HOLOMORPHIC FUNCTIONS OF EXPONENTIAL GROWTH ON ABELIAN COVERINGS OF A PROJECTIVE MANIFOLD

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Abstract

Let M be a projective manifold, $p : M_G \longrightarrow M$ a regular covering over M with a free abelian transformation group G. We describe holomorphic functions on M_G of an exponential growth with respect to the distance defined by a metric pulled back from M. As a corollary we obtain for such functions Cartwright and Liouville type theorems. Our approach brings together L_2 cohomology technique for holomorphic vector bundles on complete Kähler manifolds and geometric properties of projective manifolds.

1. Introduction.

1.1. Recently there was an essential progress in study of harmonic functions of polynomial growth on complete Riemannian manifolds (see, in particular, [CM], [Gu], [Ka], [L], [LZ], [Li], [LySu] for the results and further references). As a corollary one also obtains a description of holomorphic functions of polynomial growth on nilpotent coverings of compact Kähler manifolds (see also [Br]). On the other hand, very little is known about existence and behaviour of slowly growing harmonic (respectively holomorphic) functions on covering spaces of compact Riemannian (respectively Kähler) manifolds. The methods of the above cited papers seem to be not sufficient for application to the general situation. This paper is devoted to study of slowly growing holomorphic functions on abelian coverings of projective manifolds. Our approach is based on L_2 cohomology technique for holomorphic vector bundles on complete Kähler manifolds and geometric properties of projective manifolds and differs from the methods of the above mentioned papers.

In order to formulate the results of the paper we consider a projective manifold M and its regular covering $p : M_G \longrightarrow M$ with a free abelian transformation group

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G. Denote by r the distance from a fixed point in M_G defined by a metric pulled back from M. We study holomorphic functions f on M_G satisfying (for some $\epsilon > 0$)

$$|f(z)| \leq ce^{\epsilon r^2(z)}, \quad (z \in M_G). \tag{1.1}$$

Recall that the covering space M_G can be described as follows.

Let $\omega_1, ..., \omega_n$ be a basis of holomorphic 1-forms on M and $A : M \longrightarrow \mathbb{CT}^n$ be the Albanese map of M associated with this basis. By definition,

$$A(z) = \left(\int_{z_0}^{z} \omega_1, ..., \int_{z_0}^{z} \omega_n \right)$$

for a fixed $z_0 \in M$. Consider a free abelian quotient group G of the fundamental group $\pi_1(\mathbb{CT}^n) \cong \mathbb{Z}^{2n}$. Let $t : T_G \longrightarrow \mathbb{CT}^n$ be the regular covering over torus with the transformation group G. We can think of T_G as a locally trivial fibre bundle over $\mathbb{CT}^n$ with discrete fibres. Then $M_G = A^*T_G$ is the pullback of T_G to M. By definition the fundamental group of M_G is $H := (\pi \circ A_*)^{-1}(G) \subset \pi_1(M)$, where $\pi : \mathbb{Z}^{2n} \longrightarrow G$ denotes the quotient map. By the covering homotopy theorem there is a proper holomorphic map $A_G : M_G \longrightarrow T_G$ that covers A and such that $\widetilde{M}_G := A_G(M_G) \subset T_G$ is a covering of complex variety $A(M) \subset \mathbb{CT}^n$.

Our main result shows that if f satisfies (1.1) then there is a uniquely defined holomorphic function g on T_G with a similar growth condition such that $f = A_G^*(g)$. To its formulation we let ϕ be a smooth nonnegative function on T_G and $\tilde{\phi} = A_G^*(\phi)$. Consider the Hilbert space $\mathcal{H}_{\tilde{\phi}}(M_G)$ of holomorphic functions f on M_G with the norm

$$|f| := \int_{M_G} |f|^2 e^{-\tilde{\phi}} dV.$$

Here dV is the pullback of the volume form on M defined by a Kähler metric. Similarly we introduce the Hilbert space $\mathcal{H}_{\phi}(T_G)$ of holomorphic functions f on T_G with the norm

$$|f| := \int_{T_G} |f|^2 e^{-\phi} d\widetilde{V},$$

where $d\widetilde{V}$ is the pullback of the standard volume form on $\mathbb{CT}^n$. Let $\{dz_1, ..., dz_n\}$ be the basis of holomorphic 1-forms on $\mathbb{CT}^n$ such that $A^*(dz_i) = \omega_i$ for $i = 1, ..., n$. By the same symbol we denote the pullback of these forms to T_G. Let $\mathcal{L}(\phi) = \sum_{i,j} a_{ij}(z, \overline{z}) dz_i \wedge d\overline{z}_j$ be the Levi form of ϕ. We set

$$|\mathcal{L}(\phi)| := \sup_{i,j,z \in T_G} |a_{ij}(z)| \ .$$

Assume that there is a constant $c > 0$ such that

$$|\phi(x) - \phi(y)| \leq cd(x, y), \tag{1.2}$$

where $d(.,.)$ is the distance on T_G defined by the pullback of the flat metric on $\mathbb{CT}^n$.

Theorem 1.1 *There is a constant $C = C(M, A) > 0$ such that if $|\mathcal{L}(\phi)| < C$ and ϕ satisfies (1.2) then A_G^* maps $\mathcal{H}_{\phi}(T_G)$ isomorphically onto $\mathcal{H}_{\tilde{\phi}}(M_G)$.*

Assume now that instead of (1.2) ϕ satisfies:

for any $\epsilon > 0$, $x, y \in T_G$ with $d(x,y) \leq t$ there is a function $c(\epsilon, t) > 0$ increasing in t such that

$$\phi(x) \leq (1+\epsilon)\phi(y) + c(\epsilon, t) \ . \tag{1.3}$$

Theorem 1.2 *Let C be as in Theorem 1.1, $|\mathcal{L}(\phi)| < C' < C$ and ϕ satisfies (1.3). There is a constant $\tilde{\epsilon}(C') > 0$ such that for any $f \in \mathcal{H}_{\tilde{\phi}}(M_G)$ there exists a unique $\hat{f} \in \cap_{\epsilon < \tilde{\epsilon}(C')} \mathcal{H}_{(1+\epsilon)\phi}(T_G)$ satisfying*

$$A_G^*(\hat{f}) = f \quad \text{and} \quad |\hat{f}| \leq c(\epsilon)|f|.$$

Here we regard $\hat{f}$ as an element of $\mathcal{H}_{(1+\epsilon)\phi}(T_G)$.

In the following examples M_G is a regular covering over M with the maximal free abelian transformation group G (so $T_G = \mathbb{C}^n$).

Examples. 1. Let $\phi(z) = k \log(p + |z|^2)$ on $\mathbb{C}^n$, where $|z|$ is the Euclidean norm of the vector $z \in \mathbb{C}^n$ and $p > 0$ is so big that $|\mathcal{L}(\phi)| < C$. (Such p exists because $\mathcal{L}(\log|z|) \to 0$ when $|z| \to \infty$.) Then $\mathcal{H}_\phi(\mathbb{C}^n)$ is isomorphic to the space of holomorphic polynomials of degree $\leq k-n-1$. Therefore every holomorphic function on M_G of the corresponding polynomial growth is the pullback by A_G of a uniquely defined holomorphic polynomial on $\mathbb{C}^n$. This gives another proof for projective manifolds of the main result of [Br].

2. Let $\phi(z) = 2\sigma\sqrt{p + |z|^2}$ on $\mathbb{C}^n$, where p is such that $|\mathcal{L}(\phi)| < C$. Then $\mathcal{H}_\phi(\mathbb{C}^n)$ consists of entire functions of the exponential type $< \sigma$. Now Theorem 1.1 describes holomorphic functions f on M_G satisfying $|f(z)| < ce^{\sigma' r(z)}$, $z \in M_G$, $\sigma' < \sigma$.

3. Let $\phi(z) = 2\sigma|z|^2$ on $\mathbb{C}^n$ with $2\sigma < C$. Then the assumptions of Theorem 1.2 are fulfilled and the theorem describes holomorphic functions f on M_G satisfying $|f(z)| < ce^{\sigma' r^2(z)}$, $z \in M_G$, $\sigma' < \sigma$.

4. Assume that C is a compact complex curve of genus $g \geq 1$. Then C_G can be thought of as a submanifold in $\mathbb{C}^g$. Applying Theorem 1.2 we obtain the following Cartwright type theorem.

There is a positive number $\sigma = \sigma(C_G)$ such that any holomorphic function f on $\mathbb{C}^g$ satisfying $|f(z)| \leq ce^{\sigma'|z|^2}$, $0 < \sigma' < \sigma$, $z \in \mathbb{C}^g$, and $f|_{C_G} = 0$, equals 0 identically.

1.2. The classical Liouville theorem asserts that every bounded holomorphic function on $\mathbb{C}^n$ is a constant. Based on Theorem 1.1 we prove Liouville type theorems for holomorphic functions of slow growth on abelian coverings over a projective manifold.

Let $\Gamma \subset H_1(M, \mathbb{Z}) \cong \pi_1(M)/[\pi_1(M), \pi_1(M)]$ be the maximal free abelian subgroup of the homology group of M. Further, let $\Omega^1(M)$ be the space of holomorphic 1-forms on M. Any $\omega \in \Omega^1(M)$ determines a complex-valued linear functional on Γ by integration. For a subgroup $H \subset \Gamma$ denote by $\Lambda(H)$ the minimal complex subspace of holomorphic 1-forms vanishing on H. Assume also that the quotient group $G = \Gamma/H$ is torsion free and M_G is the regular covering over M with the transformation group G.

Theorem 1.3 *Let H be such that $\Lambda(H) = \Omega^1(M)$. Then any holomorphic on M_G function f satisfying for any $\epsilon > 0$*

$$|f(z)| \leq c(\epsilon)e^{\epsilon r(z)} \qquad (z \in M_G)$$

is a constant.

Remark 1.4 *It can be conjectured that the results of this paper are also true for abelian coverings of an arbitrary compact Kähler manifold.*

2. Preliminaries.

2.1. L_2 cohomology theory. In the proof of our main results we use L_2 cohomology technique for holomorphic vector bundles on complete Kähler manifolds. We start by reviewing some results of L_2 cohomology (see, e.g., Lárusson [La] for more details and further references).

Let X be a complex manifold of dimension n with a hermitian metric and E be a holomorphic vector bundle over X with a hermitian metric. Let $L_2^{p,q}(X,E)$ be the space of E-valued (p,q)-forms on X with the L_2 norm, and let $W_2^{p,q}(X,E)$ be the subspace of forms η such that $\overline{\partial}\eta$ is L_2. The forms η may be taken to be either smooth or just measurable, in which case $\overline{\partial}\eta$ is understood in the distributional sense. The cohomology of the resulting L_2 Dolbeault complex $(W_2^{\cdot,\cdot}, \overline{\partial})$ is the L_2-cohomology

$$H_{(2)}^{p,q}(X,E) = Z_2^{p,q}(X,E)/B_2^{p,q}(X,E),$$

where $Z_2^{p,q}(X,E)$ and $B_2^{p,q}(X,E)$ are the spaces of $\overline{\partial}$-closed and $\overline{\partial}$-exact forms in $L_2^{p,q}(X,E)$, respectively. Let E^* be the dual bundle of E with the dual metric. In our proofs we use the following result discovered by Lárusson [La].

Proposition 2.1 *Let E be a hermitian vector bundle with curvature Θ on a complex manifold X of dimension $n \geq 2$ with a complete Kähler form ω. If $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$ in the sense of Nakano, then*

$$H_{(2)}^{0,q}(X,E^*) = 0 \qquad \text{for } q < n \ .$$

Remark 2.2 *Let E satisfy conditions of Proposition 2.1. Consider linear map $\overline{\partial}$: $W_2^{0,0}(X,E^*) \longrightarrow Z_2^{0,1}(X,E^*)$ and introduce the norm in $W_2^{0,0}(X,E^*)$ by*

$$|f| := |f|_2 + |\overline{\partial}f|_2, \qquad f \in W_2^{0,0}(X,E^*) \ .$$

According to Proposition 2.1 for $q = 1$ and $q = 0$, there is a linear map s : $Z_2^{0,1}(X,E^*) \longrightarrow W_2^{0,0}(X,E^*)$ such that $s \circ \overline{\partial} = id$ and $\overline{\partial} \circ s = id$. Then by the Banach theorem, $\overline{\partial}$ is open and $s = (\overline{\partial})^{-1}$.

2.2. $\overline{\partial}$-method. Let $i : X \hookrightarrow Y$ be a complex compact submanifold of codimension 1 of an n-dimensional compact Kähler manifold Y, $n \geq 2$, with a Kähler form ω. Assume that the induced homomorphism $i_* : H_1(X, \mathbb{R}) \longrightarrow H_1(Y, \mathbb{R})$ is surjective. Let G be a free abelian quotient group of $\pi_1(Y)$. Consider the regular covering Y_G over Y with the transformation group G. From the assumption for i_* it follows that there are a regular covering X_G over X with the transformation group G (the pullback of Y_G by i) and the holomorphic embedding $i_G : X_G \hookrightarrow Y_G$ that covers i. Divisor $X \subset Y$ determines a holomorphic line bundle L over Y and a holomorphic section $s : Y \longrightarrow L$ with a simple zero along X. Further, for every $p \in X$, there is a coordinate neighbourhood (U, z) centered at p and a holomorphic frame e for L on U such that $s = z_1 e$ on U. Let h be a hermitian metric on L and ∇ be the canonical connection with curvature Θ constructed by h. By the same letters we denote the pullback of L, h, s and Θ to Y_G. Note also that if ϕ is a smooth function on Y_G then the weighted metric $e^{\phi}h$ on L has a curvature $\Theta' = -\mathcal{L}(\phi) + \Theta$.

Let U_0 be the pullback of the complement of a closed neighbourhood of $X \subset Y$ and $U_1, ..., U_N$ be the pullbacks of shrunk coordinate polydisks covering a larger neighbourhood of X. Also pull back a smooth partition of unity (ξ_i) subordinate to (U_i). Let f be a holomorphic function on X_G such that $f^2 e^{-\phi}$ is integrable on X_G. For $i \geq 1$, extend f to a holomorphic function f_i on U_i which is constant on each line $\{z_2, ..., z_n \ \text{ constant}\}$. Let $f_0 = 0$ on U_0. Since $f_i = f = f_j$ on X_G and X_G is smooth, we can define a holomorphic section of the dual bundle L^* on $U_{ij} = U_i \cap U_j$ by the formula

$$u_{ij} = (f_i - f_j)s^{-1} \ .$$

Then

$$v_i = \sum_i u_{ij}\xi_j$$

is a smooth section of L^* on U_j and $v_i - v_j = u_{ij}$. Hence $\overline{\partial} v_i = \overline{\partial} v_j$ on U_{ij}, so we get a $\overline{\partial}$-closed, L^*-valued (0,1)-form η on Y_G defined as $\overline{\partial} v_i$ on U_i. Assume that ϕ satisfies (1.2) or (1.3), where d is the distance on Y_G defined by the pullback of a metric on Y. Denote by $|f|$ the weighted L_2-norm of f with the weight $e^{-\phi}$.

Lemma 2.3 *(1) If ϕ satisfies (1.2) then $\eta \in L_2^{0,1}(Y_G, L^*)$ for L equipped with the metric $e^{\phi}h$ and $|\eta| \leq C(X, Y, h, \phi)|f|$ in the corresponding L_2-norms.*
(2) If ϕ satisfies (1.3) then $\eta \in L_2^{0,1}(Y_G, L^*)$ for L equipped with the metric $e^{(1+\epsilon)\phi}h$, $\epsilon > 0$, and $|\eta| \leq C(X, Y, h, \phi, \epsilon)|f|$.

Proof. We prove (2). The proof of (1) goes along the same lines (see also arguments in [La, Th. 3.1]).

We have to show that $|\eta|^2 e^{-(1+\epsilon)\phi}$ is integrable on Y_G. On U_0, s is bounded away from 0 and

$$\eta = \overline{\partial} v_0 = -\sum_j f_j s^{-1} \overline{\partial} \xi_j \ ,$$

so

$$|\eta|^2 \leq c \sum_j |f_j|^2,$$

where c depends only on X, Y, h. Further,

$$\int_{U_j} |f_j|^2 e^{-(1+\epsilon)\phi}\omega^n \leq c'(\epsilon, X, Y, \phi) \int_{X_G\cap U_j} |f|^2 e^{-\phi}\omega^{n-1}$$

because ϕ satisfies (1.3). Since $f^2 e^{-\phi}$ is integrable on X_G, so is $|\eta|^2 e^{-(1+\epsilon)\phi}$ on U_0. For $i \geq 1$,

$$\eta = \overline{\partial} v_i = \sum_j (f_i - f_j) s^{-1} \overline{\partial}\xi_j$$

on U_i and it remains to show that

$$\sum_{i,j\geq 1} \int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} e^{-(1+\epsilon)\phi}\omega^n < \infty \ . \tag{2.1}$$

For $x \in U_{ij}$, $i, j \geq 1$, there are $x_i \in X_G \cap U_i$ and $x_j \in X_G \cap U_j$ such that $f_i(x) = f(x_i)$, $f_j(x) = f(x_j)$ and $d(x_i, x_j) \leq c(h, X, Y)|s(x)|$. So,

$$|f_i(x) - f_j(x)||s(x)|^{-1} \leq c'(X, Y, d) \sup |df|,$$

where supremum is taken over $X_G \cap (U_i \cup U_j)$. By the Cauchy inequalities and since ϕ satisfies (1.3),

$$\begin{aligned}\int_{U_{ij}} |f_i - f_j|^2 |s|^{-2} e^{-(1+\epsilon)\phi}\omega^n &\leq c'(X, Y, d) \int_{U_{ij}} \sup |df|^2 e^{-(1+\epsilon)\phi}\omega^n \\ &\leq c''(X, Y, d, \epsilon) \int_{X_G\cap(V_i\cup V_j)} |f|^2 e^{-\phi}\omega^{n-1},\end{aligned}$$

where $V_i \supset U_i$, $V_j \supset U_j$ are pullbacks of larger polydisks. Since $f^2 e^{-\phi}$ is integrable on X_G, (2.1) follows.

The lemma is proved. $\square$

Assume now that under conditions of Lemma 2.3 there is a smooth section w of L^* such that $\overline{\partial} w = \eta$ and $|w|^2 e^{-\phi}$ (respectively, $|w|^2 e^{-(1+\epsilon)\phi}$) is integrable. Let $u_i = v_i - w$. Then u_i is a holomorphic section of L^* on U_i and $u_i - u_j = u_{ij}$, so

$$f_i - u_i \otimes s = f_j - u_j \otimes s \quad \text{on} \quad U_{ij} \ .$$

Hence we obtain a holomorphic extension F of f to Y by setting

$$F = f_i - u_i \otimes s = f_i + w \otimes s - \sum_j (f_i - f_j)\xi_j \quad \text{on} \ \ U_i.$$

The term $w \otimes s$ is L_2 with respect to $e^{-\phi}$ (respectively, $e^{-(1+\epsilon)\phi}$) by construction of w and since s is bounded. The other two terms on the right-hand side can be shown to be L_2 with respect to $e^{-\phi}$ (respectively, $e^{-(1+\epsilon)\phi}$) by arguments similar to those used for η above. Hence $F^2 e^{-\phi}$ (respectively, $F^2 e^{-(1+\epsilon)\phi}$) is integrable.

2.3. Symmetric products of curves. Let Γ be a complex compact curve of genus $g \geq 1$, $\Gamma^{\times g}$ and $S\Gamma^{\times g}$ be the direct and the symmetric products of g-copies of Γ. Then the manifold $S\Gamma^{\times g}$ is the quotient of $\Gamma^{\times g}$ by the action of the permutation group S_g. Therefore there exists a finite holomorphic surjective map $\pi : \Gamma^{\times g} \longrightarrow S\Gamma^{\times g}$. Further, $S\Gamma^{\times g}$ is birational isomorphic to $\mathbb{CT}^g$ (denote this isomorphism by j). Let $(p, ..., p) \in \Gamma^{\times g}$ be a fixed point. Denote by Γ^k, $k \leq g$, submanifold $\{(p, ..., p, z_1, ..., z_k)|\ z_1, ..., z_k \in \Gamma\} \subset \Gamma^{\times g}$.

Lemma 2.4 *For any k, image $\pi(\Gamma^k)$ is a complex submanifold of $S\Gamma^{\times g}$.*

Proof. For a point $y = (p, ..., p, z_1, ..., z_k) \in \Gamma^k$ consider its orbit $o(y) := S_g(y)$. By definition, π maps $o(y)$ to $\pi(y)$ and intersection $o(y) \cap \Gamma^k = \{(p, ..., p, S_k(z))\}$; here $z = (z_1, ..., z_k)$ and S_k is the permutation group acting on the set of k elements. The quotient by the action of S_k is manifold $X_k := (p, ..., p, S\Gamma^{\times k})$. So we have a holomorphic injective mapping $\pi_k : X_k \longrightarrow S\Gamma^{\times g}$ whose image coincides with $\pi(\Gamma^k)$. Now let y_0 be local coordinates in a neighbourhood U_0 of $p \in \Gamma$ and y_i, $1 \leq i \leq k$, be local coordinates in a neighbourhood U_i of $z_i \in \Gamma$ such that $y_0(p) = 0$, $y_i(U_i) \cap y_j(U_j) = \emptyset$ for $z_i \neq z_j$ and $y_i = y_j$ in $U_i = U_j$ for $z_i = z_j$. Denote by $\sigma_1, ..., \sigma_g$ elementary symmetric functions from g variables. For $(z_1, ..., z_g) \in U_0 \times ... \times U_0 \times U_1 \times ... \times U_k \subset \Gamma^{\times g}$ set $u_i(z) = y_0(z_i)$, $1 \leq i \leq g-k$, and $u_i(z) = y_i(z_{g-k+i})$, $1 \leq i \leq k$. By the theorem on symmetric polynomials the mapping

$$f : (w_1, ..., w_g) \mapsto (\sigma_1(u(w)), ..., \sigma_g(u(w)))$$

determines a local coordinate system on $\pi(U_0 \times ... \times U_0 \times U_1 \times ... \times U_k) \subset S\Gamma^{\times g}$ (see [GH, Ch. 2, p. 259]). Then the image of restriction $f|_{\pi(\Gamma_k)}$ belongs to $\mathbb{C}^k \subset \mathbb{C}^g$. By the same reason $f \circ \pi_k$ determines a local coordinate system in the corresponding neighbourhood on X_k. This shows that π_k is a biholomorphic embedding. Thus we proved that $\pi(\Gamma_k)$ is smooth. $\square$

2.4. Norm estimates. Let M and N be compact Riemannian manifolds and $f : M \longrightarrow N$ be a smooth surjective map. Assume that $f_* : \pi_1(M) \longrightarrow \pi_1(N)$ is a surjection. Let G be a quotient group of $\pi_1(N)$ and N_G, M_G regular coverings with the transformation group G over N and M, respectively, such that $M_G = f^*N_G$. Then there is a map $f_G : M_G \longrightarrow N_G$ that covers f. We consider M_G and N_G in the metrics pulled back from M and N, respectively. Further, if $E^p(K)$ is the space of p-forms on a Riemannian manifold K denote by $|\cdot|_x$ the norm in the vector space $E^p(K)|_x (\cong \wedge^p T_x^*)$, $x \in K$, constructed by the metric dual to the Riemannian one.

Lemma 2.5 *Let ω be a bounded differential p-form on N_G, i.e., $\sup_{x \in N_G} |\omega|_x < \infty$. Then there is $C = C(f, p) > 0$ such that*

$$|f_G^*(\omega)|_x \leq C|\omega|_{f_G(x)} \ .$$

Proof. Let us write ω in local orthogonal coordinates lifted from N. Then the compactness arguments show that the statement follows easily from a similar statement for elements of the orthogonal basis. We leave the details to the reader. $\square$

3. Proofs.

We prove Theorem 1.2 only. The proof of Theorem 1.1 is similar and can be obtained by removing ϵ in the arguments below.

3.1. We start by proving Theorems 1.1 and 1.2 for curves.

Proof of Theorem 1.2 for curves. Assume that the Albanese map $A : \Gamma \longrightarrow \mathbb{CT}^g$ is defined with respect to a basic point $p \in \Gamma$. For $X_i := \pi(\Gamma^i) \subset S\Gamma^{\times g}$ consider the flag of submanifolds $X_1 \subset ... \subset X_g = S\Gamma^{\times g}$ (see definitions in Section 2.3). The Jacobi map $j : S\Gamma^{\times g} \longrightarrow \mathbb{C}\mathbb{T}^g$ maps, by definition, X_1 biholomorphically to $A(\Gamma)$ (which we identify with Γ). Moreover, the fundamental group $\pi_1(S\Gamma^{\times g})$ is isomorphic (under j_*) to $\pi_1(\mathbb{C}\mathbb{T}^g) = \mathbb{Z}^{2g}$ and embedding $X_i \subset S\Gamma^{\times g}$ induces a surjective homomorphism of fundamental groups. Thus if G is a quotient group of $\pi_1(\mathbb{C}\mathbb{T}^g)$ one can construct regular coverings X_{iG} over X_i, $i = 1, ..., g$, with transformation group G such that $X_{1G} \subset .... \subset X_{gG}$ is a flag of complex submanifolds covering the flag $X_1 \subset ... \subset X_g$ and there is a proper surjective map with connected fibres $j_G : X_{gG} \longrightarrow T_G$ that covers j.

For any function $f \in \mathcal{H}_\phi(\Gamma_G)$ consider its pullback $f_1 := j_G^*(f)$ on X_{1G}. Then according to Lemma 2.5, f_1 belongs to the space $\mathcal{H}_{j_G^*(\phi)}(X_{1G})$ determined with respect to the pullback of the volume form of X_1. Moreover, $j_G^*(\phi)$ satisfies condition (1.3) (respectively, (1.2)) for the distance d' defined by the pullback of a Kähler metric on X_g. It follows from the inequality

$$d(j_G(x), j_G(y)) \leq C(j_G) d'(x, y) \quad (x, y \in X_{gG}).$$

Now for a sufficiently small $\epsilon > 0$ we prove that f_1 admits an extension $f_2 \in \mathcal{H}_{(1+\epsilon) j_G^*(\phi)}(X_{2G})$ satisfying conditions of Theorem 1.2; f_2 admits a similar extension $f_3 \in \mathcal{H}_{(1+2\epsilon) j_G^*(\phi)}(X_{3G})$ etc. Finally, we obtain an extension $f_g \in \mathcal{H}_{(1+(g-1)\epsilon) j_G^*(\phi)}(X_{gG})$ of f_1. Clearly, f_g is constant on fibres of j_G and thus determines a function $f' \in \mathcal{H}_{(1+(g-1)\epsilon)\phi}(T_G)$ that extends f. Our arguments will guarantee its uniqueness and fulfillment of the required norm estimates. This will finish the proof.

We use inductive arguments. Assume that we have the required extension $f_k \in \mathcal{H}_{(1+(k-1)\epsilon) j_G^*(\phi)}(X_{kG})$ of f_{k-1}. Construct now extension $f_{k+1} \in \mathcal{H}_{(1+k\epsilon) j_G^*(\phi)}(X_{(k+1)G})$.

For each k consider the regular covering Y_k over $\Gamma^k \subset \Gamma^{\times g}$ with the transformation group G. Since the map $j \circ \pi : \Gamma^k \longrightarrow \mathbb{C}\mathbb{T}^g$ is invariant with respect to the action of the permutation group S_k acting on $\Gamma^k (\cong \Gamma^{\times k})$ and $(p, ..., p) \in \Gamma^{\times g}$ is a fixed point with respect to S_k, by the covering homotopy theorem there is a covering action of S_k on Y_k. Moreover, there is a holomorphic map $\pi_G : Y_g \longrightarrow X_{gG}$ that covers $\pi : \Gamma^{\times g} \longrightarrow S\Gamma^{\times g}$ and is invariant with respect to the action of S_g. Consider the orbit $V_k = S_{k+1}(Y_k)$ in Y_{k+1}. Then V_k covers the orbit $W_k = S_{k+1}(\Gamma^k) \subset \Gamma^{k+1}$.

Lemma 3.1 *Divisor W_k determines a positive line bundle E_k over Γ^{k+1}.*

Proof. Assume without loss of generality that $\Gamma^{k+1} = \Gamma^{\times(k+1)}$. Let $P : \Gamma^{k+1} \longrightarrow \Gamma$ be the projection defined by

$$P(z_1, ..., z_{k+1}) = z_1, \quad (z_1, ..., z_{k+1}) \in \Gamma^{\times(k+1)} .$$

Then $P^{-1}(x) = (x, \Gamma^{\times k})$ for a fixed $x \in \Gamma$. Denote by E_x a positive line bundle over Γ defined by the divisor $\{x\}$ and by Θ_x its curvature (for a suitable hermitian metric on E_x) such that $\frac{\sqrt{-1}}{2\pi}\Theta_x$ is a positive (1,1)-form. Let $e_i \in S_{k+1}$, $i = 1, ..., k+1$, be such that $\cup_i e_i^{-1}(\Gamma^k) = S_{k+1}(\Gamma^k)$. Then by definition, $E_k = \otimes_i e_i^*(P^* E_x)$ is a positive line bundle over Γ_{k+1}. In fact, if in local coordinates $P^*\Theta_x = a(z_1, \overline{z_1}) dz_1 \wedge d\overline{z}_1$ with $a(z_1, \overline{z_1}) > 0$, the curvature Θ_k of E_k equals $\sum_{i=1}^k a(z_i, \overline{z_i}) dz_i \wedge d\overline{z}_i$. Clearly, $\frac{\sqrt{-1}}{2\pi}\Theta_k$ is positive implying that E_k is positive. $\square$

Let h_k be a hermitian metric on E_k with the curvature Θ_k. By the same letters we denote the pullback of h_k, E_k and Θ_k to Y_{k+1}. Let L_k be the holomorphic vector bundle on X_{k+1} defined by the divisor X_k and h'_k a hermitian metric on L_k. By the same letters we also denote the pullback of L_k and h'_k to $X_{(k+1)G}$. Below we consider L_k with the weighted metric $e^{(1+k\epsilon)j^*_G(\phi)}h'_k$. By Lemma 2.3 (2) there is a linear continuous mapping $F_{k,\epsilon} : \mathcal{H}_{(1+(k-1)\epsilon)j^*_G(\phi)}(X_{kG}) \longrightarrow Z^{0,1}_2(X_{(k+1)G}, L^*_k)$. Put $\eta_k = F_{k,\epsilon}(f_k)$. Since, by definition, $\pi^{-1}(X_k) \cap \Gamma^{k+1} = W_k$, the bundle $\pi^*_G L_k$ equals E_k on Γ^{k+1}. In particular, $\eta'_k = \pi^*_G(\eta_k)$ is a $\overline{\partial}$-closed (0,1)-form on Y_{k+1} with values in E^*_k and $\eta'_k \in L^{0,1}_2(Y_{k+1}, E^*_k)$ for E equipped with the metric $e^{(1+k\epsilon)\phi'}h_k$ where $\phi' = \pi^*_G(j^*_G(\phi))$. Further, the curvature $\mathcal{R}_k$ of E_k equals $-(1 + k\epsilon)\mathcal{L}(\phi') + \Theta_k$. Moreover, according to Lemma 2.5,

$$|\mathcal{L}(\phi')|_x \leq C(j_G \circ \pi_G, 2)||\mathcal{L}(\phi)|_{(j_G\circ\pi_G)(x)} \quad (x \in Y_g).$$

In particular, there is a positive constant C (depending on Γ only) such that

if $\sup_{x\in T_G} |\mathcal{L}(\phi)|_x < C' < C$ *and* $0 < \epsilon \leq 1/g$, $1 \leq k \leq g$, *there is an* $a = a(C') > 0$ *so that* $\mathcal{R}_k > a\Theta_k$.

Let ϕ satisfy the above condition and $\epsilon < 1/g$. Since Θ_k is a Kähler form on Γ_{k+1}, according to Proposition 2.1 and Remark 2.2 there is a linear continuous mapping $s_k : Z^{0,1}_2(Y_{k+1}, E^*_k) \longrightarrow W^{0,0}_2(Y_{k+1}, E^*_k)$ inverse to $\overline{\partial}$. Then for $r_k = s_k(\eta'_k)$ we have $\overline{\partial} r_k = \eta'_k$ and $r_k \in L_2(Y_{k+1}, E^*_k)$. Applying now arguments similar to those used in Section 2.2 (for the pullback to Y_{k+1} of local extensions of f_k) get a holomorphic function g_{k+1} on Y_{k+1} that extends $\pi^*_G(f_k)$ and belongs to $\mathcal{H}_{(1+k\epsilon)\phi'}(Y_{k+1})$.

Assume also that there is another extension $g' \in \mathcal{H}_{(1+k\epsilon)\phi'}(Y_{k+1})$ of $\pi^*_G(f_k)$. Let s' be the pullback to Y_{k+1} of a holomorphic section of the bundle E_k on Γ_{k+1} with a simple zero along W_k. (Recall that the pullback of E_k we denote by the same letter). Then $d = (g_{k+1} - g')(s')^{-1}$ is an L_2 integrable holomorphic section of E^*_k. Here E_k is taken with the weighted metric $e^{(1+k\epsilon')\phi'}h_k$, where an ϵ' satisfies $\epsilon < \epsilon' < 1/g$. The arguments are similar to those used in the proof of Lemma 2.3. Therefore according to Proposition 2.1 for $q = 0$, the function d is zero. This proves the uniqueness of the extension. Since $\pi^*_G(f_k)$ is invariant with respect to the action of the permutation group S_k, for any $e \in S_k$ the function $e^*(g_{k+1})$ is also an extension of $\pi^*_G(f_k)$ belonging to $\mathcal{H}_{(1+k\epsilon)\phi'}(Y_{k+1})$. Thus the uniqueness of extension implies that $e^*(g_{k+1}) = g_{k+1}$. So there is a uniquely defined holomorphic function f_{k+1} on $X_{(k+1)G}$ such that $\pi^*_G(f_{k+1}) = g_{k+1}$, $f_{k+1} \in \mathcal{H}_{(1+k\epsilon)j^*_G(\phi)}(X_{k+1}G)$ and f_{k+1} is an extension of f_k. In fact our arguments (based on Remark 2.2) show that we constructed a linear continuous extension operator which gives us the required norm estimates. Therefore, by induction, we get a holomorphic function f_g on X_{gG} which belongs to $\mathcal{H}_{(1+(g-1)\epsilon)j^*_G(\phi)}(X_{gG})$ and extends f_1. As it was noted at the beginning of the proof, f_g determines the required extension of f. This proves Theorem 1.2 for curves. $\square$

Proof of Theorem 1.2 for projective manifolds. Let M be a projective manifold of dimension $n \geq 2$ with a very ample line bundle L and with a Kähler form ω. We may think of M as embedded in some projective space and of L as the restriction to M of the hyperplane bundle with the standard positively curved metric. Then zero loci of sections of L are hyperplane sections of M. By Bertini's theorem, the generic linear subspace of codimension $n - 1$ intersects M transversely in a smooth curve C. By the Lefschetz hyperplane theorem, C is connected and the map $\pi_1(C) \longrightarrow \pi_1(M)$ is surjective. Let M_G be the regular covering over M with a free abelian transformation group G. Then the regular covering C_G over C with the same transformation group G is embedded into M_G. Assume that $f \in \mathcal{H}_{\tilde{\phi}}(M_G)$ with $\tilde{\phi}$ satisfying (1.3). Then $g := f|_{C_G}$ belongs to $\mathcal{H}_{(1+\epsilon)\tilde{\phi}}(C_G)$ for any positive ϵ. Indeed, let $U_1, ..., U_N$ be the pullbacks to M_G of shrunk coordinate polydisks covering an open neighbourhood of $C \subset M$ and $V_i \supset U_i$ be pullbacks of larger polydisks. We may assume that $C_G \cap V_i = \{z_1 = 0\}$, $i = 1, ..., N$, for the pullback of the corresponding local coordinates. Then application of (1.3) and subharmonicity of $|f|^2$ get

$$\int_{C_G \cap U_i} |f|^2 e^{-(1+\epsilon)\tilde{\phi}} \omega \leq c(M) \int_{V_i} |f|^2 e^{-\tilde{\phi}} \omega^n < \infty .$$

This implies $g \in \mathcal{H}_{(1+\epsilon)\tilde{\phi}}(C_G)$. Let $C = M_1 \subset M_2 \subset ... \subset M_n = M$ be a flag of projective submanifolds of M, where M_i is intersection of M with the generic linear subspace of codimension $n - i$. Let $C_G = M_{1G} \subset ... \subset M_{nG} = M_G$ be the flag of the corresponding regular coverings with the transformation group G. Then the arguments similar to those used in Section 3.1 (see also arguments in Theorem 3.1 of [La]) show that if L is very ample then g admits a unique extension $f' \in \mathcal{H}_{(1+\epsilon+\delta)\tilde{\phi}}(M_G)$ for a sufficiently small positive ϵ and $\delta = \delta(\epsilon)$. But clearly in this case $f = f'$. Thus we proved that f is uniquely determined by $f|_{C_G}$.

Let now $A : M \longrightarrow \mathbb{CT}^k$ be the Albanese map for M defined with respect to a point $p \in C$ by integration of holomorphic 1-forms $\omega_1, ..., \omega_k \in \Omega^1(M)$ (generating a basis there). Set $\eta_i := \omega_i|_C$ for $i = 1, ..., k$. Then by the Lefschetz theorem $\eta_1, ..., \eta_k$ are linearly independent in $\Omega^1(C)$. Choose 1-forms $\eta_{k+1}, ..., \eta_s \in \Omega^1(C)$ such that $\eta_1, ..., \eta_s$ generates a basis. Further, define the Albanese map $A' : C \longrightarrow \mathbb{CT}^s$ with respect to the point p by integration the forms of this basis. Then according to our construction there is a surjective map $P : \mathbb{CT}^s \longrightarrow \mathbb{CT}^k$ whose fibres are complex tori such that $P_* : \pi_1(\mathbb{CT}^s) \longrightarrow \pi_1(\mathbb{CT}^k)$ is a surjection and $A = P \circ A'$. Denote by T'_G a regular covering over $\mathbb{CT}^s$ with the transformation group G. Then there is a complex map $P_G : T'_G \longrightarrow T_G$ that covers P whose fibres are also tori. Let $A'_G : C_G \longrightarrow T'_G$ be the map covering A' and $\phi' = P^*_G(\phi)$. Note that ϕ' satisfies (1.3) on T'_G and $(A'_G)^*(\phi') = \tilde{\phi}|_{C_G}$. Applying Theorem 1.2 for curves to the map $A'_G : C_G \longrightarrow T'_G$ and the function ϕ' we obtain

there is $C = C(M, A') > 0$ such that for $|\mathcal{L}(\phi')| < C' < C$ and for sufficiently small positive numbers $\epsilon \leq \epsilon(C')$, $\delta \leq \delta(C')$ there is a uniquely defined holomorphic function $\tilde{f} \in \mathcal{H}_{(1+\epsilon+\delta)\phi'}(T'_G)$ satisfying $g = (A'_G)^*(\tilde{f})$ and $|\tilde{f}| \leq C(\epsilon, \delta)|g|$ (in the corresponding L_2-norms).

Since P_G is a proper map with connected fibres, $\tilde{f}$ determines a function $h \in \mathcal{H}_{(1+\epsilon+\delta)\phi}(T_G)$ such that $A^*_G(h)|_{C_G} = g$ and $|h| \leq \tilde{C}(\epsilon, \delta)|g|$. But as we proved, f is uniquely determined by $g = f|_{C_G}$ and $|g| \leq c(\epsilon)|f|$. Therefore $A^*_G h = f$ and h satisfies the required norm estimate. Finally, by Lemma 2.5, $|\mathcal{L}(\phi')| \leq c(P)|\mathcal{L}(\phi)|$ and so the above extension theorem is valid for any B' satisfying $|\mathcal{L}(\phi)| < B' < C/c(P)$.

This completes the proof of Theorem 1.2 for projective manifolds. $\square$

3.2. Proof of Theorem 1.3. Let M_G be a regular covering over M with the transformation group G and $A_G : M_G \longrightarrow T_G$ be the covering of the Albanese map

$A : M \longrightarrow \mathbb{CT}^n$. Assume that f is a holomorphic function on M_G satisfying

$$|f(z)| \le c(\epsilon)e^{\epsilon r(z)} \quad (z \in M_G)$$

for any $\epsilon > 0$. Let ϕ be the distance from a fixed point in T_G in the flat metric pulled back from $\mathbb{CT}^n$ and $\tilde{\phi} = A_G^*(\phi)$. Further, by ρ_G denote the distance from 0 on $G(\cong \mathbb{Z}^k)$ determined with respect to the word metric. Since by our construction growth of r and $\tilde{\phi}$ is equivalent to growth of ρ_G, the function f belongs to $\mathcal{H}_{\epsilon\tilde{\phi}}(M_G)$ for any $\epsilon > 0$. We now apply Theorem 1.1. Here we assume that $|\mathcal{L}(\phi)|$ is sufficiently small replacing if necessary ϕ by a smooth function ϕ_1 with the same growth such that $|\mathcal{L}(\phi_1)|$ is small. In fact ϕ_1 can be constructed as follows.
Note, first, that T_G is diffeomorphic to $\mathbb{T}^{2n-k} \times \mathbb{R}^k$ where second derivatives of the diffeomorphism are bounded in the flat coordinate system on T_G. Then put $\phi_1(v, x) := \sqrt{p + |x|^2}$, for $(v, x) \in \mathbb{T}^{2n-k} \times \mathbb{R}^k$, where $|x|$ is the Euclidean norm of $x \in \mathbb{R}^k$ and p is sufficiently big positive number.

Further, according to Theorem 1.1 there is a uniquely defined holomorphic function $f' \in \cap_{\epsilon>0}\mathcal{H}_{\epsilon\phi}(T_G)$ such that $A_G^*(f') = f$. Prove now that f' is a constant.

We regard the maximal free abelian subgroup $\Gamma \subset H_1(M, \mathbb{Z})$ as a lattice in $\mathbb{C}^n$ determining $\mathbb{CT}^n$ and $H \subset \Gamma$ as a sublattice such that the minimal complex vector space containing H is $\mathbb{C}^n$. Consider the pullback g of f' to $\mathbb{C}^n$. Clearly g is invariant with respect to the action (by shifts) of H and satisfies

$$|f(z)| \le c(\epsilon)e^{\epsilon|z|}$$

for any positive ϵ. For an element $e_1 \in H$ let X_1 be a minimal complex vector space containing $\{ne_1\}_{n\in\mathbb{Z}}$. For any $z \in \mathbb{C}^n$ consider restriction $g' = g|_{z+X_1}$. We identify $z + X_1$ with $\mathbb{C}$ and $\{ne_1\}_{n\in\mathbb{Z}}$ with $\mathbb{Z}$. Then g' is a holomorphic function on $\mathbb{C}$ of an arbitrary small exponential type which is constant on $\mathbb{Z}$. Therefore by Cawrtright's theorem [Ca], g' is constant on $\mathbb{C}$. This implies that $g(z + v) = g(z)$ for any $z \in \mathbb{C}^n$ and $v \in X_1$. In particular, there is a holomorphic function g_1 on the quotient $\mathbb{C}^n/X_1 = \mathbb{C}^{n-1}$ of an arbitrary small exponential type whose pullback to $\mathbb{C}^n$ coincides with g. Denote by H_1 image of H in $\mathbb{C}^n/X_1 = \mathbb{C}^{n-1}$. By definition, g_1 is invariant with respect to the action of H_1 and the minimal complex vector space containing H_1 is $\mathbb{C}^{n-1}$. Choose $e_2 \in H_1$ and denote by X_2 the minimal complex subspace containing $\{ne_2\}_{n\in\mathbb{Z}}$. Applying the very same arguments we get $g_1(z + v) = g_1(z)$ for any $z \in \mathbb{C}^{n-1}$ and $v \in X_2$. Continuing by induction we finally obtain that the initial function g is constant.

This completes the proof of the theorem. $\square$

Note that our arguments give a more general statement.

Theorem 3.2 *Let H be such that $\Lambda(H) = \Omega^1(M)$. Then there is a positive constant $\sigma = \sigma(M)$ such that any holomorphic on M_G function f satisfying*

$$|f(z)| \le ce^{\sigma' r(z)} \qquad (0 < \sigma' < \sigma,\ z \in M_G)$$

is a constant.

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