sequence of R-linear maps with $t \ge 0$ such that*

$$f_t(V) \subseteq A(x_0, \cdots, x_{n-1})_t$$

for each t. Then there exists a unique sequence of homogeneous maps $P_f^t \colon A(x_0, \cdots, x_n) \to A(x_0, \cdots, x_{n-1})$ such that

1) *$P_f^t|_V = f_t$ for each $t \ge 0$;*
2) *The anti-Cartan formula*

$$P_f^t(xy) = \sum_{i+j=t} P_f^i(y) P_f^j(x)$$

hold for any $x, y \in A(x_0, \cdots, x_n)$.

The proof is straight forward.

Let $\chi\colon A(x_0, \cdots, x_n)$ be the convolution, that is, χ is the anti-automorphism with $\chi(x_i) = -x_i$ for each i. Let

$$\bar{d}_0 = d_0 \circ \chi = \chi \circ d_0.$$

Then we have $\bar{d}_0(x_j) = x_{j-1}$ for $j > 0$ and

$$\bar{d}_0(x_0) = (1 + x_0)(1 + x_1) \cdots (1 + x_{n-1}) - 1.$$

Let

$$\Delta_{s-1} = \sum_{0 \le l_1 < l_2 < \cdots < l_s \le n-1} x_{l_1} x_{l_2} \cdots x_{l_s}.$$

for $1 \le s \le n$. For $0 \le s \le n-1$, let $\delta_s\colon V \to A(x_0, \cdots, x_{n-1})$ be the R-linear map defined by

$$\delta_i(x_j) = \begin{cases} x_{j-1} & \text{for} & i = 0, j > 0, \\ 0 & \text{for} & i > 0, j > 0 \\ \Delta_i & \text{for} & j = 0 \end{cases}$$

Let

$$\delta = (\delta_0, \delta_1, \cdots, \delta_{n-1}, 0, \cdots, 0).$$

Proposition 3.6. *The map* $\bar{d}_0\colon A(x_0, \cdots, x_n) \to A(x_0, \cdots, x_{n-1})$ *is decomposed as*

$$\bar{d}_0 = \sum_{i=0}^{\infty} P_\delta^i.$$

Proof. By the anti-Cartan formula, the map

$$\sum_{i=0}^{\infty} P_\delta^i$$

is a well-defined anti-homomorphism of algebras. Since

$$\bar{d}_0(x_j) = \sum_{i=0}^{\infty} P_\delta^i(x_j)$$

for each $0 \le j \le n$, the assertion follows. $\square$

Let $q_i\colon A(x_0, \cdots, x_n) \to A(x_0, \cdots, x_n)$ be the composite

$$A(x_0, \cdots, x_n) \xrightarrow{\text{proj.}} A(x_0, \cdots, x_n)_i \hookrightarrow A(x_0, \cdots, x_n)$$

for $0 \le i < \infty$. Let

$$\Gamma_t F(S^1)_{n+1} = \{w \in F(S^1)_{n+1} | q_i(w) = 0 \quad \text{for} \quad 0 < i < t\}$$

for $1 \leq t < \infty$. It is well-known [6] that $\{\Gamma_t\}$ is the descending central series if $R = \mathbb{Z}$ and descending p-central series if $R = \mathbb{Z}/p$. Let

$$L(S^1) = \bigoplus_{t=1}^{\infty} \Gamma_t(F(S^1))/\Gamma_{t+1}(F(S^1)).$$

If $R = \mathbb{Z}$, then $L(S^1)$ is the free simplicial Lie algebra over $\mathbb{Z}$ generated by S^1. If $R = \mathbb{Z}/p$, then $L(S^1)$ is the free restricted simplicial Lie algebra over $\mathbb{Z}/p$ generated by S^1. The simplicial structure $L(S^1)$ is as follows.

1) $L(S^1)_{n+1} = L(x_0, \cdots, x_n)$;
2) $d_j(x_k) = x_k$ for $k < j-1$, $d_j(x_{j-1}) = 0$ and $d_j(x_k) = x_{k-1}$ for $k > j-1$, where $x_{-1} = -(x_0 + x_1 + \cdots + x_{n-1}) \in L^u(S^1)_n$ or $L(S^1)_n$.
3) $s_j x_k = x_k$ for $k < j-1$, $s_j x_{j-1} = x_{j-1} + x_j$ and $s_j x_k = x_{k+1}$ for $k > j-1$.

A Lie monomial $w = [[x_{i_1}, x_{i_2}, \cdots]^{p^s}$ in $L^u(S^1)_{n+1}$ or $L(S^1)_{n+1}$ is called *non-degenerate* if the set

$$\{i_1, \cdots, i_t\} = \{0, 1, \cdots, n\}.$$

By Theorem 3.3, we have

Corollary 3.7. *The Moore chain complexes $NL(S^1)_{n+1}$ is the submodules of $L(S^1)_{n+1}$ spanned by non-degenerate Lie monomials, respectively.*

The spectral sequence, which is denoted by $\{E^r(F(S^1))\}$, induced by the descending p-central (integral descending central) series of $F(S^1)$ is called (integral) *Adams spectral sequence* of $F(S^1)$. By Curtis Theorem , this spectral sequence is convergent to $\pi_*(F(S^1); p) = \pi_{*+1}(S^2; p)$ (or $\pi_{*+1}(S^2)$ if we use integral descending central series). A description of higher differentials in the Adams spectral sequence is as follows.

Let $w \in F(S^1)_{n+1}$. Then

$$e(w) = 1 + \sum_{i=1}^{\infty} q_i(w)$$

in $A(x_0, \cdots, x_n)$. We simply write $(w)_i$ for $q_i(w)$. Let

$$\alpha \in E^1_{t,*}(F(S^1)) = \pi_*(L_t(S^1))$$

and let z^α be a cycle in the simplicial group $L(S^1)$ such that the homotopy class of z^α is α, that is z^α is a cycle representative of α. Since the map

$$g \colon \Gamma_t(F(S^1)) \to L_t(S^1) = \Gamma_t(F(S^1))/\Gamma_{t+1}(F(S^1)),$$

is a simplicial epimorphism, there is an element

$$w^\alpha \in N(\Gamma_t(F(S^1))) = N(F(S^1)) \cap F(S^1)$$

such that

$$g(w^\alpha) = z^\alpha.$$

The element w^α is called a *Moore representative* of α.

Theorem 3.8. *Let* $\alpha \in E^1_{t,*}(F(S^1_{n+1}))$ *and let* w_α *be a Moore representative in* $F(S^1)_{n+1}$. *Let* $1 \le r \le \infty$. *Then* $d^j(\alpha) = 0$ *for* $j < r$ *if and only if the following linear equations holds in* $A(x_0, \cdots, x_{n-1})$:

$$\sum_{i=0}^{j} P^i_\delta w^\alpha_{t+j-i} = 0.$$

for $0 \le j < r$. *Furthermore, if* $r < \infty$, *then*

$$\sum_{i=0}^{r} P^i_\delta w^\alpha_{t+r} \in L_{t+r}(S^1)_n \subseteq A(x_0, \cdots, x_{n-1}),$$

which is a cycle representative of $-d^r(\alpha)$.

Proof. Since $d^r(\alpha) = 0$ if and only if

$$d_0(w^\alpha) \in \Gamma_{t+r} F(S^1)_n,$$

the assertion follows from Proposition 3.6. $\square$

Let w be a word in $F(S^1)_{n+1}$. We call w is a *basic non-degenerate commutator* of weight $s = l(w)$ if w can be written down as a commutator

$$[\cdots [y_{i_1}, y_{i_2}], \cdots], y_{i_s}]$$

such that the set

$$\{i_1, \cdots, i_s\} = \{0, \cdots, n\},$$

that is each generator y_j occurs in the commutator w at least once. Let $N_t F(S^1)_{n+1}$ be the subgroup of $F(S^1)_{n+1}$ generated by basic nondegenerate commutators w with $l(w) \ge t$ and let $N^{(p)}_t F(S^1)_{n+1}$ be the subgroup of $F(S^1)_{n+1}$ generated by w^{p^r}, where w runs over all basic non-degenerated commutators with $l(w) \cdot p^r \ge t$. Note that

$$N_t F(S^1)_{n+1} \subseteq N(F(S^1))_{n+1} \cap \Gamma_t(F(S^1)_{n+1})$$

if $R = \mathbb{Z}$ and

$$N^{(p)}_t F(S^1)_{n+1} \subseteq N(F(S^1))_{n+1} \cap \Gamma_t(F(S^1)_{n+1})$$

if $R = \mathbb{Z}/p$.

By Corollary 3.7, we have

Proposition 3.9. *Let* r *be any non-negative integer. If* $R = \mathbb{Z}$, *there is an isomorphism*

$$N_t F(S^1)_{n+1}/(N_t F(S^1)_{n+1} \cap \Gamma_{t+r} F(S^1)_{n+1})$$
$$\cong N(F(S^1))_{n+1} \cap \Gamma_t(F(S^1))_{n+1}/(N(F(S^1))_{n+1} \cap \Gamma_{t+r}(F(S^1))_{n+1}).$$

If $R = \mathbb{Z}/p$, then there is an isomorphism

$$N_t^{(p)}F(S^1)_{n+1}/(N_t^{(p)}F(S^1)_{n+1} \cap \Gamma_{t+r}F(S^1)_{n+1})$$
$$\cong N(F(S^1))_{n+1} \cap \Gamma_t(F(S^1))_{n+1}/(N(F(S^1))_{n+1} \cap \Gamma_{t+r}(F(S^1))_{n+1}).$$

Remark 3.10. By Theorem 3.8 and Proposition 3.9, $\pi_*(L_t(S^1))$ are represented by those words w in $N_tF(S^1)$ (or $N_t^{(p)}F(S^1)$) with $P_\delta^0 w_t = 0$ and the higher differentials in the Adams spectral sequence are related to higher formal Steenrod operations on $N_tF(S^1)$.

3.3. E^1-**terms of The Integral Adams Spectral Sequence.** In this subsection, the ground ring R is a subring of $\mathbb{Q}$. Let L be the free functor from free R-modules to Lie algebras. Let X be a pointed simplicial set. Let $\bar{R}(X) = R(X)/R(*)$ be the reduced free simplicial R-module generated by X. In particular, $\bar{R}(S^n) = K(R, n)$. Let $L(X) = L(\bar{R}(X))$ for any pointed simplicial set X. Let V be a free R-module and let L' be the kernel of the abelianizer

$$L(V) \rightarrow V.$$

Then L' is functor from free R-modules to graded free R-modules. Let $Q_n(L'(V))$ be the set of indecomposable elements of degree n of $L'(V)$. Let $S_n(V)$ be the set of monomials of degree n in the polynomial algebra $S(V)$. Note that $Q_1(L'(V)) = 0$.

Lemma 3.11. *For each $n \geq 2$, there is a functorial short exact sequence*

$$0 \rightarrow Q_n(L'(V)) \rightarrow S_{n-1}(V) \otimes V \xrightarrow{\text{mult.}} S_n(V) \rightarrow 0.$$

Proof. Let $K(V)$ be the kernel of $S_{n-1}(V) \otimes V \xrightarrow{\text{mult}} S_n(V)$ and let $\phi \colon T_n(V) \rightarrow S_n(V)$ be the composite

$$T_n(V) = T_{n-1}(V) \otimes V \xrightarrow{\text{proj.}} S_{n-1}(V) \otimes V \xrightarrow{\text{mult.}} S_n(V).$$

Then $\phi|_{L_n'(V)} \colon L_n'(V) \rightarrow S_n(V)$ is zero which gives a functorial map

$$\tilde{\phi} \colon L_n'(V) \rightarrow K(V).$$

It is a routine work to show that the map $\tilde{\phi}$ is onto and $\tilde{\phi}|_{D_n(L'(V))} \colon D_n(L'(V)) \rightarrow K(V)$ is zero and so $\tilde{\phi}$ factors through $Q_n(L'(V))$, where $D(L'(V))$ is the set of decomposable elements of $L'(V)$. Thus $K(V)$ is a functorial quotient of $Q_n(L'(V))$. By checking Poincare series, the quotient map

$$Q_n(L'(V)) \rightarrow K(V)$$

is an isomorphism and hence the result. $\qquad\square$

Note: There is no functorial cross-section from $Q(L')$ to L' (See [8]).

Proposition 3.12. *There is a homotopy equivalence*

$$L(S^1) \simeq \bar{R}(S^1) \oplus L(S^2).$$

Proof. Since $K(\mathbb{Z}, 1) \simeq S^1$, we have

$$Q(L') \simeq \bar{R}(S^2).$$

Let $\phi\colon L(L_2(S^1)) \to L'(S^1)$ be the inclusion. Then ϕ is a homotopy equivalence by checking the spectral sequence induced by Lie filtrations. The assertion follows. $\square$

Let V be a free simplicial R-module and let C be a (pointed) simplicial coalgebra. Let $f, g\colon C \to T(V)$ be pointed simplicial coalgebra maps. We call f is *coalgebra homotopic* to g if there is a pointed homotopy $F_t\colon C \to T(V)$ such that $F_0 = f$, $F_1 = g$ and F_t is a coalgebra map for each t. Let

$$[T(V), T(V)]^{\text{coalg}}$$

be the set of coalgebra homotopy classes. If $f\colon T(V) \to T(V)$ is a simplicial coalgebra map, then we have the restriction

$$f|_{L(V)}\colon L(V) \to L(V).$$

If f is coalgebra homotopic to g, then

$$f|_{L(V)} \simeq g|_{L(V)}.$$

This defines a map

$$\theta\colon [T(V), T(V)]^{\text{coalg}} \to [L(V), L(V)].$$

Theorem 3.13. *Suppose that $R = \mathbb{Z}_{(p)}$. Then the homotopy groups $\pi_*(L(S^{2n}))$ has the following exponents.*

    1) *If $n = 1, 3$, then*

$$p \cdot \pi_*(L(S^{2n})) = 0$$

       for $* > 2n$ and any prime p;

    2) *If $p = 2$, then*

$$4 \cdot \pi_*(L_t(S^{2n})) = 0$$

       for $t > 2$ and any n.

Proof. Consider the Cohen representation

$$\theta\colon H_\infty \to [T(S^{2n}), T(S^{2n})]^{\text{coalg}} \to [L(S^{2n}), L(S^{2n})],$$

where H_∞ is the Cohen group. First we assume that $n = 1, 3$. Then the Samelson product

$$S^{2n} \wedge S^{2n} \to F(S^{2n})$$

is null homotopic. Since $\pi_*(\Gamma_2(F(S^{2n}))) \to \pi_*(F(S^{2n}))$ is a monomorphism, the Samelson product

$$S^{2n} \wedge S^{2n} \to \Gamma_2(F(S^{2n}))$$

is null homotopic and so the composite

$$S^{2n} \wedge S^{2n} \to \Gamma_2(F(S^{2n})) \to \Gamma_2(F(S^{2n})/\Gamma_3(F(S^{2n})))$$

is null homotopic. Let $J_t T(S^{2n}) = \mathbb{Z}_{(p)}(J_t(S^{2n}))$. Then there is a monomorphism

$$[J_t T(S^{2n}), T(S^{2n})]^{\mathrm{coalg}} \to [\mathbb{Z}_{(p)}((S^{2n})^{\times t}), T(S^{2n})]^{\mathrm{coalg}}$$

for any t. Since the Samelson product is trivial, the group

$$[\mathbb{Z}_{(p)}((S^{2n})^{\times t}), T(S^{2n})]^{\mathrm{coalg}}$$

is abelian. Let $\mathrm{id}^{*p} \colon T(V) \to T(V)$ be the p-fold convolution product of the identity and let $T(p) \colon T(V) \to T(V)$ be the morphism of Hopf algebras induced by the map $p \colon V \to V, \quad , x \to px$. Then $\mathrm{id}^{*p}|_{J_t T(S^{2n})}$ is coalgebra homotopic to $T(p)|_{J_t T(S^{2n})}$ in

$$[\mathbb{Z}_{(p)}((S^{2n})^{\times t}), T(S^{2n})]^{\mathrm{coalg}}$$

for each t and so

$$\mathrm{id}^{*p}|_{J_t T(S^{2n})} \simeq T(p)_{J_t T(S^{2n})}$$

for each t. This id^{*p} is coalgebra homotopic to $T(p)$. Let $x \in L_t(S^{2n})$. Then

$$\mathrm{id}^{*p}(x) = px \quad T(p)(x) = p^t x.$$

Thus

$$p^t - p = p(p^{t-1} - 1) = 0$$

in $[L_t(S^{2n}), L_t(S^{2n})]$ and so

$$p \cdot \pi_*(L_t(S^{2n})) = 0$$

for $t > 1$. Since $L_1(S^{2n}) = K(Z_{(p)}, 2n)$, we proves assertion 1.

The proof of assertion 2 is similar to assertion 1, where one needs the fact that the Samelson product

$$S^{2n} \wedge S^{2n} \to F(S^{2n})$$

is of order 2 up to homotopy and the higher Samelson products are null homotopic localized at 2. $\qquad\square$

Remark 3.14. $\pi_*(L(S^n) \otimes \mathbb{Z}/p)$ is known as a specific module over the Λ-algebra [1, 4, 10]. By considering the Bockstein spectral sequence for $\pi_*(L(S^n))$, this theorem shows that $\pi_*(L(S^{2n}))$ is the kernel of the Bockstein on $\pi_*(L(S^{2n}) \otimes \mathbb{Z}/p)$ when $n = 1, 3$.

Theorem 3.15. *Let* $R = \mathbb{Z}_{(p)}$. *If* t *is not a power of* p, *then*

$$\pi_*(L_t(S^{2n})) = 0.$$

Proof. By a result in [8], there exist functors $A^{\min}$ and $Q_t^{\max}$ for $t \geq 2$ such that

1) $A^{\min}$ is a (smallest) coalgebra retract of T with $V \subseteq A^{\min}(V)$ for each V;
2) $Q_t^{\max}$ is a subfunctor of L_t and $Q_t^{\max}$ is a retract of T_t;
3) There is a functorial coalgebra decomposition

$$T(V) \cong T(\bigoplus_{t=2}^{\infty} Q_t^{\max}(V)) \otimes A^{\min}(V).$$

Let $L^{\min}(V)$ be the primitives of $A^{\min}(V)$. Then there is a functorial decomposition

$$L(V) \cong L(\bigoplus_{t=2}^{\max} Q^{\max}(V)) \oplus L^{\min}(V).$$

Since $Q_t^{\max}$ is a retract of T_t, $Q^{\max}(S^{2n})$ is a retract of $T_t(S^{2n})$ and so $Q_t^{\max}(S^{2n})$ is either contractible or homotopic to $K(\mathbb{Z}_{(p)}, 2tn)$. Now $Q_t^{\max}(S^{2n})$ is contractible because $Q_t^{\max}$ is subfunctor of $L_t(V)$. It follows that

$$L(\bigoplus_{t=2}^{\infty} Q_t^{\max}(S^{2n}))$$

is contractible. The assertion follows from a result in [8] that L_t is a functorial retract of

$$L(\bigoplus_{t=2}^{\infty} Q_t^{\max}(S^{2n}))$$

if t is not a power of p. $\square$

By Theorems 3.13 and 3.15, we have

Theorem 3.16. *Let* $\{E^r\}_{r \geq 1}$ *be the integral Adams spectral sequence of* $F(S^1)$. *Then*

1) $E_{t,*}^r = 0$ *unless* $t = 2p^s$ *some prime* p *and some non-negative integer* s.
2) $p \cdot E_{2p^s,*}^r = 0$ *for any prime* p *and any integer* $s > 0$.
3) *Let* $\alpha \in E_{2p^s,*}^1$ *with* $s > 0$. *Then the only differentials* d^r, *which are possibly non-trivial on* α, *are*

$$d^{2p^t - 2p^s}$$

with $t > s$.

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